FATIGUE FAILURE CRITERIA FOR MATERIALS UNDER RANDOM TRIAXIAL STATE OF STRESS

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SUMMARY

Five fatigue failure criteria are formulated for a random triaxial state of stress whose components have zero mean values. It is assumed that fatigue failure is determined by the stress and strain components that act on an expected fracture plane. It is shown that in some special cases the proposed failure theories reduce to classical theories applied for sinusoidal stresses.

KEYWORDS

Fatigue failure criteria; stochastic loadings; triaxial state of stress.

1. INTRODUCTION

Loadings that change randomly with time result in random states of stress. It means that material undergoes fatigue under the influence of stress components that are stochastic processes and the principal stresses change their magnitudes and directions in a random way.

The available test results for materials under complex states of stress are limited to a harmonic in-phase [2,3,6,7] and out-of-phase loading [1,8]. The only stochastic loadings experimentally studied so far do not extend beyond the uniaxial case [4,5].

It is clear therefore that any prospective fatigue research attempts involving a multiaxial stochastic loading should be based on sound theoretical foundations. With such an aim in mind the present author has formulated five fatigue failure criteria for isotropic materials under a random triaxial state of stress with zero mean value components. It is assumed that random stress tensor is a 6-dimensional stationary and ergodic Gaussion process with low-band frequency. The location of a fracture plane will be given by the mean values of direction cosines $\hat{l}_n$, $\hat{m}_n$, $\hat{n}_n$ (n=1,2,3) of the principal stress axes $\sigma_1 \geq \sigma_2 \geq \sigma_3$. The following notation will be used: σ_{az} will stand for the fatigue limit normal stress determined in zero mean load uniaxial push-pull tests and τ_{az} will denote the fatigue limit shear stress determined under alternating torsion loading conditions.

2. CRITERION OF THE MAXIMUM NORMAL STRESS ACTING ON A FRACTURE PLANE

We shall assume that:

1. Fatigue fracture is caused by the normal stress $\sigma_\eta(t)$ acting on a fracture plane (t-time).
2. A fracture plane is perpendicular to the mean direction along which the maximum principal stress $\sigma_1(t)$ acts.

According to the above assumptions a fracture plane is defined by the unit normal vector

$$\bar{\eta} = \hat{l}_1 \bar{i} + \hat{m}_1 \bar{j} + \hat{n}_1 \bar{k} ,$$

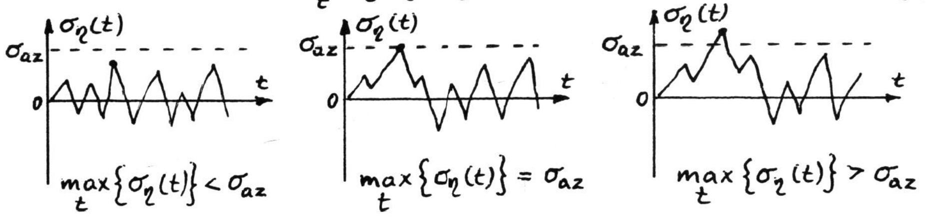
where $\bar{i}, \bar{j}, \bar{k}$ are cartesian basis vectors.

The normal stress $\sigma_\eta(t)$ acting in the plane is

$$\sigma_\eta(t) = \hat{l}_1^2 \sigma_{xx}(t) + \hat{m}_1^2 \sigma_{yy}(t) + \hat{n}_1^2 \sigma_{zz}(t) + 2\hat{l}_1 \hat{m}_1 \sigma_{xy}(t) + 2\hat{l}_1 \hat{n}_1 \sigma_{xz}(t) + 2\hat{m}_1 \hat{n}_1 \sigma_{yz}(t) \quad (1)$$

If $\sigma_\eta(t)$ exceeds σ_{az} then structural damage will accumulate in a material, resulting in fracture. For a limit state corresponding to the fatigue strength the mathematical form of the criterion is (Fig.1)

$$\max_t \{\sigma_\eta(t)\} = \sigma_{az} \quad (2)$$



no fatigue fracture | limit state | fatigue fracture appear

Fig.1 Results of acting $\sigma_\eta(t)$ with various maximum values in time

In oder to show that in a special case the above criterion reduces to the classical maximum principal stress theory under sinusoidal stresses [5]

$$\sigma_{a1} = \sigma_{az} \quad (3)$$

it is sufficient to assume that normal stresses $\sigma_{ii}(t), (i = x,y,z)$ have constant directions along the x,y,z axes, respectively, and are given by the formula

$$\sigma_{ii}(t) = \sigma_{an} \sin\omega t \ , (n = 1,2,3), (i = x,y,z) \quad (4)$$

Taking additionally $\hat{l}_1 = 1$, we obtain

$$\max_t \{\sigma_\eta(t)\} = \sigma_{a1} = \sigma_{az}$$

3. CRITERION OF THE MAXIMUM STRAIN IN THE DIRECTION PERPENDICULAR TO A FRACTURE PLANE

Here we will assume that:

1. Fatigue fracture is caused by the strain $\varepsilon_\eta(t)$ appearing in the direction $\bar{\eta}$ perpendicular to a fracture plane.
2. A fatigue fracture plane is perpendicular to a mean direction which is the direction of the principal strain $\varepsilon_1(t)$.

Since for an isotropic body the principal stress and strain directions coincide the two fracture planes determined by the mean directions of the maximum principal stress and strain will also coincide.

The strain $\varepsilon_\eta(t)$ in the direction $\bar{\eta}$ perpendicular to a fracture plane may be expressed, similarly as $\sigma_\eta(t)$, in the following way

$$\varepsilon_\eta(t) = \hat{l}_1^2 \varepsilon_{xx}(t) + \hat{m}_1^2 \varepsilon_{yy}(t) + \hat{n}_1^2 \varepsilon_{zz}(t) + 2\hat{l}_1 \hat{m}_1 \varepsilon_{xy}(t) + 2\hat{l}_1 \hat{n}_1 \varepsilon_{xz}(t) + 2\hat{m}_1 \hat{n}_1 \varepsilon_{yz}(t) \quad (5)$$

Making use of Hooke's law

$$\varepsilon_{ij} = \frac{1+\nu}{E}\left(\sigma_{ij} - \frac{\nu}{1+\nu}\sigma_{kk}\delta_{ij}\right), \ (i,j,k = x,y,z) \quad (6)$$

where E - Young's modulus, ν - Poisson ratio and

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

we obtain

$$E\varepsilon_\eta(t) = [\hat{l}_1^2(1+\nu) - \nu]\sigma_{xx}(t) + [\hat{m}_1^2(1+\nu) - \nu]\sigma_{yy}(t) + [\hat{n}_1^2(1+\nu) - \nu]\sigma_{zz}(t) + 2(1+\nu)\hat{l}_1\hat{m}_1\sigma_{xy}(t) + 2(1+\nu)\hat{l}_1\hat{n}_1\sigma_{xz}(t) + 2(1+\nu)\hat{m}_1\hat{n}_1\sigma_{yz}(t) \quad (7)$$

For a limit state corresponding to the fatigue strength the criterion will take the following form

$$\max_t \{E\varepsilon_\eta(t)\} = \sigma_{az} \quad (8)$$

In a particular case when normal stresses are given by formula (4) and $\hat{l}_1 = 1$ we get in view of (7) and (8)

$$\sigma_{a1} - \nu(\sigma_{a2} + \sigma_{a3}) = \sigma_{az} \quad (9)$$

So, we arrive at the classical maximum principal strain criterion for a sinusoidal loading [5].

4. CRITERION OF THE MAXIMUM SHEAR STRESS ON A FRACTURE PLANE

This criterion is based on the following assumptions:

1. Fatigue fracture is caused by the shear stress $\tau_{\eta s}(t)$ acting in the $\bar{s}$ direction on a fracture plane.
2. The direction of $\bar{s}$ coincides with the mean direction of the maximum shear stress $\tau_1(t)$ and the fracture plane is determined by a mean location of either of two planes in which $\tau_1(t)$ is present.

As it is known the maximum shear stress τ_1 in the coordinate system of principal axes 1,2,3 acts on a plane coming through the axis 2, inclined at $\pi/4$ to the axes 1 and 3.
The stress direction is perpendicular to the axis 2 (Fig.2).
If the directions 1,2,3 are mean directions of principal axes

then, according to Fig.2, the unit vectors $\bar{s}, \bar{\eta}$ and $\bar{\xi}$ (directed along axis 2) form a new constant set of axes rotated with respect to the x, y, z coordinate system. Cosines of the rotation angles expressed by the mean cosines of principal axes are given in Table 1.

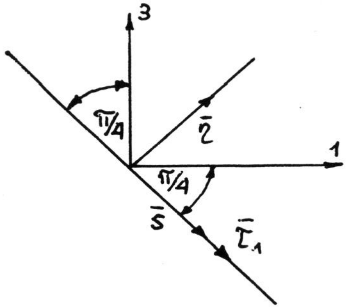


Fig.2. Location of τ_1 in a plane perpendicular to the principal axis 2

Table 1. Rotation angle cosines of s, η, ξ system of coordinates relative to x, y, z system

	x	y	z
s	$\frac{\hat{l}_1 - \hat{l}_3}{\sqrt{2}}$	$\frac{\hat{m}_1 - \hat{m}_3}{\sqrt{2}}$	$\frac{\hat{n}_1 - \hat{n}_3}{\sqrt{2}}$
η	$\frac{\hat{l}_1 + \hat{l}_3}{\sqrt{2}}$	$\frac{\hat{m}_1 + \hat{m}_3}{\sqrt{2}}$	$\frac{\hat{n}_1 + \hat{n}_3}{\sqrt{2}}$
ξ	$\hat{l}_2$	$\hat{m}_2$	$\hat{n}_2$

The shear stress $\tau_{\eta s}(t)$ may be found by using the transformation law for stress tensor components

$$\sigma_{\gamma\beta}(t) = \alpha_{\gamma i}\,\alpha_{\beta j}\,\sigma_{ij}(t), \quad (\gamma, \beta = s, \eta, \xi), \quad (i, j = x, y, z) \tag{10}$$

where α are cosines of axis rotation angles.

Making use of the values given in Table 1, we get the following form of $\tau_{\eta s}(t)$

$$\begin{aligned}\tau_{\eta s}(t) = {} & 0.5(\hat{l}_1^2 - \hat{l}_3^2)\sigma_{xx}(t) + 0.5(\hat{m}_1^2 - \hat{m}_3^2)\sigma_{yy}(t) + \\ & + 0.5(\hat{n}_1^2 - \hat{n}_3^2)\sigma_{zz}(t) + (\hat{l}_1\hat{m}_1 - \hat{l}_3\hat{m}_3)\sigma_{xy}(t) + \\ & + (\hat{l}_1\hat{n}_1 - \hat{l}_3\hat{n}_3)\sigma_{xz}(t) + (\hat{m}_1\hat{n}_1 - \hat{m}_3\hat{n}_3)\sigma_{yz}(t)\end{aligned} \tag{11}$$

In a limit state corresponding to the fatigue strength the maximum value of shear stress $\tau_{\eta s}(t)$ is equal to the corresponding amplitude of the maximum shear stress associated with a sinusoidal push-pull test performed at the σ_{az} level. Hence, the criterion takes the form

$$\max_t \{2\tau_{\eta s}(t)\} = \sigma_{az} \tag{12}$$

It is to be noted that the magnitude of stress vector $\bar{\tau}_1(t)$ cannot be regarded as the only factor governing fatigue fracture. This can be proved by taking a hypothetical state of stress where a plane of the maximum shear stress with a normal $\bar{\eta}$ (Fig.2) remains constant with time. Let the maximum shear stress τ_1 acting on this plane have a constant amplitude but let its direction vary in a random fashion. Such conditions are sufficient for fatigue fracture to occur provided the stress is high enough.

But it would mean that under a uniaxial loading fracture should occur in a specimen subjected to a stationary stress τ_1 which is not true.

If normal stresses are assumed to vary sinusoidally according to (4) and $\hat{l}_1 = \hat{n}_3 = 1$ then in view of (11) and (12) we get

$$\sigma_{a1} - \sigma_{a3} = \sigma_{az} \tag{13}$$

So, we arrive at the classical maximum shear stress criterion for a sinusoidal load [5].

5. CRITERION OF THE SHEAR STRESS ENERGY IN THE DIRECTION OF MAXIMUM SHEAR STRESS ON A FRACTURE PLANE

Here we assume that:

1. Fatigue fracture is governed by the specific elastic energy of shear deformation equal to the specific work done on the shear strain in the direction $\bar{s}$ on a fracture plane.
2. The $\bar{s}$ direction coincides with the mean direction of maximum shear stress $\tau_1(t)$ and the fracture plane is given by a mean location of either of the two planes containing $\tau_1(t)$.

The specific work done on the shear strain $\Phi_{sf}(t)$ in the direction of $\bar{s}$ (Fig.2) is given by the expression

$$\Phi_{sf}(t) = \tau_{\eta s}(t)\,\varepsilon_{\eta s}(t) = \frac{1+\nu}{E}\tau^2_{\eta s}(t) \tag{14}$$

For a uniaxial stress system where $\sigma_{xx}(t) = \sigma_{az}\sin\omega t$, $\Phi_{\eta s}$ is given as

$$\Phi_{sf}(t) = \frac{1+\nu}{4E}\sigma_{az}^2\sin^2\omega t \tag{15}$$

Assuming that in a limit state corresponding to the fatigue strength the maximum value of energy $\Phi_{sf}(t)$ in a random state of stress is equal to the maximum value of energy in a uniaxial state of sinusoidal stress, we arrive at the following form of the criterion

$$\max_t \{2\tau_{\eta s}(t)\} = \sigma_{az} \tag{16}$$

Expression (16) turns out to be the same as (12), i.e. the criterion of maximum shear stress.

It should be noted that the specific work done on shear strains on the whole fracture plane with a normal $\bar{\eta}$ (Fig.2) is given by

$$\Phi_{s\eta f}(t) = \frac{1+\nu}{E}\left[\tau^2_{\eta s}(t) + \tau^2_{\eta\xi}(t)\right] \tag{17}$$

If we assume such a state of stress that

$$\tau_{\eta s}(t) = \tau_a\sin\omega t\,; \quad \tau_{\eta\xi}(t) = \tau_a\sin(\omega t + \pi/2) \tag{18}$$

then from (17) we get

$$\Phi_{s\eta f}(t) = \frac{1+\nu}{E}\tau_a^2 = const., \tag{19}$$

i.e. a time-independent amount of shear strain energy that can be safely absorbed by a material subjected to a one-dimensional loading.

The total shear strain energy

$$\begin{aligned}\Phi_f(t) = \frac{1+\nu}{6E}\Big\{ & [\sigma_{xx}(t) - \sigma_{yy}(t)]^2 + [\sigma_{yy}(t) - \sigma_{zz}(t)]^2 + \\ & + [\sigma_{zz}(t) - \sigma_{xx}(t)]^2 + 6[\sigma_{xy}^2(t) + \sigma_{xz}^2(t) + \sigma_{yz}^2(t)]\Big\}\end{aligned} \tag{20}$$

cannot be treated as a reliable basis for random fatigue life

predictions either, since it is easy to present an example where stresses changing with time will cause fatigue fracture although the energy $\Phi_f(t)$ will remain constant.
To show that it is sufficient to take

$$\sigma_{xx}(t) = \sigma_a \sin\omega t \; ; \; \sigma_{yy}(t) = \sigma_a \sin(\omega t + \pi/3)$$

and the remaining components $\sigma_{ij}(t) = 0, \; (i,j = x,y,z)$.
After substituting to (20), we get

$$\Phi_f(t) = \frac{1+\nu}{4E}\,\sigma_a^2 = \mathrm{const.} \tag{21}$$

Likewise, if we take $\sigma_{xx}(t) = \sigma_a \sin\omega t$, $\sigma_{xy}(t) = [\sigma_a/\sqrt{2(1+\nu)}]\sin(\omega t + \pi/2)$ and the remaining components $\sigma_{ij}(t) = 0, \; (i,j = x,y,z)$ then the total strain energy

$$\Phi(t) = \frac{1}{2E}\Big\{\sigma_{xx}^2(t) + \sigma_{yy}^2(t) + \sigma_{zz}^2(t) - 2\nu[\sigma_{xx}(t)\,\sigma_{yy}(t) + \sigma_{yy}(t)\,\sigma_{zz}(t) + \sigma_{xx}(t)\sigma_{zz}(t)] + 2(1+\nu)[\sigma_{xy}^2(t) + \sigma_{xz}^2(t) + \sigma_{yz}^2(t)]\Big\} \tag{22}$$

becomes constant and is equal to

$$\Phi(t) = \frac{\sigma_a^2}{2E} = \mathrm{const.} \tag{23}$$

6. CRITERION OF THE MAXIMUM SHEAR AND NORMAL STRESSES ACTING ON A FRACTURE PLANE

Here the following assumptions are made:

1. Fatigue fracture is caused by normal $\sigma_\eta(t)$ and tangent $\tau_{\eta s}(t)$ stresses acting in a fixed direction $\bar{s}$ on a fracture plane with a normal $\bar{\eta}$.
2. The $\bar{s}$ direction on a fracture plane coincides with a mean direction of the maximum shear stress $\tau_{\eta\,max}(t)$.
3. In a limit state corresponding to the fatigue strength the maximum value of a combination of stresses $\tau_{\eta s}(t)$ and $\sigma_\eta(t)$ under a complex random loading is constant and can be written as

$$\max_t\left\{\tau_{\eta s}(t) + K\sigma_\eta^r(t)\right\} = F\,, \tag{24}$$

where K, r, F - material constants determined in sinusoidal loading tests.

Criterion (24) is a general form of a number of criteria applied for sinusoidal loads. They differ in various values of the adopted K, r, F parameters and the assumed location of fracture plane (or critical shearing plane).
Consider a special case by taking the $\bar{s}$ direction to coincide with a mean direction of maximum shear stress $\tau_1(t)$ and by assuming a fracture plane to be determined by a mean location of either of the two planes containing $\tau_1(t)$. Then the following form of the criterion is obtained

$$\max_t\Big\{0.5(\hat{l}_1^2 - \hat{l}_3^2)\sigma_{xx}(t) + 0.5(\hat{m}_1^2 - \hat{m}_3^2)\sigma_{yy}(t) + 0.5(\hat{n}_1^2 - \hat{n}_3^2)\sigma_{zz}(t) + (\hat{l}_1\hat{m}_1 - \hat{l}_3\hat{m}_3)\sigma_{xy}(t) + (\hat{l}_1\hat{n}_1 - \hat{l}_3\hat{n}_3)\sigma_{xz}(t) + (\hat{m}_1\hat{n}_1 - \hat{m}_3\hat{n}_3)\sigma_{yz}(t) + K[0.5(\hat{l}_1 + \hat{l}_3)^2\sigma_{xx}(t) + 0.5(\hat{m}_1 + \hat{m}_3)^2\sigma_{yy}(t) + 0.5(\hat{n}_1 + \hat{n}_3)^2\sigma_{zz}(t) + (\hat{l}_1 + \hat{l}_3)(\hat{m}_1 + \hat{m}_3)\sigma_{xy}(t) + (\hat{l}_1 + \hat{l}_3)(\hat{n}_1 + \hat{n}_3)\sigma_{xz}(t) + (\hat{m}_1 + \hat{m}_3)(\hat{n}_1 + \hat{n}_3)\sigma_{yz}(t)]^r\Big\} = F \tag{25}$$

Taking additionally that normal stresses follow formula (4) and that $\hat{l}_1 = \hat{n}_3 = 1$, we have

$$0.5(\sigma_{a1} - \sigma_{a3}) + K[0.5(\sigma_{a1} + \sigma_{a3})]^r = F \tag{26}$$

Further, if we take the constants to be $K = (2\tau_{az}/\sigma_{az}) - 1$, $F = \sigma_{az}$ (after Matake [6] and Stanfield [8]) and $r = 1$, then we arrive at the so called "ellipse arc" criterion due to Gough [2].

It is possible to formulate a criterion based on octahedral stresses, thus implying that fatigue fracture is caused by octahedral normal $\sigma_{oct.\eta}(t)$ and octahedral shearing $\tau_{oct.\eta s}(t)$ stresses acting in a fixed direction $\bar{s}$ on an octahedral plane with a normal $\bar{\eta}$, similarly as in criterion (24). Such a condition may be useful if the principal directions remain constant relative to a fixed x,y,z coordinate system. Otherwise, the octahedral plane with a normal $\bar{\eta}$ will not maintain a permanent location. In order to fix its location one should assume the mean principal directions. Then, however, the normal and shearing stresses in a chosen direction on such a plane would not be octahedral at each instant t. It is not possible therefore to use an octahedral stress criterion if a random stress system is involved.

7. CONCLUSIONS

1. Out of the five proposed criteria for fatigue failure two have the same mathematical form.
2. The proposed criteria call for appropriate experimental verification.
3. It was analytically shown that scalar quantities of the total specific deformation energy or only shear deformation energy cannot be responsible for fatigue fracture if the principal directions vary. This applies both to sinusoidal and random loadings. The octahedral stresses should be also avoided in random fatigue life predictions.

8. REFERENCES

[1] Dietmann H., Issler L., Festigkeitsberechnung bei mehrachsiger phasenverschobener Schwingbenspruchung mit korperfesten Hauptspannungsrichtungen, Konstruktion 28, H.1, 1976, ss. 23-30.

[2] Gough H.J., Engineering steels under combined cyclic and static stresses, Trans. ASME, 17(2), 1950, pp. 113-125.

[3] Gough H.J., Pollard H.V., The strength of metals under combined alternating stresses, Proc. Inst. Mech. Engng. (London), 131, 1935, p. 103.

[4] Macha E., Simulation of a Material Life-Time under Random Triaxial Stress State, in: Simulation Systems 79, L.Dekker, G.Savastano, G.C.Vansteenkiste, Ed. North-Holland Publishing Company, 1980, pp. 425-435.

[5] Macha E., Mathematical Models of the Life to Fracture for Materials Subcject to Random Complex Stress systems, Scientific Papers of the Institute of Materials Science and Applied Mechanics of Wrocław Technical University No 41, Monographs No 13, 1979, p. 99 (in Polish).

[6] Matake T., An Explanation on Fatigue Limit under Combined Stress, Bulletin of the JSME, Vol. 20, No 141, March 1977, pp. 257-263.

[7] Mc Diarmid D.L., A new analysis of fatigue under combined bending and twisting, Aeronautical Journal, July 1974, pp. 325-329.

[8] Rotvel F., Biaxial fatigue tests with zero mean stresses using tabular specimens, Int. J. Mech. Sci., Vol. 12, 7, 1970, pp. 597-613.