FRACTURE STATISTICS OF GLASS CYLINDERS BROKEN BY COMPRESSION

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ABSTRACT

A statistical analysis of the compression-fracture stress of commercial glass cylinders was made, and the parameters of Weibull's normal and logarithmic-normal frequency functions are estimated by means of maximum likelihood method. The dispersion of these parameters is determined using the information Fischer matrix, and simulation. The influence of the specimen size is determined experimentally and is qualitatively in keeping with Weibull's predictions. The χ^2 is acceptable for the different frequency functions used. But a theoretical analysis shows that the Weibull Frequency function is the only one valid in fracture statistics.

KEYWORDS

Fracture statistics; Weibull frequency function; maximum likelihood; dispersion of parameters; specimen size.

INTRODUCTION

In uniaxial compression of brittle bodies, Jayatilaka and Trustrum [1] predicted a normal distribution with mean independent of volume and variance inversely proportional thereto. A comparison of this prediction using compacted cement cylinders broken by compression was made by Kittl and Aldunate [2], and the result thereof was that the χ^2 test does not reject the Weibull or the normal distributions. But the normal distribution is a little better than the Weibull one. The mean was independent of the volume, as was the variance. In the case of the same material subjected to traction [3] the same thing happens as under compression. The justification of the Jayatilaka and Trustrum prediction is that the final failure of a brittle material, under compression, occurs only after a certair proportion of the cracks have failed, where this proportion is a material property. The general view of the problem is complex because though it is possible to justify the independence of the volume adopting the Jayatilaka and Trustrum model, this is not true for the case of traction. In keeping with the preceding observation it is possible that glass is more like a Weibull body under compression. The Kittl and Gunther justification [3] for the independence of the volume under traction is that, in the cement paste, the size of the defects does not increase with the volume and the Weibull statistics must be modified, and it is assumed that this is not true for glass. This work deals with these last points and with uncertainty in the determination of the parameters.

THEORETICAL ESTIMATION OF THE PARAMETERS OF THE DISTRIBUTIONS.

Under compression the Weibull cumulative fracture probability is [2]:

$$F(\sigma) = 1 - \exp\left\{-\frac{V}{V_0}\left(\frac{\sigma-\sigma_L}{\sigma_0}\right)^m\right\} \tag{1}$$

where σ is the failure stress, σ_L, σ_0 and m are Weibull's parameters and V_0 is the volume unity. An estimation of m, σ_0 is made by Trustrum and Jayatilaka [5] but with $\sigma_L = 0$. We assumed here that $\sigma_L \neq 0$.

The likelihood function is

$$L(m, \sigma_L, \sigma_0) = \prod_{i=1}^{n} f(\sigma_i; m, \sigma_L, \sigma_0) \tag{2}$$

where $f(\sigma)$ is the probability density function. Then the maximum-likelihood estimators of the parameters m, σ_L, σ_0 are the random variables $\hat{m} = d_1(\sigma_1, \ldots, \sigma_n)$; $\hat{\sigma}_L = d_2(\sigma_1, \ldots, \sigma_n)$; $\hat{\sigma}_0 = d_3(\sigma_1, \ldots, \sigma_n)$ such that the values $\hat{m}_L$, $\hat{\sigma}_{L_L}$, $\hat{\sigma}_{0L}$ maximize $L(m, \sigma_L, \sigma_0)$.

The point where likelihood reaches a maximum is a solution of the three equations

$$\frac{\partial L}{\partial m} = \frac{\partial \ln L}{\partial m} = 0 \;;\quad \frac{\partial L}{\partial \sigma_L} = \frac{\partial \ln L}{\partial \sigma_L} = 0 \;;\quad \frac{\partial L}{\partial \sigma_0} = \frac{\partial \ln L}{\partial \sigma_0} = 0 \tag{3}$$

A random sample of size n from Weibull distribution has the density

$$L = \prod_{i=1}^{n} \frac{m\,V}{\sigma_0\,V_0}\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right)^{m-1} \exp\left\{-\frac{V}{V_0}\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right)^m\right\} \tag{4}$$

To find the location of its maximum we compute

$$\frac{\partial \ln L}{\partial m} = \frac{n}{m} + \sum_{i=1}^{n} \ln\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right) - \frac{V}{V_0}\sum_{i=1}^{n}\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right)^m \ln\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right) = 0 \tag{5}$$

$$\frac{\partial \ln L}{\partial \sigma_0} = \frac{m}{\sigma_0}\left[-n + \frac{V}{V_0}\sum_{i=1}^{n}\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right)^m\right] = 0$$

$$\frac{\partial \ln L}{\partial \sigma_L} = -(m-1)\sum_{i=1}^{n}(\sigma_i-\sigma_L)^{-1} + \frac{m\,V}{\sigma_0\,V_0}\sum_{i=1}^{n}\left(\frac{\sigma_i-\sigma_L}{\sigma_0}\right)^{m-1} = 0$$

These equations can be solved by the Newton-Raphson method.

Maximum-likelihood estimators are asymptotically efficient estimators and BAN (Best Asymptotically Normal) estimators, consistent estimators and squared-error consistent estimators, functions of the minimal sufficient statistics, and the estimators of the functions are a function of the estimators. For large samples the maximum-likelihood estimators approximately distributed by the multivariate normal distribution with means m, σ_L, σ_0, have a Fischer information matrix for n elements nR in the quadratic form [4], where

$$r_{ij} = -E\left(\frac{\partial^2 \ln f(\sigma; m, \sigma_L, \sigma_0)}{\partial \theta_i\, \partial \theta_j}\right), \quad \{\theta\} = \{m, \sigma_0, \sigma_L\} \tag{6}$$

and E means the expected value operator. The variances and covariances of the estimators are $1/n\ R^{-1}$. The r_{ij} matrix elements are:

$$r_{11} = \frac{1}{m^2}\left(1.82379 - 0.84555 \ln\frac{V}{V_0} + \ln^2\frac{V}{V_0}\right)$$

$$r_{12} = -\frac{1}{\sigma_0}\left(0.42277 - \ln\frac{V}{V_0}\right)$$

$$r_{13} = \frac{1}{\sigma_0}\left(\frac{V}{V_0}\right)^{1/m}\left\{\frac{1}{m}\Gamma\left(1-\frac{1}{m}\right) - J(m) + \left(1-\frac{1}{m}\right)\Gamma\left(1-\frac{1}{m}\right)\times\cdots \right. \tag{7}$$

$$r_{22} = \frac{m^2}{\sigma_0^2} \qquad\qquad \left.\cdots\times \ln\frac{V}{V_0}\right\}$$

$$r_{23} = \left(\frac{m}{\sigma_0}\right)^2\left(1-\frac{1}{m}\right)\left(\frac{V}{V_0}\right)^{1/m}\Gamma\left(1-\frac{1}{m}\right)$$

$$r_{33} = \left(\frac{m-1}{\sigma_0}\right)^2\left(\frac{V}{V_0}\right)^{2/m}\Gamma\left(1-\frac{2}{m}\right)$$

where

$$\Gamma(m) = \int_0^\infty t^{m-1} e^{-t}\, dt \,, \quad J(m) = \int_0^\infty t^{1-\frac{1}{m}} \ln t\, e^{-t}\, dt$$

For the estimation using the method of least squares we take into account equation (1). Hence

$$Y = \ln\left\{\ln\left[\frac{1}{1-F(\sigma)}\right]\right\} = m \ln(\sigma-\sigma_L) + \ln\left(\frac{V}{\sigma_0^{\,m} V_0}\right) \tag{8}$$

if, $\sigma_1 \le \sigma_2 \le \dots \le \sigma_n$. Then as shown by Jayatilaka [5] the expected value of $F(\sigma_i)$ with less biasing is (i - 1/2)/n, the linear model is

$$Y_i = \ln\left\{\ln\left[\frac{1}{1-(i-0.5)/n}\right]\right\} = m \ln(\sigma_i - \sigma_L) + \ln\left(\frac{V}{V_o\,\sigma_o^m}\right) + \varepsilon_i \quad (9)$$

where $E(\varepsilon_i) = 0$, ε_i is the error arising from the difference between the observed value of $F(\sigma_i)$ and its expected value. Then the least square estimators that maximize the correlations coefficient ρ are $\hat{m}_s$, $\hat{\sigma}_{os}$, $\hat{\sigma}_{Ls}$. Therefore:

$$\hat{m}_s = \frac{\sum_i Y_i \ln(\sigma_i - \hat{\sigma}_{Ls}) - \frac{1}{n}\sum_i Y_i \sum_i \ln(\sigma_i - \hat{\sigma}_{Ls})}{\sum_i \ln^2(\sigma_i - \hat{\sigma}_{Ls}) - \frac{1}{n}\left\{\sum_i \ln(\sigma_i - \hat{\sigma}_{Ls})\right\}^2}$$

$$\hat{\sigma}_{os} = \left\{\frac{V}{V_o}\exp\left[-\frac{1}{n}\sum_i Y_i + \frac{\hat{m}_s}{n}\sum_i \ln(\sigma_i - \hat{\sigma}_{Ls})\right]\right\}^{1/\hat{m}_s} \quad (10)$$

$$\hat{\sigma}_{Ls} = \hat{\sigma}_{LL}$$

where

$$\rho(\hat{m}_s, \hat{\sigma}_{os}, \hat{\sigma}_{Ls}) \quad \text{maximum} \quad (\rho^2 \le 1.0)$$

Under the assumptions of the linear model, that is to say $E(\varepsilon) = 0$ and $Var(\varepsilon_i) = \sigma^2$, the properties of least square estimators are that they are minimum-variance unbiased. But eqn (9) does not comply with the linear-model hypothesis. For the estimation by the method of moments we proceed as follows: The first moment μ_1 is called the mean of σ, and the second moment is called the variance

$$\mu_1 = \int_{\sigma_L}^{\infty} \sigma f(\sigma)\, d\sigma = \sigma_o \left(\frac{V}{V_o}\right)^{-1/m} \Gamma\left(1 + \frac{1}{m}\right) + \sigma_L$$

$$\mu_2 = \int_{\sigma_L}^{\infty} (\sigma - \mu_1)^2 f(\sigma)\, d\sigma = \sigma_o^2 \left(\frac{V}{V_o}\right)^{-2/m} \left\{\Gamma\left(1 + \frac{2}{m}\right) - \Gamma^2\left(1 + \frac{1}{m}\right)\right\} \quad (11)$$

The set moments are $\hat{\mu}_1 = 1/n \sum_i^n \sigma_i$, $\hat{\mu}_2 = 1/n-1 \sum_{i=1}^n (\sigma_i - \hat{\mu}_1)^2$; hence the equations of moment are $\hat{\mu}_1 = \mu_1$, $\hat{\mu}_2 = \mu_2$. The third moment μ_3, called a measure of asymmetry, is not used. Therefore the third condition consists of using m obtained by least square.

Under quite general conditions it can be shown that the estimators derived by the method of moments are consistent and asymptotically normal.

In the normal distribution the cumulative probability of failure is

$$F(\sigma) = \frac{1}{S_\sigma \sqrt{2\pi}} \int_{-\infty}^{\sigma} \exp\left\{-\frac{(\sigma - \bar{\sigma})^2}{2 S_\sigma^2}\right\} d\sigma \quad (12)$$

S_σ and $\bar{\sigma}$ are estimated by $\bar{\sigma} = 1/n \sum_{i=1}^n \sigma_i$ and $S_\sigma^2 = 1/n-1 \sum_{i=1}^n (\sigma_i - \bar{\sigma})^2$.

In the logarithmic normal distributions

$$F(\sigma) = \frac{1}{\beta\sqrt{2\pi}} \int_0^{\sigma} \frac{1}{\sigma} \exp\left\{-\frac{(\ln\sigma - \alpha)^2}{2\beta^2}\right\} d\sigma \quad (13)$$

Maximum-likelihood estimators of the parameters α and β are

$$\hat{\alpha} = \frac{1}{n}\sum_{i=1}^{n} \ln\sigma_i \quad , \quad \hat{\beta}^2 = \frac{1}{n-1}\sum_{i=1}^{n} (\ln\sigma_i - \hat{\alpha})^2 \quad (14)$$

Regarding the log normal displacement distribution, the cumulative probability is

$$F(\sigma) = \frac{1}{\delta\sqrt{2\pi}} \int_{\psi}^{\sigma} \frac{1}{\sigma - \psi} \exp\left\{-\frac{[\ln(\sigma - \psi) - \gamma]^2}{2\delta^2}\right\} d\sigma \quad , \quad \sigma > \psi \quad (15)$$

$$F(\sigma) = 0 \quad , \quad \sigma \le \psi$$

The estimator of the parameters ψ, γ, δ was calculated by the following method. Calculating the three empirical moments $\hat{\mu}_1$, $\hat{\mu}_2$ and $\hat{\mu}_3 = 1/n \sum_{i=1}^n (\sigma_i - \hat{\mu}_1)^3$ the skewness is $S_K = \hat{\mu}_3\, \hat{\mu}_2^{-3/2}$. Defining

$$\eta = \left\{\frac{1}{2} S_K + \sqrt{1 + \frac{1}{4} S_K^2}\right\}^{1/3} + \left\{\frac{1}{2} S_K - \sqrt{1 + \frac{1}{4} S_K^2}\right\}^{1/3} \quad (16)$$

then the estimators of the parameters are $\hat{\delta}^2 = \ln(1 + \eta^2)$, $\hat{\psi} = \hat{\mu}_1 - \sqrt{\hat{\mu}_2}/\eta$ and $\hat{\gamma} = \ln(\hat{\mu}_1 - \hat{\psi}) - 1/2\, \hat{\delta}^2$.

EXPERIENCES CARRIED OUT, AND ESTIMATION OF THE PARAMETERS

For making a statistical comparison the test was performed for different distributions (Weibull, Normal, Log-Normal, Log-Normal displacement) of the 30-sample set.

Commercial glass rod was used for cutting sets of 30 elements, one set being 0.014 m high and 0.007 m in diameter, and the other set being 0.010 m high and 0.005 m in diameter. The faces of the cylinders were polished to get a uniform compression stress and then the cylinders were broken using a Monsanto Manual Machine.

The statistical parameter for different distributions was estimated by the method developed above (see tables 3 and 4), and the χ^2 test was performed for 95% confidence, with V = K - 1 - P where K is the number of intervals and P is the number of parameters estimated (K = 5 and 6 with interval width of 39.2 MPA for sets I and II) (table 5).

TABLE 1 - Failure Stress of Glass Cylinder Broken by Compression. Set I is 0.014 m high and 0.007 m in diameter (volume $v = 0.539$ cm^3) (1 MPA = $9.807 \cdot 10^{-2}$ kg/cm^2)

σ (MPA)	n = 30	
113.3	173.2	229.2
113.8	175.7	235.6
117.2	182.1	236.9
117.7	183.4	239.4
137.5	184.6	245.8
140.1	188.5	247.0
140.6	191.0	248.3
152.8	191.5	249.6
163.0	198.6	257.2
168.1	219.0	298.0

TABLE 2 - Failure Stress of Glass Cylinder Broken by Compression. Set II is 0.010 m high and 0.005 m in diameter (volume $v = 0.196$ cm^3)

σ (MPA)	n = 30	
141.3	194.7	247.1
142.3	199.7	259.6
144.8	204.7	262.1
147.3	208.7	267.1
152.3	209.7	287.1
162.3	214.7	289.6
174.7	217.2	297.0
177.2	224.7	304.5
179.7	229.6	339.5
189.7	239.6	344.5

TABLE 3 - Statistical Parameters of Weibull Distribution for Sets I, II. The dispersion for estimators by maximum likelihood is shown in brackets.

Set I

Method	m	σ_o(MPA)	σ_1(MPA)
maximum likeli-hood	2.19 (0.37)	85.8 (12.4)	90.5 (7.0)
moment	2.19	86.4	89.8
least square	2.19	92.7	83.3

Set II

Method	m	σ_o(MPA)	σ_1(MPA)
maximum likeli-hood	1.33 (0.20)*	26.6 (3.2)*	138.0 (4.4)*
moment	1.47	29.8	139.4
least square	1.47	34.1	138.0

* calculated by simulation

TABLE 4 - Statistical Parameters of Normal, Log-Normal, Log-Normal Displacement for Set I, II.

Distribu tion	Para-meter	Set I	Set II
Normal	$\bar{\sigma}$ (MPA)	191.3	221.7
	S_σ(MPA)	48.9	57.1
Log-Normal	α **	7.542	7.691
	β **	0.267	0.257
Log Nor-mal dis-placed	ψ **	-14461.7	-1652.9
	δ **	0.030	0.148
	γ **	9.705	8.262

** calculated with failure stress in kg/cm^2

TABLE 5 - Statistical Comparison with χ^2 Test for 95% of Confidence. (For Weibull distribution the parameter obtained by maximum likelihood was used) V is the number of degrees of freedom.

Distri bution	Actual χ^2		V		χ^2 95%	
	I	II	I	II	I	II
Weibull	2.40	1.64	1	2	3.84	5.99
Log-N displac	2.47	2.50	1	2	3.84	5.99
Normal	2.48	4.10	2	3	5.99	7.81
Log-N	2.94	1.73	2	3	5.99	7.81

THE UNICITY OF THE FRACTURE STATISTICS AND EXPERIMENTAL RESULTS

Before discussing the results we would like to prove the unicity of the fracture statistics. If we suppose that in a body with brittle fracture we make a hypothetical subdivision in two volumes $V_1 \cup V_2 = V$,

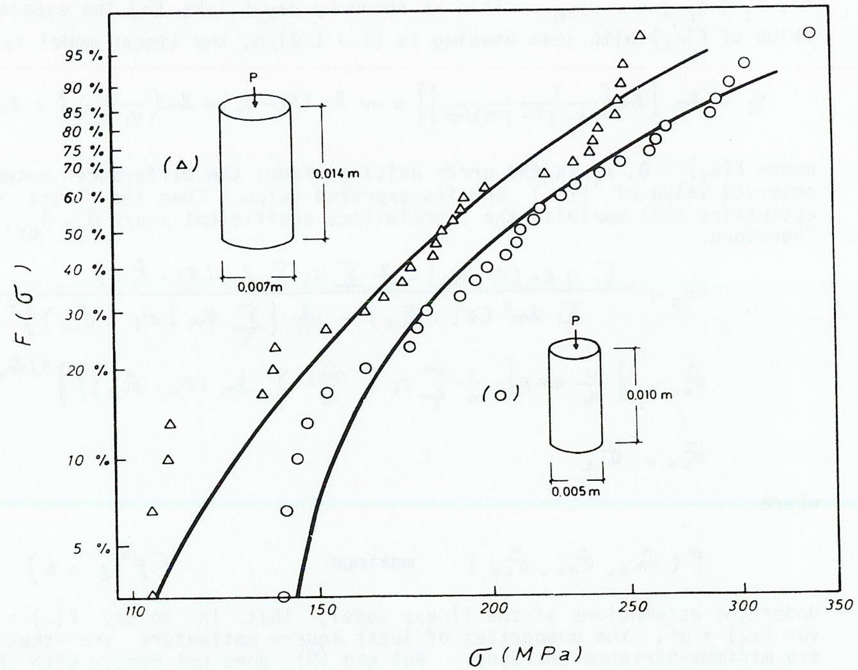

Fig. 1. Weibull plot ln (ln (1/1-F (σ)) as a function of lnσ for glass cylinders broken by compression.

$V_1 \cap V_2 = \emptyset$ without common points and that there are m and n cracks with p and q survivals in a field stress σ then the probability of survival of part V_1 is $\tilde{F}(V_1) = P/m$ and that of part V_2 is $\tilde{F}_2(V_2) = q/n$. The total possibilities of propagation are mxn and the number of possibilities of non-fracture in the total body is pxq. Hence the probability of non-fracture is $\tilde{F}_{12}(V_1 + V_2) = \tilde{F}_1(V_1) \cdot \tilde{F}_2(V_2)$. But it is necessary that $\tilde{F}_{12}(0) \equiv \tilde{F}_1(0) \equiv \tilde{F}_2(0) = 1$, which means that the probability of fracture of a very small body is null. If $V_1 = 0$ or $V_2 = 0$ then we get $\tilde{F}_{12}(V_1) = \tilde{F}_1(V_1)$ and $\tilde{F}_{12}(V_2) = \tilde{F}_2(V_2)$ from the preceding formulas, or in a more general expression $\tilde{F}_{12}(V) = \tilde{F}_1(V) = \tilde{F}_2(V)$.

The preceding paragraph means that there is only one function with the property $\tilde{F}(V_1 + V_2) = \tilde{F}(V_1) \cdot \tilde{F}(V_2)$. But this functional equation with the conditions $\tilde{F}(0) = 1$ and $\tilde{F}(\infty) = 0$ has only one analytical solution, well-known from Euler's times:

$$F(\sigma, V) = 1 - \exp\left\{-\frac{V\,\phi(\sigma)}{V_0}\right\} \qquad (17)$$

where $F = 1 - \tilde{F}$ and $\phi(\sigma)$ is the specific risk of fracture. Equation (17) is the celebrated Weibull expression for the cumulative probability of fracture in a constant stress field. The common expression of $\phi(\sigma)$ is:

$$\phi(\sigma) = \left(\frac{\sigma - \sigma_L}{\sigma_0}\right)^m, \quad \sigma > \sigma_L$$

$$\phi(\sigma) = 0, \quad \sigma \leqslant \sigma_L \qquad (18)$$

In accordance with the preceding theorem the only possibilities for a departure from the Weibull law (17) are: 1) The body is not isotropic; 2) The actual stress field is not the one that is used in the computations; 3) The cracks can propagate without fracture of the body; 4) $\phi(\sigma)$ is not appropriate. It is possible that some combination of slight departures of these possibilities may give for the normal distribution a more appropriate adaptation to the experimental results. In order to improve the investigations on compression, a better knowledge of the stress field is necessary, because the isotropy is insured in this case. In the case of glass the influence of the volume follows the Weibull prediction in a proportion of 30% (see Fig. 1); hence, when some crack propagates in the stress field the glass body is broken.

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