NON-STATIONARY PROBLEMS OF LINEAR FRACTURE MECHANICS BY THE BOUNDARY ELEMENT METHOD

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ABSTRACT

This work investigates the dynamic stress intensity factor for a crack in an infinite elastic medium. Two classes of special problems of thermoelasticity are considered. These are the problems of classical elastodynamics and quasi-static problems of uncoupled thermoelasticity. The boundary value problems in these fields are recasted in an integral form. The boundary integro-differential equations are solved by the boundary element method in the Laplace transformed domain. The method is illustrated by two numerical examples for harmonic and impact load in elastodynamics and harmonically varying temperature load in thermoelasticity.

KEYWORDS

Crack; stress-intensity factor; elastodynamics; thermoelasticity; boundary integro-differential equations.

INTRODUCTION

The boundary integral equation method /BIEM/ appears to be an efficient method of numerical calculation for many practical problems in the field of engineering /Brebbia and Walker, 1980/.

The investigation of the stress intensity factor /SIF/ in an infinitely extending elastic medium occurs often in linear fracture mechanics. For dynamic SIF problem, the finite element method /FEM/ and the finite difference method /FDM/ do not provide a satisfactory solution because of the fact that an infinite medium is represented by a finite size model. Even the construction of special non-reflecting boundaries to be placed at the ends of the mesh does not fully alleviate this problem. Difficulties of this kind can be avoided by the boundary integro-differential equation formulation /BIDEM/ of crack problems. The BIEM and BIDEM in the Laplace transformed domain for general boundary value problems of thermoelasticity has been developed by the authors /Sládek, V. and Sládek, J. 1983; Sládek, J. and V.Sládek, 1984a/. The numerical solution of the non-stationary problems when the Laplace transform is employed essentially consists of series of solutions to a static-like problem for a number of discrete values of the transformed parameter p. The final solution is obtained by a numerical inversion to the time domain.

FORMULATION OF THE PROBLEM

Consider a homogeneous, isotropic, perfectly elastic body occupying the region V and bounded by the surface S. For this medium the linearized equations of thermoelasticity /Nowacki, 1975/ hold

$$\mu u_{i,kk} + (\lambda + \mu) u_{k,ki} + X_i = \rho \ddot{u}_i + \gamma \Theta_{,i} \qquad (1)$$

$$\Theta_{,kk} - \frac{1}{\varkappa}\dot{\Theta} - \alpha \dot{u}_{k,k} = -\frac{Q}{\varkappa} \qquad (2)$$

In these equation $\Theta = T - T_o$ denotes the increase of temperature with respect to the natural state T_o for which the stresses and deformations are equal to zero; next, u_i and X_i denote the components of the displacement vector and the body vector respectively. The function $Q(\bar{x},t)$ describes the intensity of the heat sources. The physical meaning of the coefficients in eqs. /1/ and /2/ is given elsewhere /Nowacki, 1975/. The dot denotes the derivative with respect to time. Eqs. /1/ and /2/ represent a system of equations for coupled problems of thermoelasticity.

The Laplace transform of eqs. /1/ and /2/ becomes

$$\mu \bar{u}_{i,kk} + (\lambda + \mu) \bar{u}_{k,ki} + \bar{X}_i = \rho p^2 \bar{u}_i + \gamma \bar{\Theta}_{,i} \qquad (3)$$

$$\bar{\Theta}_{,kk} - \frac{p}{\varkappa}\bar{\Theta} - \alpha p \bar{u}_{k,k} = -\frac{\bar{Q}}{\varkappa} \qquad (4)$$

where p is Laplace transform parameter.

In the case of steady state problems of thermoelasticity the displacement vector and temperature vary harmonically in time

$$u_i(\bar{x},t) = u_i^*(\bar{x},\omega) e^{-i\omega t}, \qquad \Theta_i(\bar{x},t) = \Theta_i^*(\bar{x},\omega) e^{-i\omega t} \qquad (5)$$

Inserting /5/ into eqs. /1/ and /2/ one can see that the problem of evaluating the amplitudes u_i^* and Θ^* is equivalent to that of evaluating the Laplace transforms $\bar{u}_i$ and $\bar{\Theta}$, bearing in mind the change $p \rightarrow -i\omega$.

In what follows we shall be interested in two special classes of problems.

Classical Elastodynamics

Classical dynamic elasticity is based on the assumption that the motion may be treated as adiabatic. The equations of motions are reduced /Nowacki, 1975/ to

$$\mu u_{i,kk} + (\lambda_s + \mu) u_{k,ki} + X_i = \rho \ddot{u}_i \qquad (6)$$

where $\lambda_s = \lambda + \gamma\alpha\varkappa$ referes to the adiabatic state, while μ, λ, γ and α to the isothermal state.

It is well known that a system of partial differential equations along with the appropriate boundary and initial conditions may be recast in an integral form. The general formulation and solution of the transient elastodynamic problem by combining the BIEM with the Laplace transform with respect to time was done by Cruse and Rizzo /1968/ and Cruse /1968/. The BIEM formulation of the boundary value problem for a crack in an infinite body does not lead to a unique treatment. For the boundary value problems of fracture mechanics there has been developed the formulation through the BIDE /Sládek, V. and J. Sládek, 1984; 1983/.

Let us consider a crack in an infinite elastic medium under the load $t_i\,(\bar{\eta},t)|_{S^+_{cr}} = -t_i\,(\bar{\eta},t)|_{S^-_{cr}}$. Then the Laplace transform of the displacement field can be calculated /Sládek, V. and J. Sládek, 1983; 1984; Sládek, J. and V. Sládek, 1984b/ by

$$\bar{u}_k(\bar{x},p) = -\int_{S^+_{cr}} \Delta\bar{u}_i(\bar{\eta},p)\,\bar{T}_{ik}(\bar{\eta}-\bar{x},p)\,dS_\eta \tag{7}$$

where the crack opening displacements $\Delta\bar{u}_i\,(\bar{\eta},p)$ are determined by the BIDE

$$c_{lpjr}\,n^+_p(\bar{\zeta})\Big[c_{iskt}\int_{S^+_{cr}} \Delta\bar{\varkappa}_{rsi}\,\partial'_t U_{kj}(\bar{\eta}-\bar{\zeta},p)\,dS_\eta + \rho p^2\int_{S^+_{cr}} \Delta\bar{u}_i(\bar{\eta},p)\,n_r(\bar{\eta})\,U_{ij}(\bar{\eta}-\bar{\zeta},p)\,dS_\eta\Big] = \bar{t}^+_l(\bar{\zeta},p) \tag{8}$$

where

$$\Delta\bar{\varkappa}_{rsi} = \left[n_r(\bar{\eta})\,\partial'_s - n_s(\bar{\eta})\,\partial'_r\right]\Delta\bar{u}_i(\bar{\eta},p)\,, \qquad \partial'_k = \partial/\partial\eta_k$$

The kernels $\bar{U}_{ik}(\bar{r},p)$ and $\bar{T}_{ik}(\bar{r},p)$ represent the fundamental displacements and traction vectors respectively /Sládek, V. and J. Sládek, 1983; 1984/.

Quasi-static Problem of Uncoupled Thermoelasticity

Neglecting the coupling term $\alpha\dot{u}_{k,k}$ in /2/ one obtains the uncoupled set of equations. In what follows we shall consider harmonically varying fields. Then the amplitudes follow the governing equations

$$\mu u^*_{i,kk} + (\lambda+\mu)\,u^*_{k,ki} + X^*_i = -\rho\,\omega^2 u^*_i + \gamma\Theta^*_{,i} \tag{9}$$

$$\Theta^*_{,kk} + i\,\frac{\omega}{\varkappa}\,\Theta^* = -\frac{Q}{\varkappa} \tag{10}$$

As the heat conduction process is markedly slower than the elastic wave propagation, the characteristic frequency, $\omega_1 = \varkappa/a^2$, for thermal processes is much smaller than that of elastic processes $\omega_2 = c_2/a$ where $c_2 = \sqrt{\mu/\rho}$ is the propagation velocity of the shear wave (S) and a is a characteristic length of the problem.

For instance, for iron and $a \sim 1$cm: $\omega_1 \sim 0.17\,s^{-1}$, $\omega_2 \sim 3\cdot 10^5\,s^{-1}$. In order that the characteristic frequencies ω_1 and ω_2 could be comparable the length of a crack in the infinite body should be about 60 Å. Such small distances are typical for microscopic world where continuum mechanics is not valid. That is why the inertia terms in eq. /9/ may be omitted in problems with a macroscopic crack under a harmonically varying thermal load. The boundary value problem is considered as quasi-static one. The boundary value problem

$$t_i(\bar{\eta},t)|_{S_{cr}} = 0, \quad \Theta(\bar{\eta},t)|_{S^+_{cr}} = \Theta(\bar{\eta},t)|_{S^-_{cr}} = \Theta^*(\bar{\eta},\omega)\,e^{-i\omega t} \tag{11}$$

for a crack in an infinite body leads to the solution of the BIDE for the heat flux $q^*(\bar{\eta},\omega)$ /Sládek, V. and J. Sládek, 1983; Sládek, J. and V. Sládek, 1984a/

$$\int_{S^+_{cr}} q^*(\bar{\eta},\omega)\,\Theta'^*(r,\omega)\,dS_\eta = \frac{1}{4\varkappa}\,\Theta^*(\bar{\zeta},\omega)\,, \qquad r = |\bar{\eta}-\bar{\zeta}| \tag{12}$$

and the BIDE for crack opening displacements

$$c_{lpjr}\,c_{iskt}\,n^+_p(\bar{\zeta})\int_{S^+_{cr}} \Delta\varkappa^*_{rsi}\,\partial'_t U_{kj}(\bar{r})\,dS_\eta = \gamma\Theta^*(\bar{\eta},\omega) - \gamma\,\hat{T}_{lk}(n_\zeta,\partial_\zeta)\int_V \Theta^*_{,i}(\bar{x},\omega)\,U_{ik}(\bar{r})\,dV_x \tag{13}$$

where the temperature gradients in the body are given by

$$\Theta^*_{,i}(\bar{x},\omega) = 2\varkappa\int_{S^+_{cr}} q^*(\bar{\eta},\omega)\,\Theta'^*_{,i}(r,\omega)\,dS_\eta \tag{14}$$

Fundamental solutions U_{kj}, Θ'^* and $\Theta'^*_{,i}$ may be found elsewhere /Sládek, V. and J. Sládek, 1983/.

NUMERICAL EXAMPLES

Example 1: Consider an infinitely extending linear elastic medium with a penny-shaped crack under the uniform loading over all the crack surface and pointing at opposite to the normal direction, $t^+_i\,(\bar{\eta},t) = \delta_{i3}q(t)$. In the case of a flat crack $(n^+_i(\bar{\eta}) = -\delta_{i3})$ the normal and tangential components of the crack opening displacement in /8/ are decoupled. As we are interested in the calculation of the stress intensity factor K_I, the solution of the BIDE for $\Delta\bar{u}_3\,(\bar{\eta},p)$ is sufficient. Making use the BEM the BIDE /8/ for $\Delta\bar{u}_3$ are converted into a system of algebraic equations for unknown crack opening displacements in nodal points $\Delta\bar{u}_3^b$

$$\sum_{b=1}^{N-1}\left[(\Delta\bar{u}_3^{b+1}-\Delta\bar{u}_3^{b})(Q^{ba}+P^{ba})+\Delta\bar{u}_3^{b}R^{ba}\right]=\bar{q}(p), \qquad (a=1,2,\ldots,N) \tag{15}$$

where

$$Q^{ba}=-2\int_0^{\pi}\int_0^{1} s^{-3}\left[\zeta_a\cos\varphi-\eta(\xi)\right]E(s)\,\eta(\xi)\,d\xi\, d\varphi$$

$$E(s)=\lambda^2(U_3+5U_2+U_4)-4\mu^2U_2-\mu(\lambda+2\mu)\,U_3$$

$$s=\left[\zeta_a^2-\eta^2(\xi)-2\zeta_a\eta(\xi)\cos\varphi\right]^{1/2}, \qquad \eta(\xi)=\eta_b+(\eta_{b+1}-\eta_b)\xi$$

$$P^{ba}=2\rho p^2(\lambda+2\mu)\,a\int_0^{\pi}\int_0^{1}\frac{\xi}{s}U_1(\xi)(\eta_{b+1}-\eta_b)\,d\xi\, d\varphi$$

$$R^{ba}=2\rho p(\lambda+2\mu)\,a\int_0^{\pi}\int_0^{1}\frac{U_1}{s}\eta(\xi)(\eta_{b+1}-\eta_b)\,d\xi\, d\varphi \tag{16}$$

Symbols U_1, U_2, U_3, U_4 are defined in fundamental solutions, and a stands for the radius of the crack. The nodal points η_b and ζ_a are chosen on the radius of the crack.

The stress intensity factor K_I is calculated by

$$K_I=\frac{\mu\sqrt{2\pi}}{4(1-\nu)}\;\frac{\Delta u_3(a-\varepsilon,t)}{\sqrt{\varepsilon}} \tag{17}$$

where ν is Poisson ratio. The dimensionless SIF f_I is defined as the ratio of K_I and K_I^{stat}, i.e. $f_I = K_I\sqrt{\pi}\,/2q\sqrt{a}$. For harmonically varying load $q(t) = qe^{-i\omega t}$ the time dependence of K_I is given by

$$K_I=K_I^{*}e^{-i\omega t}=2q\sqrt{\frac{a}{\pi}}\,f_I^{*}e^{-i\omega t}=2q\sqrt{\frac{a}{\pi}}\,|f_I|e^{-i\omega(t-\delta)} \tag{18}$$

where $\omega\delta$ stands for the phase angle

$$\omega\delta=tg^{-1}\left(\frac{Im\{f_I^{*}\}}{Re\{f_I^{*}\}}\right) \tag{19}$$

From the peaks in Fig. 1 a resonant effect for S-wave is seen. Figure 2 shows the dependence of the phase angle $\omega\delta$ on the dimensionless parameter $\omega a/c_1$, where c_1 is the propagation velocity of the pressure (P) wave.

The transient response of a penny-shaped crack to impact load is illustrated in Fig.3. The numerical Laplace inversion according to Manolis and Beskos /1981/ has been employed.

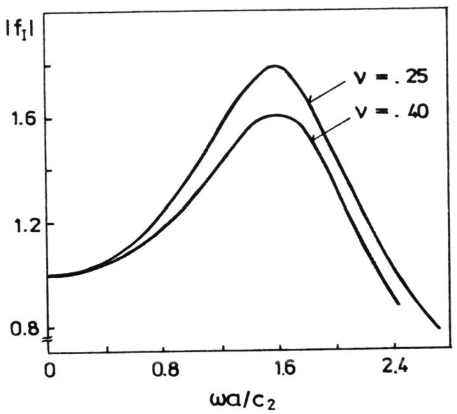

Fig. 1. Normal SIF versus frequency for a penny-shaped crack

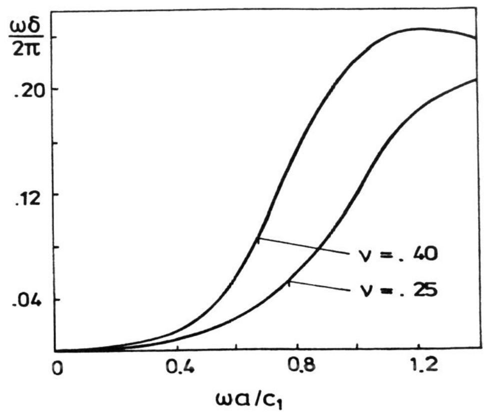

Fig. 2. Phase shift versus frequency for a penny-shaped crack

<u>Example 2</u>: In this example the geometry of the problem is the same as that in Example 1. Only the temperature loading on the crack surfaces is assumed in the form /11/ for $\theta^{*}(\bar{\eta},\omega)=\theta^{*}=$ const . For a flat crack eqs. /12/ and /14/ take place and eq. /13/ becomes

$$\int_{S_{cr}}\Delta u^{*}_{3,\alpha}(\bar{\eta},\omega)\frac{r_\alpha}{r^3}E(r)\,dS_\eta=\gamma\theta^{*}-\gamma\int_V\theta^{*}_{,i}(\bar{x},\omega)(2\mu\delta_{k3}\partial_3+\lambda\partial_k)\,U_{ik}(\bar{x}-\bar{\zeta})\,dV_x \tag{20}$$

The BIDE /12/ and /20/ are solved by the BEM in the same way as in Example 1.

The magnitude of the dimensionless SIF f_I-$K_I(\omega)$ /$K_I(0)$ and phase angle $\omega\delta$ as functions of the dimensionless parameter $\omega a^2/\varkappa$ are shown in Fig. 4. Material parameters for iron have been used: $\mu = \lambda = 7.9\ 10^4$ MPa, the coefficient of linear thermal expansion $\alpha_t = 1.67\ 10^{-5}$ deg^{-1}, $\gamma = (3\lambda + 2\mu)\alpha_t$, and the coefficient of temperature conduction $\varkappa = 1.7\ 10^{-5}$ m^2/s.

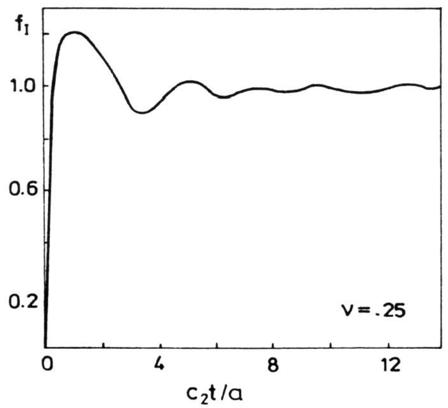

Fig. 3 Dynamic SIF as a function of time for normal impact on a penny-shaped crack

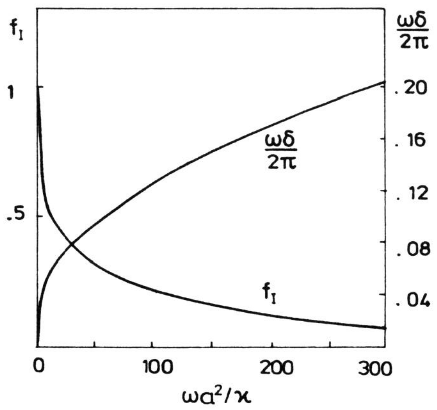

Fig. 4. Normal SIF and phase angle versus frequency for a penny-shaped crack under thermal load

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