NEAR-TIP ANALYSIS OF PLANE-STRESS MODE–I STEADY CRACK GROWTH IN LINEAR HARDENING MATERIAL WITH BAUSCHINGER EFFECT

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ABSTRACT

Most of the work on near-tip analysis are based on the assumption that the hardening of the material is isotropic. However, for most of engineering materials, which is isotropic in its virginal state the hardening is anisotropic with Bauschinger effect. In this paper the constitutive law for anisotropic hardening suggested by Kadaschevich and Novozhilov (1958) is used to obtain the near-tip fields for plane-stress mode-I steady crack growth, with the plastic reloading zone being considered. The numerical results are compared with those of Amazigo and Hutchinson (1977) for linear isotropic-hardening material with the reloading zone being neglected.

KEYWORDS

Steady crack growth; linear hardening material; Anisotropic hardening; Bauschinger effect; Near-tip fields.

INTRODUCTION

Most of the work on near-tip analysis of hardening materials are based on the assumption that the hardening is isotropic. For example, Amazigo and Hutchinson (1977) obtained the singularity fields at the tip of a steadily growing crack, with the plastic reloading along the flank behind the crack tip being neglected. For most of engineering materials the hardening is anisotropic with Bauschinger effect. It was pointed by Xie and Hwang (1983) for mode-III crack in power-law hardening material and by Zhang's and Hwang (1983) for plane-strain mode-I crack that the Bauschinger effect has a nonneglible effect on the near-tip fields for growing cracks. In this paper the Bauschinger effect is considered in the near-tip analysis of plane-stress mode-I steady crack growth, based on the constitutive law for anisotropic hardening suggested by Kadaschevich and Novozhilov (1958), with the plastic reloading zone being considered. Comparison of the numerical results with those of Amazigo and Hutchinson (1977) confirmed the nonneglible role of the Bauschinger effect in near-tip fields for growing cracks.

BASIC EQUATIONS AND BOUNDARY CONDITIONS

The constitutive equations for linear anisotropic hardening material, as suggested by Kadaschevich and Novozhilov (1958) can be written in the form

$$\dot{\varepsilon}^{p}_{ij} = \frac{1}{2h\,\overset{\circ}{\sigma}_e}\,\dot{\overset{\circ}{\sigma}}_e\,\overset{\circ}{\sigma}{}^{*}_{ij} \tag{1}$$

$$\alpha_{ij} = 2g\,\varepsilon^{p}_{ij} \tag{2}$$

Here α_{ij} denote stresses corresponding to the center of the yielding surface, "p" plastic strain components, superdot "." the time-derivative d/dt, supercirclet "o" active stress components ($\overset{\circ}{\sigma}_{ij} = \sigma_{ij} - \alpha_{ij}$), star "*" the deviator components ($\overset{\circ}{\sigma}{}^{*}_{ij} = \overset{\circ}{\sigma}_{ij} - \frac{1}{3}\overset{\circ}{\sigma}_{kk}\delta_{ij}$), and $\overset{\circ}{\sigma}_e$ the equivalent active stress,

$$\overset{\circ}{\sigma}_e = \left(\frac{3}{2}\,\overset{\circ}{\sigma}{}^{*}_{ij}\,\overset{\circ}{\sigma}{}^{*}_{ij}\right)^{1/2} \tag{3}$$

h and g are material constants, namely

$$\frac{1}{h} = \frac{3}{\beta}\left(\frac{1}{E_t} - \frac{1}{E}\right), \quad \frac{1}{g} = \frac{3}{1-\beta}\left(\frac{1}{E_t} - \frac{1}{E}\right) \tag{4}$$

where E denotes Young's modulus, E_t tangent modulus following yield and β parameter related to anisotropy of hardening with the extreme value $\beta = 1$ for isotropic hardening and $\beta = 0$ for ideal Bauschinger effect. Here and hereafter sum convention is adopted for repeating indices, with the Latin indices i, j, ... ranging over 1, 2, 3 and Greek indices λ , ω , ... over 1, 2 only. The time-derivative of eq. (3) gives

$$\dot{\overset{\circ}{\sigma}}_e = \frac{3}{2}\,\overset{\circ}{\sigma}{}^{*}_{ij}\,\dot{\overset{\circ}{\sigma}}_{ij} / \overset{\circ}{\sigma}_e$$

Replacing $\dot{\overset{\circ}{\sigma}}_{ij}$ by $\dot{\sigma}_{ij} - 2g\,\dot{\varepsilon}^{p}_{ij}$ and making use of eqs. (1) and (4), we can reduce the above eq. to the form

$$\dot{\overset{\circ}{\sigma}}_e = \frac{3}{2}\,\beta\,\overset{\circ}{\sigma}{}^{*}_{ij}\,\dot{\sigma}_{ij} / \overset{\circ}{\sigma}_e \tag{5}$$

Let x_1, x_2 be the moving cartesian coordinates with origin at the crack-tip. Denote by $\underset{\sim}{\sigma}$ the stress tensor, $\underset{\sim}{\varepsilon}$ the strain tensor and $\underset{\sim}{s} = \dot{\underset{\sim}{\sigma}}$, $\underset{\sim}{\mathcal{E}} = \dot{\underset{\sim}{\varepsilon}}$ their time-rates. Then the components of $\underset{\sim}{s}$ can be expressed in terms of the rate of stress function $\dot{\varphi}$

$$s_{11} = \frac{\partial^2\dot{\varphi}}{\partial x_2^2}, \quad s_{22} = \frac{\partial^2\dot{\varphi}}{\partial x_1^2}, \quad s_{12} = -\frac{\partial^2\dot{\varphi}}{\partial x_1 \partial x_2} \tag{6}$$

or, in polar coordinates (r,θ) centered at the tip.

$$s_{rr} = \frac{1}{r}\frac{\partial\dot{\varphi}}{\partial r} + \frac{1}{r^2}\frac{\partial^2\dot{\varphi}}{\partial\theta^2}, \quad s_{\theta\theta} = \frac{\partial^2\dot{\varphi}}{\partial r^2}, \quad s_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\dot{\varphi}}{\partial\theta}\right) \tag{6'}$$

And the components of $\underset{\sim}{\mathcal{E}}$ can be expressed in terms of components of velocity vector

$$\mathcal{E}_{11} = \frac{\partial v_1}{\partial x_1}, \quad \mathcal{E}_{22} = \frac{\partial v_2}{\partial x_2}, \quad \mathcal{E}_{12} = \frac{1}{2}\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right) \tag{7}$$

or, in polar coordinates,

$$\mathcal{E}_{rr} = \frac{\partial v_r}{\partial r}, \quad \mathcal{E}_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial\theta} + v_r\right), \quad \mathcal{E}_{r\theta} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}v_\theta\right) \tag{7'}$$

Referring to the results for isotropic hardening(β= 1) of Amazigo and Hutchinson (1977), we shall look for plane-stress solutions (s_{33} = 0, σ_{33} = 0) corresponding to dominant singularity

$$\dot{\varphi} = A_0\,r^{s+1} f_0(\theta) \tag{8}$$

$$\left\{s_{\lambda\omega},\ \dot{\overset{\circ}{\sigma}}_e\right\} = A_0\,r^{s-1}\left\{t_{\lambda\omega}(\theta),\ t^{\circ}(\theta)\right\} \tag{9}$$

$$\left\{\sigma_{\lambda\omega},\ \overset{\circ}{\sigma}_e,\ \overset{\circ}{\sigma}{}^{*}_{ij}\right\} = A_0\,r^{s}\left\{\Sigma_{\lambda\omega}(\theta),\ \Sigma^{\circ}(\theta),\ S_{ij}(\theta)\right\} \tag{10}$$

$$\left\{v_1,\ v_2\right\} = A_0\,r^{s}\left\{g_0(\theta),\ h_0(\theta)\right\} \tag{11}$$

$$\left\{u_1,\ u_2\right\} = A_0\,r^{s+1}\left\{G_0(\theta),\ H_0(\theta)\right\} \tag{12}$$

$$\mathcal{E}_{ij} = A_0\,r^{s-1}\,\psi_{ij}(\theta), \quad \varepsilon_{ij} = A_0\,r^{s}\,E_{ij}(\theta) \tag{13}$$

where u_1, u_2 are displacement components, A_0 is an amplitude factor. The exponent s and the functions of θ appearing in the right sides of eqs. (8)-(13) are to be determined. From (3) and (5), we have

$$\Sigma^\circ(\theta)=\left\{\frac{3}{2}S_{ij}(\theta)S_{ij}(\theta)\right\}^{1/2},\quad \text{with}\quad S_{33}(\theta)=-S_{\lambda\lambda}(\theta)$$
$$t^\circ(\theta)=\frac{3}{2}\beta S_{\lambda\omega}(\theta)t_{\lambda\omega}(\theta)/\Sigma^\circ(\theta) \tag{14}$$

Plastic incompressibility requires

$$E_{33}(\theta)=-E_{\lambda\lambda}(\theta)+(1-2\nu)\Sigma_{\lambda\lambda}(\theta)/E\ ,$$
$$\psi_{33}(\theta)=-\psi_{\lambda\lambda}(\theta)+(1-2\nu)t_{\lambda\lambda}(\theta)/E \tag{15}$$

With (8),(9), eqs (6') lead to

$$t_{rr}(\theta)=(s+1)f_0(\theta)+f_0''(\theta)\ ,$$
$$t_{\theta\theta}(\theta)=s(s+1)f_0(\theta)\ ,\qquad t_{r\theta}(\theta)=-sf_0'(\theta) \tag{16}$$

and with (11) and (13), eqs (7) lead to

$$\psi_{11}(\theta)=s\cos\theta\, g_0(\theta)-\sin\theta\, g_0'(\theta)\ ,$$
$$\psi_{22}(\theta)=s\sin\theta\, h_0(\theta)+\cos\theta\, h_0'(\theta)\ , \tag{17}$$
$$\psi_{12}(\theta)=\frac{1}{2}\left\{(g_0'(\theta)+sh_0(\theta))\cos\theta+(sg_0(\theta)-h_0'(\theta))\sin\theta\right\}$$

where $'=d/d\theta$.
Identify the time parameter t with the increase of crack length, so that in steady state we have for any scalar or tensor fields ( )

$$(\dot{\ \ })=\frac{d}{dt}(\ \ )=-\frac{\partial}{\partial x_1}(\ \ ) \tag{18}$$

Applied to stress tensor $\underset{\sim}{\sigma}$ and strain tensor $\underset{\sim}{\varepsilon}$, (18) gives, respectively,

$$\sin\theta\,\Sigma_{\lambda\omega}'(\theta)=s\cos\theta\,\Sigma_{\lambda\omega}(\theta)+t_{\lambda\omega}(\theta) \tag{19}$$

$$\sin\theta\,E_{\lambda\omega}'(\theta)=s\cos\theta\,E_{\lambda\omega}(\theta)+\psi_{\lambda\omega}(\theta) \tag{20}$$

Some of the equations in (19) and (20) are integrable after substituting (16) and (17) into them, and lead to

$$\Sigma_{12}(\theta)=(s+1)\sin\theta\, f_0(\theta)+\cos\theta\, f_0'(\theta)\ ,$$
$$\Sigma_{22}(\theta)=-(s+1)\cos\theta\, f_0(\theta)+\sin\theta\, f_0'(\theta)\ , \tag{21}$$
$$E_{11}(\theta)=-g_0(\theta)$$

The remaining equations in (19) and (20) are

$$\sin\theta\,\Sigma_{11}'(\theta)=s\cos\theta\,\Sigma_{11}(\theta)+t_{11}(\theta) \tag{22}$$
$$\sin\theta\,E_{12}'(\theta)=s\cos\theta\,E_{12}(\theta)+\psi_{12}(\theta) \tag{23}$$
$$\sin\theta\,E_{22}'(\theta)=s\cos\theta\,E_{22}(\theta)+\psi_{22}(\theta) \tag{24}$$

From (9), (10), (13) and Hooke's law, the constitutive eq. (1)is reduced to

$$\psi_{\lambda\omega}(\theta)=\frac{1+\nu}{E}t_{\lambda\omega}(\theta)-\frac{\nu}{E}t_{\pi\pi}(\theta)\delta_{\lambda\omega}+\frac{\mu}{2h}t^\circ(\theta)S_{\lambda\omega}(\theta)/\Sigma^\circ(\theta) \tag{25}$$

where ν — Poisson's ratio, $\mu=1$ for plastic loading, and $\mu=0$ for elastic response, $t_{\lambda\omega}(\theta)$ can be obtained by transformation from (16), $\psi_{\lambda\omega}(\theta)$ are substituted from (17) and

$$S_{\lambda\omega}(\theta)=(1+2g\frac{1+\nu}{E})\Sigma_{\lambda\omega}(\theta)-(\frac{1}{3}+2g\frac{\nu}{E})\Sigma_{\pi\pi}(\theta)\delta_{\lambda\omega}-2gE_{\lambda\omega}(\theta) \tag{26}$$

where $\Sigma_{12}(\theta)$, $\Sigma_{22}(\theta)$ and $E_{11}(\theta)$ are substituted from (21). Eqs. (22)—(25) are the six governing equations for the six unknown functions $f_0(\theta)$, $g_0(\theta)$, $h_0(\theta)$, $\Sigma_{11}(\theta)$, $E_{12}(\theta)$ and $E_{22}(\theta)$ for plastic zone (μ=1) as well as for unloading zone (μ=0). The functions $G_0(\theta)$, $H_0(\theta)$ for displacements can be determined throug the following relations obtained from (11), (12) and (18):

$$\sin\theta\,G_0'(\theta)=(s+1)\cos\theta\,G_0(\theta)+g_0(\theta)$$
$$\sin\theta\,H_0'(\theta)=(s+1)\cos\theta\,H_0(\theta)+h_0(\theta) \tag{27}$$

The crack-tip geometry is shown in Fig. 1. Since for hardening materials stresses and strains should be continuous across boundary Γ between neighboring zones, we have the contiguity condition

$$[f_0(\theta)]_\Gamma=[f_0'(\theta)]_\Gamma=[g_0(\theta)]_\Gamma=[h_0(\theta)]_\Gamma=0 \tag{28}$$

where $[P]_\Gamma$ denotes the jump of P across Γ .

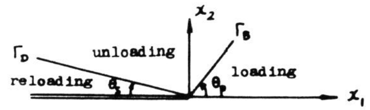

Fig. 1. Crack-tip geometry.

At unloading boundary an additional contiguity condition should be added to (28):

$$\dot{\sigma}_e^\circ(\theta_p+0) = \dot{\sigma}_e^\circ(\theta_p-0) = 0 \qquad (29)$$

The location of reloading boundary Γ_D is determined from

$$\sigma_e^\circ(x_2)\big|_{\Gamma_D} = \sigma_e^\circ(x_2)\big|_{\Gamma_B} \quad \text{for same } x_2 \qquad (30)$$

By symmetry the boundary conditions at $\theta = 0$ are

$$f_o'(0) = 0 \ , \quad g_o'(0) = 0 \ , \quad h_o(0) = 0 \qquad (31)$$

The traction-free conditions at $\theta = \pi$ requires

$$f_o(\pi) = f_o'(\pi) = 0 \qquad (32)$$

In the unloading zone eq. (25) (with $\mu=0$) can be intergrated in closed form. The basic equations are integrated numerically over the loading and reloading plastic zones. The normalizing condition is taken as $\Sigma^\circ(0)=1$, which coincides with the normalizing condition in the work of Amazigo and Hutchinson (1977) in case of isotropic hardening (β=1). The values of f"(0) and the exponent of singularity s are assumed to start the numerical integration from θ=0, and the values of these two parameters are refined by iteration until the boundary conditions (32) at $\theta = \pi$ are satisfied with a prescribed accuracy.

NUMERICAL RESULTS AND DISCUSSIONS

The numerical results exhibit no dependence on the Poisson's ratio ν. Fig.2 shows the variation of the singularity exponent s with the tangent modulus ratio $\alpha = E_t/E$ and the parameter β of hardening anisotropy. The angles θ_p and θ_s subtended by the loading and the reloading plastic zone are tabulated in Tables 1 and 2. The structures of the near-tip zones are shown in Fig.3, in which the points "x" and "o" denote computed cases which turn out with and without reloading zone, respectively. From Fig.3 it follows that the neglect of the reloading zone by Amazigo and Hutchinson(1977) is justified except for the case of very low hardening (i.e. for small α). For α=0.01 the angular distribution of stress components is shown in Fig.4(a),(b) with β as parameters, and that of plastic strain-rate components is shown in Fig.5. The results for the case of isotropic hardening (β=1) agree quite well with those of Amazigo and Hutchinson (1977). These figures show the significant role of the plastic anisotropic hardening.

TABLE 1 Values of θ_p

α \ β	1	0.9	0.7	0.5	0.3	0.1
0.75	1.4099	1.4522	1.5921	1.9818	3.1086	3.1368
0.25	1.3714	1.4086	1.5319	1.9233	3.0192	3.1232
0.10	1.2854	1.3216	1.4438	1.8624	3.0038	3.1187
0.01	1.0662	1.1088	1.2524	1.7846	3.0303	3.1228

TABLE 2 Values of θ_s

α \ β	1	0.9	0.7	0.5	0.3	0.1
0.75	0	0	0	0	0	0
0.25	0	0	0	0	0.215×10^{-2}	0
0.10	0	0	0	0	0.155×10^{-1}	0
0.10	0.552×10^{-5}	0.813×10^{-5}	0.211×10^{-6}	0.397×10^{-2}	0.515×10^{-1}	0.896×10^{-3}

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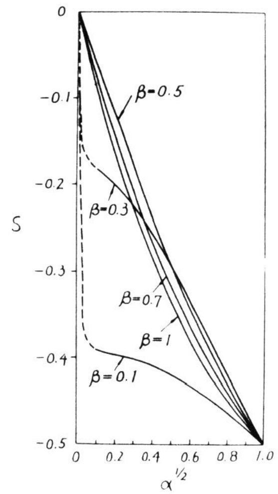

Fig. 2. Variation of singularity exponent s with α and β.

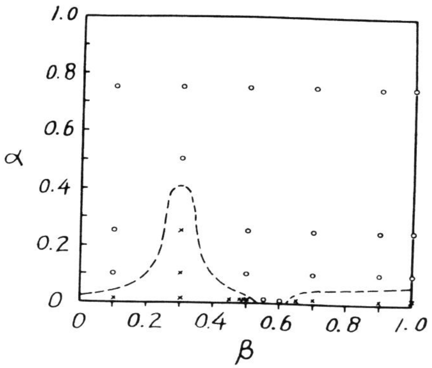

Fig. 3. Structure of the near-tip zone ("x" denotes the case when reloading zone exists. "o" the case without reloading zone, the dotted curve is the partition estimated).

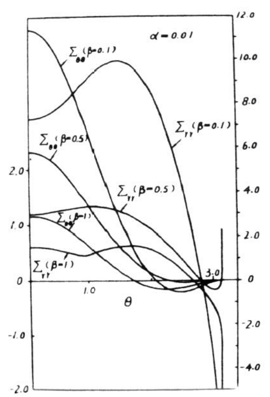

Fig. 4(a). Angular distribution of stress components Σ_{rr}, $\Sigma_{\theta\theta}$ for α =0.01 and different β . (The right-side ordinate scale for β=0.1)

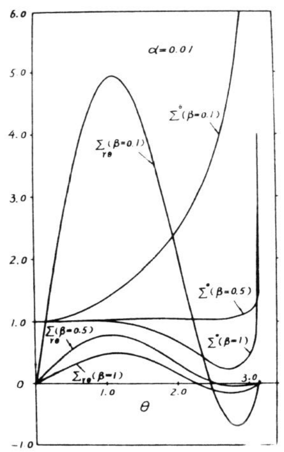

Fig. 4(b). Angular distribution of stress component $\Sigma_{r\theta}$ and active equivalent stress Σ^{o} for α =0.01 and different β .

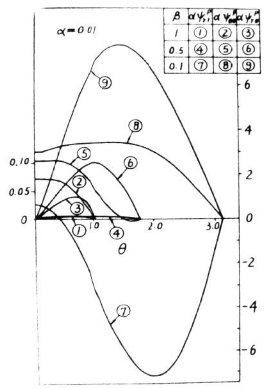

Fig. 5. Angular distribution of plastic strain-rate components $\alpha\psi^{p}_{rr}$, $\alpha\psi^{p}_{\theta\theta}$, $\alpha\psi^{p}_{r\theta}$ for α = 0.01 and different β . (The right-side ordinate scale for β = 0.1).