THE STRESS-STRAIN FIELDS AT CRACK TIP IN CRACKED SPHERICAL SHELL

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ABSTRACT

Based on shallow shell theory, taking into account of lateral shear deformation, the governing equations for cracked spherical shell expressed in displacement and stress functions f, F and φ are proposed. They are reduced to equivalent two-order differential equations and then a general solution including Mode I,II, III for stress-strain fields at crack tip in a spherical shell is obtained.

INTRODUCTION

The study of cracked shell is an important problem in practical engineering constructions. Since curvature exists in shells, extension and bending are coupled, which makes the problem very difficult. In earlier literatures the classical shell theory (Folias,1965) was used. In recent years more investigators began to study the problem with Reissner's theory. By using Reissner's shell theory a ten-order differential equation is proposed (Sih and Hagendorf,1973). Since the problem is complicated, only the first term of expression is given. In order to calculate stress intensity factors (especially for mixed mode), it is neccessary to search the general solution for stress-strain fields at crack tip, which is the aim of this paper.

THE GOVERNING EQUATIONS OF A CRACKED SPHERICAL SHELL AND THEIR SIMPLIFIED FORMS

A spherical shell containing a through crack is shown in Fig.1 with the crack tip as the origin of the coordinates. The shallow shell theory, taking into account shear deformation, could be expressed as follows(**Hu,1981**): Relations between stress and strain:

$$
\begin{aligned}
M_x &= -D\left(\frac{\partial \psi_x}{\partial x} + \nu \frac{\partial \psi_y}{\partial y}\right) \\
M_y &= -D\left(\frac{\partial \psi_y}{\partial y} + \nu \frac{\partial \psi_x}{\partial x}\right) \\
M_{xy} &= -\frac{D}{2}(1-\nu)\left(\frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\right)
\end{aligned} \qquad (1)
$$

$$
Q_x = C\left(\frac{\partial w}{\partial x} - \psi_x\right), \quad Q_y = C\left(\frac{\partial w}{\partial y} - \psi_y\right) \qquad (2)
$$

$$\frac{\partial u}{\partial x} = \frac{1}{B}(N_x - \nu N_y) + k w$$
$$\frac{\partial v}{\partial y} = \frac{1}{B}(N_y - \nu N_x) + k w \qquad (3)$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1+\nu}{2B} N_{xy}$$

Where M_x, M_y, M_{xy}, Q_x, Q_y are bending moments, torsional moment and shear forces respectively; N_x, N_y, N_{xy} are membranous forces. u,v,w are the tangential and transverse displacements respectively. B is extension-compression stiffness; **D** is bending stiffness; **C** is shearing stiffness.
The equibrium equations are:

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0, \qquad \frac{\partial N_{xy}}{\partial x} + \frac{\partial N_y}{\partial y} = 0 \qquad (4)$$

$$\frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} - Q_x = 0, \qquad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = 0 \qquad (5)$$
$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + k(N_x + N_y) + q = 0$$

Where q is the lateral load per unit area. **k is the curvature.**
Introducing stress function φ, let

$$N_x = \frac{\partial^2 \varphi}{\partial y^2}, \quad N_y = \frac{\partial^2 \varphi}{\partial x^2}, \quad N_{xy} = -\frac{\partial^2 \varphi}{\partial x \partial y} \qquad (6)$$

From eq. (3), we have compatibility equation.

$$\frac{1}{B}\nabla^2\nabla^2\varphi + k\nabla^2 w = 0 \qquad (7)$$

Substituting eq. (1), (2) into eq. (5), we have:

$$D\left(\frac{\partial^2 \psi_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \psi_x}{\partial y^2} + \frac{1+\nu}{2}\frac{\partial^2 \psi_y}{\partial x \partial y}\right) + C\left(\frac{\partial w}{\partial x} - \psi_x\right) = 0 \qquad (8)$$

$$D\left(\frac{1+\nu}{2}\frac{\partial^2 \psi_x}{\partial x \partial y} + \frac{1-\nu}{2}\frac{\partial^2 \psi_y}{\partial x^2} + \frac{\partial^2 \psi_y}{\partial y^2}\right) + C\left(\frac{\partial w}{\partial y} - \psi_y\right) = 0 \qquad (9)$$

$$C\left(\nabla^2 w - \frac{\partial \psi_x}{\partial x} - \frac{\partial \psi_y}{\partial y}\right) + k\nabla^2\varphi + q = 0 \qquad (10)$$

Boundary conditions along the free edge are:

$$\theta = \pm\pi: \quad M_y = 0, \quad M_{xy} = 0, \quad Q_y = 0 \qquad (11)$$

$$N_y = 0, \quad N_{xy} = 0 \qquad (12)$$

Introducing displacement functions f, F, let

$$\psi_x = \frac{\partial F}{\partial x} + \frac{\partial f}{\partial y}, \qquad \psi_y = \frac{\partial F}{\partial y} - \frac{\partial f}{\partial x} \qquad (13)$$

Substituting eq. (13) into eq. (8), (9), we have:

$$\frac{\partial}{\partial x}[D\nabla^2 F + C(W - F)] + \frac{\partial}{\partial y}\left[\frac{D}{2}(1-\nu)\nabla^2 f - Cf\right] = 0 \qquad (14)$$

$$\frac{\partial}{\partial y}[D\nabla^2 F + C(W - F)] - \frac{\partial}{\partial x}\left[\frac{D}{2}(1-\nu)\nabla^2 f - Cf\right] = 0 \qquad (15)$$

From it we have:

$$\frac{D}{2}(1-\nu)\nabla^2 f - Cf + i[D\nabla^2 F + C(W-F)] = C\Phi(x + iy) \qquad (16)$$

Where Φ (x + iy) is an analytic function. **Hu(1981) assumed that Φ=0. It is** correct for cases without singularity. As the crack tip is a singular point, generally speaking, $\Phi \neq 0$.
Separating real part and imaginary part in eq. (16), we have:

$$D\nabla^2 F + C(W - F) = C\, Im\Phi \qquad (17)$$

$$\frac{D}{2}(1-\nu)\nabla^2 f - Cf = C\, Re\Phi \qquad (18)$$

From eq. (17), we have:

$$W = F - \frac{D}{C}\nabla^2 F + Im\Phi \qquad (19)$$

Substituting eq. (13), (19) into eq. (10), we have:

$$D\nabla^2\nabla^2 F - k\nabla^2\varphi = q \qquad (20)$$

Substituting eq. (19) into eq. (7), we have:

$$\frac{1}{B}\nabla^2\nabla^2\varphi + k\nabla^2 F - k\frac{D}{C}\nabla^2\nabla^2 F = 0 \qquad (21)$$

The governing equations can be reduced to three equations (18), (20) and (21) in terms of f, F and φ. The function f, which is similar to that in bending plate case, is uncoupled. The function F and φ should satisfy two four-order differential equations, we could reduce them further.
If q = 0, from eq. (20), (21), we have:

$$\nabla^2\nabla^2\nabla^2 F - \frac{k^2 B}{C}\nabla^2\nabla^2 F + \frac{k^2 B}{D}\nabla^2 F = 0 \qquad (22)$$

It may be proved that, function F in eq. (22) is the sum of three functions F_0, F_1 and F_2, which should satisfy the following equations respectively.

$$\nabla^2 F_0 = 0 \qquad (23)$$

$$\nabla^2 F_1 - 4\lambda_1^2 F_1 = 0 \qquad (24)$$

$$\nabla^2 F_2 - 4\lambda_2^2 F_2 = 0 \qquad (25)$$

$$F = F_0 + F_1 + F_2 \qquad (26)$$

$$4\lambda_1^2 = \frac{k^2 B}{2C} + \sqrt{\frac{k^4 B^2}{4C^2} - \frac{k^2 B}{D}}, \quad 4\lambda_2^2 = \frac{k^2 B}{2C} - \sqrt{\frac{k^4 B}{4C^2} - \frac{k^2 B}{D}} \qquad (27)$$

With the F known, from eq. (20) the φ could be obtained.

$$\varphi = \varphi_0 + \frac{4D}{k}(\lambda_1^2 F_1 + \lambda_2^2 F_2) \qquad (28)$$

Where φ_0 is harmonic function, which should satisfy $\nabla^2\varphi_0 = 0$. From eq. (18), function f could be found

$$f = f_0 - Re\Phi \qquad (29)$$

f_0 should satisfy following equation

$$\nabla^2 f_0 - 4\mu^2 f_0 = 0 \qquad (30)$$

Where $4\mu^2 = \frac{2C}{D(1-\nu)} = \frac{10}{h^2}$

With functions f, F and φ known, the generalized displacements and generalized stresses could be obtained. They should satisfy the boundary conditions of free edges:

$$\theta = \pm\pi, \quad M_\theta = 0, \quad M_{r\theta} = 0, \quad Q_\theta = 0 \tag{31}$$

$$N_\theta = 0 \quad N_{r\theta} = 0 \tag{32}$$

THE EIGENFUNCTION OF THE DISPLACEMENT FUNCTIONS AND STRESS FUNCTION

Expand the analytic function Φ in series

$$\Phi(x+iy) = \sum_\lambda (\beta_{\lambda-1} + i\alpha_{\lambda-1}) z^{\lambda-1} = \sum_\lambda (\beta_{\lambda-1} + i\alpha_{\lambda-1}) r^{\lambda-1}[\cos(\lambda-1)\theta + i\sin(\lambda-1)\theta] \tag{33}$$

Function f and w could be expressed as

$$f = f_0 + \sum_\lambda r^{\lambda-1}[\alpha_{\lambda-1}\sin(\lambda-1)\theta - \beta_{\lambda-1}\cos(\lambda-1)\theta] \tag{34}$$

$$w = F - \frac{D}{C}\nabla^2 F + \sum_\lambda r^{\lambda-1}[\alpha_{\lambda-1}\cos(\lambda-1)\theta + \beta_{\lambda-1}\sin(\lambda-1)\theta] \tag{35}$$

Function F could be found from eq. (26)

$$F = F_0 + F_1 + F_2 \tag{26}$$

Functions f_0, F_1 and F_2 should satisfy the eq. (30), (24) and (25) respectively. These equations are Helmholtz's equations. Their solutions could be expressed in modified Bessel functions. From the condition of finite strain energy, we must drop out the modified Bessel function of second kind. Keeping in mind that λ will take positive, zero as well as negative values, the solutions could be expressed in modified Bessel function of first kind only.

For symmetric case

$$f_\lambda = \sin\lambda\theta\, I_\lambda(2\mu r) = \sin\lambda\theta \sum_{m=0,1,\cdots} \frac{\mu^{2m} r^{\lambda+2m}}{m!\,\varphi(\lambda,m)} \tag{36}$$

$$F_{1,\lambda} = \cos\lambda\theta\, I_\lambda(2\lambda_1 r) = \cos\lambda\theta \sum_{m=0,1,\cdots} \frac{\lambda_1^{2m} r^{\lambda+2m}}{m!\,\varphi(\lambda,m)} \tag{37}$$

$$F_{2,\lambda} = \cos\lambda\theta\, I_\lambda(2\lambda_2 r) = \cos\lambda\theta \sum_{m=0,1,\cdots} \frac{\lambda_2^{2m} r^{\lambda+2m}}{m!\,\varphi(\lambda,m)} \tag{38}$$

For anti-symmetric case

$$\tilde{f}_\lambda = \cos\lambda\theta\, I_\lambda(2\mu r) = \cos\lambda\theta \sum_{m=0,1,\cdots} \frac{\mu^{2m} r^{\lambda+2m}}{m!\,\varphi(\lambda,m)} \tag{39}$$

$$\tilde{F}_{1,\lambda} = \sin\lambda\theta\, I_\lambda(2\lambda_1 r) = \sin\lambda\theta \sum_{m=0,1,\cdots} \frac{\lambda_1^{2m} r^{\lambda+2m}}{m!\,\varphi(\lambda,m)} \tag{40}$$

$$\tilde{F}_{2,\lambda} = \sin\lambda\theta\, I_\lambda(2\lambda_2 r) = \sin\lambda\theta \sum_{m=0,1,\cdots} \frac{\lambda_2^{2m} r^{\lambda+2}}{m!\,\varphi(\lambda,m)} \tag{41}$$

Where

$$\varphi(\lambda,m) = \frac{\Gamma(\lambda+m+1)}{\Gamma(\lambda+1)} = \begin{cases} (\lambda+1)(\lambda+2)\cdots(\lambda+m) \\ 1 \end{cases} \tag{42}$$

The linear combination of them is also a solution of these equations, so the general solutions of eq. (30), (24) and (25) are expressed by

$$f_0 = \sum_\lambda \sum_{n=0,1,\cdots} (A_{\lambda-1+2n} f_{\lambda-1+2n} + B_{\lambda-1+2n} \tilde{f}_{\lambda-1+2n}) \tag{43}$$

$$F_1 = \sum_\lambda \sum_{n=0,1,\cdots} (K^{(1)}_{\lambda-1+2n} F_{1,\lambda-1+2n} + L^{(1)}_{\lambda-1+2n} \tilde{F}_{1,\lambda-1+2n}) \tag{44}$$

$$F_2 = \sum_\lambda \sum_{n=0,1,\cdots} (K^{(2)}_{\lambda-1+2n} F_{2,\lambda-1+2n} + L^{(2)}_{\lambda-1+2n} \tilde{F}_{2,\lambda-1+2n}) \tag{45}$$

Substituting eq (43) - (45) into eq. (26) , (28) , (32) and (33) the functions f, F, φ and w could be expressed by

$$\begin{aligned} f = {} & \sum_\lambda \sum_{n=0,1,\cdots} [A_{\lambda-1+2n}\sin(\lambda-1+2n)\theta + B_{\lambda-1+2n}\cos(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\mu^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \\ & + \sum_\lambda r^{\lambda-1}[\alpha_{\lambda-1}\sin(\lambda-1)\theta - \beta_{\lambda-1}\cos(\lambda-1)\theta] \end{aligned} \tag{46}$$

$$\begin{aligned} F = {} & \sum_\lambda r^{\lambda+1}[K^{(0)}_{\lambda+1}\cos(\lambda+1)\theta + L^{(0)}_{\lambda+1}\sin(\lambda+1)\theta] \\ & + \sum_\lambda \sum_{n=0,1,\cdots} [K^{(1)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta + L^{(1)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\lambda_1^{2m} r^{\lambda-1+2(m+n)}}{m!\,\varphi(\lambda-1+2n,m)} \\ & + \sum_\lambda \sum_{n=0,1,\cdots} [K^{(2)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta + L^{(2)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\lambda_2^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \end{aligned} \tag{47}$$

$$\begin{aligned} \varphi = {} & \sum_\lambda r^{\lambda-1}[M_{\lambda-1}\cos(\lambda-1)\theta + N_{\lambda-1}\sin(\lambda-1)\theta] \\ & + \frac{4D}{k}\lambda_1^2 \sum_\lambda \sum_{n=0,1,\cdots} [K^{(1)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta + L^{(1)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \\ & \times \sum_{m=0,1,\cdots} \frac{\lambda_1^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} + \frac{4D}{k}\lambda_2^2 \sum_\lambda \sum_{n=0,1,\cdots} [K^{(2)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta \\ & + L^{(2)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\lambda_2^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \end{aligned} \tag{48}$$

$$\begin{aligned} W = {} & \sum_\lambda r^{\lambda+1}[K^{(0)}_{\lambda+1}\cos(\lambda+1)\theta + L^{(0)}_{\lambda+1}\sin(\lambda+1)\theta] \\ & + \left(1 - \frac{4D}{C}\lambda_1^2\right) \sum_\lambda \sum_{n=0,1,\cdots} [K^{(1)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta + L^{(1)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\lambda_1^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \\ & + \left(1 - \frac{4D}{C}\lambda_2^2\right) \sum_\lambda \sum_{n=0,1,\cdots} [K^{(2)}_{\lambda-1+2n}\cos(\lambda-1+2n)\theta + L^{(2)}_{\lambda-1+2n}\sin(\lambda-1+2n)\theta] \sum_{m=0,1,\cdots} \frac{\lambda_2^{2m} r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \end{aligned}$$

$$+\sum_{\lambda} r^{\lambda-1}\left[\alpha_{\lambda-1}\cos(\lambda-1)\theta+\beta_{\lambda-1}\sin(\lambda-1)\theta\right] \tag{49}$$

According to the singularity analysis, the generalized stresses should be of $O(r^{\frac{1}{2}})$. These condition mean that functions f, F, and φ should be of $O(r^{\frac{3}{2}})$, the generalized displacements ψ_x , ψ_y , w, u, v should be of $O(r^{\frac{1}{2}})$. From the condition of singularity, f, F, φ and w should satisfy the following conditions:

1. In order to insure that f is of $O(r^{\frac{3}{2}})$, in eq. **(46)** it is neccessary that

When $$\lambda-1<3/2, \quad A_{\lambda-1}=-\alpha_{\lambda-1}, \quad B_{\lambda-1}=\beta_{\lambda-1} \tag{50}$$

2. In order to insure that F is of $O(r^{\frac{3}{2}})$, in eq. **(47)** it is neccessary that

When $$\lambda-1<3/2, \quad K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1}=0, \quad L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1}=0 \tag{51}$$

3. In order to insure that φ is of $O(r^{\frac{3}{2}})$, in eq. **(48)** it is neccessary that

When $$\begin{aligned}\lambda-1<3/2, \quad & M_{\lambda-1}=-\frac{4D}{C}(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\\ & N_{\lambda-1}=-\frac{4D}{C}(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\end{aligned} \tag{52}$$

4. In order to insure that w is of $O(r^{\frac{1}{2}})$, in eq. **(49)** it is neccessary that

When $$\begin{aligned}\lambda-1<1/2, \quad & \alpha_{\lambda-1}=(\frac{D}{C}4\lambda_1^2-1)K^{(1)}_{\lambda-1}+(\frac{D}{C}4\lambda_2^2-1)K^{(2)}_{\lambda-1}\\ & \beta_{\lambda-1}=(\frac{D}{C}4\lambda_1^2-1)L^{(1)}_{\lambda-1}+(\frac{D}{C}4\lambda_2^2-1)L^{(2)}_{\lambda-1}\end{aligned}$$

Introducing new coefficients

$$\begin{aligned}\tilde{\alpha}_{\lambda-1}&=\alpha_{\lambda-1}+(1-\frac{D}{C}4\lambda_1^2)K^{(1)}_{\lambda-1}+(1-\frac{D}{C}4\lambda_2^2)K^{(2)}_{\lambda-1}\\ \tilde{\beta}_{\lambda-1}&=\beta_{\lambda-1}+(1-\frac{D}{C}4\lambda_1^2)L^{(1)}_{\lambda-1}+(1-\frac{D}{C}4\lambda_2^2)L^{(2)}_{\lambda-1}\end{aligned} \tag{53}$$

When $$\lambda-1<1/2, \quad \tilde{\alpha}_{\lambda-1}=0, \quad \tilde{\beta}_{\lambda-1}=0$$

The boundary conditions are:

$$\theta=\pm\pi: \quad M_\theta=M_{r\theta}=Q_\theta=0, \quad N_\theta=N_{r\theta}=0$$

The boundary conditions in terms of functions f, F and φ could be expressed as

$$\nabla^2F-(1-\nu)\frac{\partial^2F}{\partial r^2}-(1-\nu)\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial f}{\partial\theta})=0 \tag{54}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial F}{\partial\theta})+(2\mu f_0-\frac{\partial^2 f}{\partial r^2})=0 \tag{55}$$

$$\frac{\partial f_0}{\partial r}-\frac{D}{C}\frac{1}{r}\frac{\partial}{\partial\theta}\nabla^2F=0 \tag{56}$$

$$\frac{\partial^2\phi}{\partial r^2}=0 \tag{57}$$

$$\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial\phi}{\partial\theta})=0 \tag{58}$$

Substituting eq. **(46)** - **(49)** into eq. **(54)** - **(58)** , linear equations whose unknowns are coefficients of the expansions could be obtained. In order to satisfy these equations, let

$$\lambda=\pm\frac{n}{2}, \qquad n=0,1,2,\cdots \tag{59}$$

From the condition of finite strain energy, λ should be positive only. The relations between coefficients in eigenfuction expansion could be found from these linear equations.

THE STRESS AND DISPLACEMENT FUNCTIONS AND STRESS-STRAIN FIELDS AT CRACK TIP

Using the relation between coefficients in eigenfunction expansion, functions f, F, φ and w given in eq. **(46)** - **(49)** could be expressed as

1. If λ is fractional

$$\begin{aligned}f=&\sum_{\lambda} r^{\lambda-1}[-(K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1})\sin(\lambda-1)\theta+(L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1})\cos(\lambda-1)\theta]\\ &+\sum_{\lambda}\sum_{n=0,1,\cdots}[-\frac{4D}{C}(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}(\tilde{\beta}_{\lambda-1}+\frac{4D}{C}\lambda_1^2L^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2L^{(2)}_{\lambda-1})\\ &\cos(\lambda-1+2n)\theta]\frac{(-\mu^2)^n}{n!\varphi(\lambda+n-2,n)}\sum_{m=0,1,\cdots}\frac{\mu^{2m}r^{\lambda-1+2(n+m)}}{m!\varphi(\lambda-1+2n,m)}\end{aligned} \tag{60}$$

$$\begin{aligned}f_0=&\sum_{\lambda} r^{\lambda-1}[-\frac{4D}{C}(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1)\theta+(\tilde{\beta}_{\lambda-1}+\frac{4D}{C}\lambda_1^2L^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2L^{(2)}_{\lambda-1})\cos(\lambda-1)\theta]\\ &+\sum_{\lambda}\sum_{n=0,1,\cdots}[-\frac{4D}{C}(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}(\tilde{\beta}_{\lambda-1}+\frac{4D}{C}\lambda_1^2L^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2L^{(2)}_{\lambda-1})\cos(\lambda-1+2n)\theta]\\ &\times\frac{(-\mu^2)^n}{n!\varphi(\lambda+n-2,n)}\sum_{m=0,1,\cdots}\frac{\mu^{2m}r^{\lambda-1+2(n+m)}}{m!\varphi(\lambda-1+2n,m)}\end{aligned} \tag{61}$$

$$\begin{aligned}F=&\sum_{\lambda} r^{\lambda-1}[(K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1})\cos(\lambda-1)\theta+(L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1})\sin(\lambda-1)\theta]\\ &+\sum_{\lambda} r^{\lambda+1}\frac{4}{\lambda(\lambda+1)(1-\nu)}[-(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\cos(\lambda+1)\theta+(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\sin(\lambda+1)\theta]\\ &+\sum_{\lambda}\sum_{n=0,1,\cdots}[\frac{\lambda-1}{\lambda-1+2n}K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta]\frac{(-\lambda_1^2)^n}{n!\varphi(\lambda+n-2,n)}\sum_{m=0,1,\cdots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\varphi(\lambda-1+2n,m)}\end{aligned}$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[\frac{\lambda-1}{\lambda-1+2n}K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (62)$$

$$W=\sum_{\lambda}r^{\lambda-1}\tilde{\beta}_{\lambda-1}\sin(\lambda-1)\theta+\sum_{\lambda}r^{\lambda+1}\frac{4}{\lambda(\lambda+1)(1-\nu)}\left[-(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\cos(\lambda+1)\theta+(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\sin(\lambda+1)\theta\right]$$

$$+(1-\frac{4D}{C}\lambda_1^2)\sum_{\lambda}\sum_{n=0,1,\ldots}\left[\frac{\lambda-1}{\lambda-1+2n}K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_1^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)}$$

$$+(1-\frac{4D}{C}\lambda_2^2)\sum_{\lambda}\sum_{n=0,1,\ldots}\left[\frac{\lambda-1}{\lambda-1+2n}K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (63)$$

$$\varphi=\sum_{\lambda}\sum_{n=0,1,\ldots}\left[\frac{\lambda-1}{\lambda-1+2n}K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{4D\lambda_1^2}{k}\frac{(-\lambda_1^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)}$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[\frac{\lambda-1}{\lambda-1+2n}K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{4D\lambda_2^2}{k}\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (64)$$

2. If λ is integer

$$f=\sum_{\lambda}r^{\lambda-1}\left[-(K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1})\sin(\lambda-1)\theta+(L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1})\cos(\lambda-1)\theta\right]$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[-\frac{\lambda-1}{\lambda-1+2n}(\tilde{\alpha}_{\lambda-1}+\frac{4D}{C}\lambda_1^2K^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1+2n)\theta+\frac{4D}{C}(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\cos(\lambda-1+2n)\theta\right]$$

$$\cdot\frac{(-\mu^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\mu^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (65)$$

$$f_0=\sum_{\lambda}r^{\lambda-1}\left[-(\tilde{\alpha}_{\lambda-1}+\frac{4D}{C}\lambda_1^2K^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1)\theta+\frac{4D}{C}(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\cos(\lambda-1)\theta\right]$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[-\frac{\lambda-1}{\lambda-1+2n}(\tilde{\alpha}_{\lambda-1}+\frac{4D}{C}\lambda_1^2K^{(1)}_{\lambda-1}+\frac{4D}{C}\lambda_2^2K^{(2)}_{\lambda-1})\sin(\lambda-1+2n)\theta+\frac{4D}{C}(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\cos(\lambda-1+2n)\theta\right]$$

$$\cdot\frac{(-\mu^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\mu^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (66)$$

$$F=\sum_{\lambda}r^{\lambda-1}\left[(K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1})\cos(\lambda-1)\theta+(L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1})\sin(\lambda-1)\theta\right]$$

$$+\sum_{\lambda}r^{\lambda+1}\frac{4}{\lambda(\lambda+1)(1-\nu)}\left[(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\cos(\lambda-1)\theta-(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\sin(\lambda-1)\theta\right]$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_1^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)}$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (67)$$

$$\varphi=\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{4D\lambda_1^2}{k}\cdot\frac{(-\lambda_1^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)}$$

$$+\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{4D\lambda_2^2}{k}\cdot\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (68)$$

$$W=\sum_{\lambda}r^{\lambda-1}\tilde{\alpha}_{\lambda-1}\cos(\lambda-1)\theta+\sum_{\lambda}r^{\lambda+1}\frac{4}{\lambda(\lambda+1)(1-\nu)}\left[(\lambda_1^2K^{(1)}_{\lambda-1}+\lambda_2^2K^{(2)}_{\lambda-1})\cos(\lambda-1)\theta-(\lambda_1^2L^{(1)}_{\lambda-1}+\lambda_2^2L^{(2)}_{\lambda-1})\sin(\lambda-1)\theta\right]$$

$$+(1-\frac{4D}{C}\lambda_1^2)\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(1)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(1)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_1^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_1^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)}$$

$$+(1-\frac{4D}{C}\lambda_2^2)\sum_{\lambda}\sum_{n=0,1,\ldots}\left[K^{(2)}_{\lambda-1}\cos(\lambda-1+2n)\theta+\frac{\lambda-1}{\lambda-1+2n}L^{(2)}_{\lambda-1}\sin(\lambda-1+2n)\theta\right]\frac{(-\lambda_2^2)^n}{n!\,\varphi(\lambda+n-2,n)}\sum_{m=0,1,\ldots}\frac{\lambda_2^{2m}r^{\lambda-1+2(n+m)}}{m!\,\varphi(\lambda-1+2n,m)} \qquad (69)$$

When $\lambda-1<3/2$ $\quad K^{(1)}_{\lambda-1}+K^{(2)}_{\lambda-1}=0\,,\quad L^{(1)}_{\lambda-1}+L^{(2)}_{\lambda-1}=0$

When $\lambda-1<1/2$ $\quad \tilde{\alpha}_{\lambda-1}=0\,,\quad \tilde{\beta}_{\lambda-1}=0$

Substituting F, f, φ and w into eq. (1) - (3) and (13) the generalized stresses and generalized displacements could be found. These are the stress-strain fields at crack tip in a cracked spherical shell. The singularity of first term is the same to **the results (Sih and Hagendorf, 1973).**

CONCLUSION

1. Similar to the Williams' expansion in plane fracture problem, a general solution of the stress-strain fields including Mode I, II, III at crack-tip for Reissner's shell is given.

2. The governing equation in terms of displacement functions f, F and **stress function φ given by Hu(1981)** is only valid in analysis without crack. For cracked shell the corresponding equations are given in this paper and they could be reduced to equivalent two-order differential equations.

3. The general solution for stress-strain fields at the tip of a crack in a shell provides a better mechanical foundation for calculation of stress intensity factors such as boundary collocation, variational method, asymptotic method and adaptation to special crack tip element in a finite element formulation etc. The analytical methods for plane fracture problem could be applied to the Reissner's shell fracture analysis.

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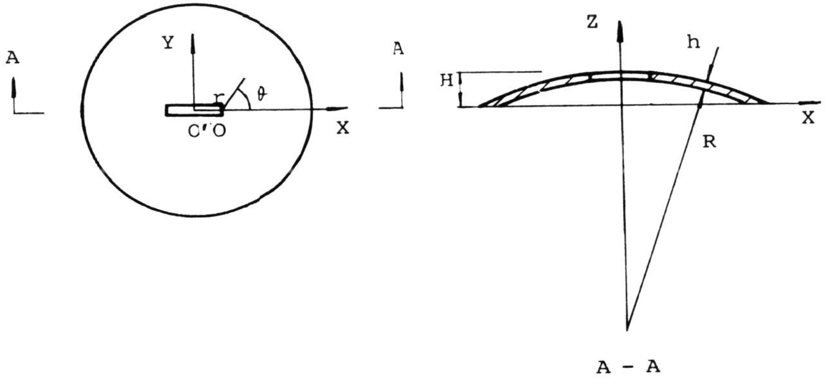

Fig. 1