FINITE ELEMENT EVALUATION OF FRACTURE MECHANICS PARAMETERS USING RAPID MESH REFINEMENT

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ABSTRACT

This paper presents a simple and effective technique for the evaluation of fracture mechanics parameters. Finite elements are applied as a reliable tool for singularity problem solution. Because of the limitations on application on material and geometrical nonlinearity special elements are avoided. It is the rapid mesh refinement technique that enables accurate modelling of a cracked body response on the applied load. All classical elements are admissible. Simplex isoparameter triangular elements are applied here due to simplicity and universality in their formulation. A simple procedure, based on displacement field, is given for the evaluation of stress intensity factor and J-integral. The procedure requires linear variation of displacements inside an element. An analysis of the errors appearing during the solution process is given, enabling the optimized choice of the error influencing parameters. A final result is a possibility to use simple, commercially available programme and simplex elements for an accurate evaluation of fracture mechanics parameters with relatively small number of degrees of freedom and without limitation on material nature.

KEYWORDS

Finite elements, Rapid mesh refinement, C^0 convergency, Displacement formulation, Discretization and computational errors, Linear and nonlinear elastic behaviour, Elasto-plastic behaviour.

INTRODUCTION

Fracture mechanics parameters can be evaluated if the displacement or stress field in a certain domain containing a crack is known. The calculation of the displacement and stress field around crack is well-known problem in the theory of elasticity. The singularity problems are described through partial differential equations of 2m-th order and appropriate boundary conditions (Whiteman and Akin, 1979). The attention is restricted here on the second order (m=1) equations and prescribed displacements as a boundary condition. Therefore, the displacement formulation of finite elements is used and C^0 convergency requirements are essential.

Analytical solutions of the aforementioned problem are restricted on a simple shapes of problem domain and a certain class of the prescribed displacements. In any other case numerical methods are more efficient. Today the finite element method is the most widely used for obtaining the solution of many problems of the theory of elasticity. However, an early attempt of finite element application on the evaluation of the stress intensity factors was unsatisfactory, although large number of elements were used (Chan and co-workers, 1970, used classical elements and uniform mesh refinemnet). As a consequence special elements were developed (Barsoum, 1976 and 1977). The special elements are efficient in many fracture mechanics problems, but they suffer from two main disadvantages:

- an additional effort is needed for their formulation
- real constitutive relations and/or large displacement formulation is not directly applicable.

On the other hand it was found that the efficiency of the classical elements applied on the singularity problems can be much improved using technique named rapid mesh refinement (Fried, 1971a, Fried and Yang, 1972). The application of this technique on the stress factor intensity evaluation has given excellent results (Johnson, 1981, used constant and linear strain triangular elements and obtained errors both in the strain energy and stress intensity factor less than 1% with only 138 degrees of freedom. The problem solved was a double edge cracked tension plate). However, one should consider the magnitude of all errors appearing during the solution process. Therefore, besides the discretization error, the computational error should be also discussed.

DISCRETIZATION AND COMPUTATIONAL ERRORS

If an interpolation function inside an element includes a complete set of a polynomial of the degree p, the discretization error of the strain energy for the 2m-th order problems can be expressed as (Fried and Yang, 1972) $O(h^{2(p+1-m)})$, where h is the diameter of an element in the uniform mesh. For the fixed value of m (m=1 for the second order problems), the discretization error can be influenced by p and h, the order and the size of an element. In this paper p will be chosen due to a procedure for J-integral evaluation, getting the fixed value p=1. For the problem containing a singularity, whose solution is not a polynomial, uniform mesh of the elements with polynomial interpolation functions produces the largest discretization error in the elements around a singularity. Therefore, the only way to reduce the discretization error is to reduce the size of the elements around singularity. The principle of this reduction is the same discretization error in all elements (Fried and Yang, 1972).

The special type of the singularity is considered now - a crack in a body under the given load. The displacement field around a crack tip is given in the form $u=r^{\alpha}$, where r is the radial distance from a crack tip and α is the parameter characterizing singularity. If the diameter of the smallest element in the nonuniform mesh is denoted by h, the diameters of elements in a radial direction are $h_i=\alpha_i h$, where α_i is the coefficient determined from the principle of the same discretization error in all elements:

$$\alpha_i = i^{(p-\alpha+1/2)/\alpha} \qquad (1)$$

It was proved (Fried and Yang, 1972) that the full rate of convergence is regained in this way. Consequently, the discretization error is $O(h^2)$ for m=1, p=1 and α=1/2 (Linear elastic fracture mechanics).

The computational error mainly appears as a consequence of the round-off processes in the computer memory. The effect of numerical integration and curved boundaries does not exist here because numerical integration is exact and there are no curved boundaries in the chosen example (for details of this effect see Ciarlet and Raviart, 1972). Therefore, we concentrate here on the round-off error, given as (Fried, 1971b):

$$\frac{\|\delta x\|}{\|x\|} = c \cdot 10^{-s} \cdot C_n(K) \qquad (2)$$

where s is a number of decimals in the computer word, $C_n(K)$ is a spectral condition number of the global stiffness matrix and c is a numerical constant. Using the relation for spectral condition number in the case of a non-uniform mesh (Fried, 1972), the round-off error can be expressed as:

$$\frac{\|\delta x\|}{\|x\|} = c_1 \cdot 10^{-s} \cdot \frac{h_{max}}{h_{min}} \cdot N \qquad (3)$$

where N is a number of elements, h_{max} and h_{min} are the maximum and minimum values for the diameters, respectively, and c_1 is a numerical constant.

It is usefull to discuss now the influence of the parameters on the magnitude of the errors. It is clear that the larger number of elements reduces the discretization error, the reduction being larger for the higher order elements (the larger p). However, the larger p causes the larger diameter ratio (see eqn 1), what is, together with the larger number of elements, undesirable from the computational error point of view. It is impossible to find the "universally" best combination of the parameters influencing the magnitude of errors (p, N, h_{max}/h_{min}, s). The criteria for satisfying choice is (Fried, 1972):

$$10^{-s} \cdot C_n(K) < 1 \qquad (4)$$

EVALUATION OF STRESS INTENSITY FACTORS AND J-INTEGRAL

Knowing the displacement or the stress field in a domain containing a crack there are few possibilities for the stress intensity factor and J-integral evaluation. It is optional to choose between the displacement or the stress field. Since the finite element formulation here is based on the displacements, it is more accurate to use the displacement field for any further calculation. A standard technique for the stress intensity factor evaluation is an extrapolation of $K_I=K_I(r)$ curve to obtain the value for K_I at r=0. The curve is obtained from the nodes along the radial direction $\theta=\pi$, using the relation for the plane problem:

$$K_I = \frac{Gv}{k+1}\sqrt{\frac{2\pi}{r}} \qquad (5)$$

where G is the shear modulus, v is a displacement in y direction and k is a parameter characterizing the type of a problem: k=3-4ν for the plane strain and k=(3-ν)/(1+ν) for the plane stress, ν being Poisson´s Ratio.

The starting point for J-integral evaluation is its definition as a path independent integral on the contour Γ:

$$J = \int_\Gamma \left(W dy - \sigma^{ij} n_j \frac{\partial u_i}{\partial x} ds\right) \tag{6}$$

where W is the strain energy given by $W=\frac{1}{2}\sigma^{ij}\varepsilon_{ij}$, σ^{ij} and ε_{ij} are components of the stress and strain tensors, n_j is an outward normal on Γ and $\frac{\partial u_i}{\partial x}$ is a gradient of displacements.

It is essential to transform the relation (6) into the form dependent on the displacement field only. Such a transformation involves constitutive relations which are taken here as for a homogeneous linear isotropic body, as well as strain-displacement relationship, which is taken here as for the small displacement gradients. For the plane problem the final relation for J-integral is the function of displacement gradients only (the relation which is not restricted on the plane problem can be find in Sedmak and co-workers, 1981):

$$J=\frac{G}{2}\int_\Gamma \left\{\left[k_1\left(\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right)\left(\frac{\partial v}{\partial y}+\frac{\partial u}{\partial x}\right)+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)\right] dy+2\left(\frac{\partial u \partial u}{\partial x \partial y}+k_2\frac{\partial u \partial v}{\partial x \partial x}+k_1\frac{\partial v \partial v}{\partial x \partial y}\right)dx\right\} \tag{7}$$

where k_1 and k_2 are constants characterizing the type of the problem:

- for the plane strain $\quad k_1=\frac{2(1-\nu)}{1-2\nu} \quad k_2=\frac{1}{1-2\nu}$ (8)

- for the plane stress $\quad k_1=\frac{2}{1-\nu} \quad k_2=\frac{1+\nu}{1-\nu}$ (9)

The displacement gradients can be calculated directly from the known displacement field. Under the assumption of a linear variation of displacements, what is satisfied in the simplex finite elements, it is possible to transform the integral relation (7) into a summ suitable for the further calculation:

$$J = \frac{G}{2}\sum_{J,K=1,2}^{N-1,N}(FY\cdot y_{KJ}+FX\cdot x_{KJ}) \tag{10}$$

where FY and FX are the expressions multiplying dy and dx in the relation (7), respectively, being constants inside an element with a linear variation of displacements, y_{KJ} and x_{KJ} are the differencies of the coordinates of points K and J along the integration path and N is a number of elements along the integration path. This technique was tested (see Berković, 1980) in the case of a centrally cracked plate using eight different paths. The obtained values for J-integral varied only 1% from the mean value.

In the case of linear elastic problem J-integral reduces to the strain energy release rate and can be directly related with the stress intensity factor. In the case of the plane problem the relation between them is:

$$K_I = \sqrt{\frac{8G}{k+1}\cdot J} \tag{11}$$

RESULTS

The double edge cracked tension plate under plane strain condition was tested (Fig. 1). Due to the symmetry only a quarter of the plate was considered. The basic finite element mesh is presented at Fig. 2.

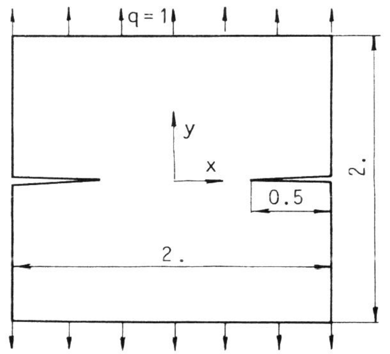

Fig. 1. The double edge cracked plate

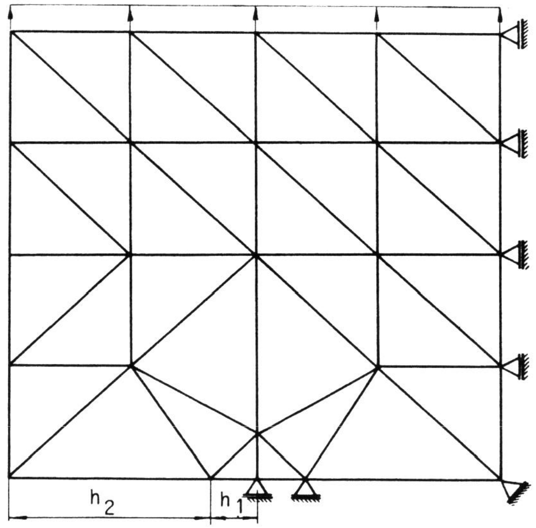

Fig. 2. The basic finite element mesh

Five meshes were tested here, all of them obtained by adding one new "layer" of elements in the lower half of the domain. K_I was calculated by the extrapolation technique and via J-integral. Results are given in Table 1.

TABLE 1 Results of U, K_I and J

Mesh	I	II	III	IV	V
Number of elements N	32	44	60	84	108
Strain energy U	2.593	2.712	2.780	2.825	2.842
K_I (extrapolation)	1.27	1.38	1.45	1.50	1.52
J	1.92	2.18	2.30	2.36	2.40
K_I (via J)	1.45	1.55	1.59	1.61	1.62

The exact value for the strain energy is given for a quarter of the plate (Babuška and Szabo, 1982) as U=0.73422. Hellen has reported K_I=1.669 using virtual crack extension method (Hellen, 1973).

The largest computational error belongs to the fifth mesh. According to eqn (3), for N=108 and h_{max}/h_{min}=45.5, using single precision (s=7.2), the computational error is:

$$\frac{\|\delta x\|}{\|x\|} = c_1 \cdot 10^{-7.2} \cdot 45.5 \cdot 108 \approx 5 \cdot 10^{-4}$$

DISCUSSION

It is clear that relatively small number of elements give an accurate prediction for both the strain energy and stress intensity factor evaluated via J-integral. The extrapolation technique was not so successfull. It is also clear that the single precision is sufficient for this type of a problem. If the more complicated problems should be solved, with much larger number of elements and ratio h_{max}/h_{min}, the double precision is to be used. This situation can be avoided using higher order elements, but they require the more complicate and less accurate relations for J-integral evaluation.

CONCLUSIONS

Rapid mesh refinement is very simple and effective technique for the evaluation of fracture mechanics parameters. The formulation itself is not limited regarding material properties - it is therefore possible to treat linear elastic, nonlinear elastic and elasto-plastic behaviour. However the care is to be taken with J-integral evaluation for elasto-plastic behaviour, since its path independency is doubtfull in that case. Numerical experiments indicate large error of J-integral along the paths close to the crack tip, but the average value is in good agreement with predicted values (Shiratori and Miyoshi, 1980). Many authors have tried to reformulate J-integral in order to regain the path independency (Hellen, 1980). It is also suggested to simulate the elasto-plastic behaviour by an equivalent nonlinear elastic behaviour. This is satisfactory provided that no energy release rate is attached to

J-integral (Knott, 1980).

The possibility of using simple, standard programme with the classical elements is a great benefit of this technique. Commercially available programme is used here (Hinton and Owen, 1977), modified only to accept the simplex triangular isoparametric elements. In the case of nonlinear elastic or elasto-plastic behaviour an excellent extension is the other book of the same authors (Hinton and Owen, 1980).

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