APPLICATION OF THE EIGENFUNCTION EXPANSION METHOD TO DETERMINE CRACK-TIP STRESSES IN THIN PLATES SUBJECTED TO BENDING

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ABSTRACT

Expressions for the crack-tip stress field in thin plates subjected to bending are derived using the three-dimensional eigenfunction expansion solutions developed by Hartranft and Sih. The approach presented in this paper avoids errors introduced by the Kirchoff boundary conditions in earlier work by Williams. Expressions for the maximum shearing stress, τ_{max}, obtained from the present approach are compared with those from Williams' solution and the respective trajectories are illustrated for the case of skew-symmetric bending moments applied at infinity.

KEYWORDS

Crack-tip stresses, thin plates, skew-symmetric bending, eigenfunction expansion.

INTRODUCTION

The eigenfunction expansion method was first applied by Williams (1957,1961) to obtain series solutions for the crack-tip stresses and displacements in cracked bodies subjected to extensional and bending loads. Williams' solutions for the bending problem were, however, limited by approximations associated with the application of Kirchoff's boundary conditions along the crack border. Subsequently, Hartranft and Sih (1968), Knowles and Wang (1961), and Wang (1970) analyzed this problem using Reissner's bending theory. Hartranft and Sih (1968) and Knowles and Wang (1961) considered the case of symmetric bending moments at infinity, while Wang (1970) analyzed the case of skew-symmetric (twisting) moments at infinity. Hartranft and Sih's (1968) analysis showed that the transverse shear stresses for symmetric bending at infinity do not contain singular terms. Wang's (1970) analysis indicates that for skew-symmetric bending at infinity the transverse shear stresses contain singular terms. A detailed study of Wang's (1970) analysis by Subramonian (1980) identified certain truncation errors in Wang's evaluation of the transverse shear stresses. The corrected solutions indicated that Wang's results are correct except for the existence of singular terms for the transverse shear stresses. In the present analysis, similar conclusions are reached using a simpler approach based on the three-dimensional analysis of Hartranft and Sih (1969) and by introducing the conventional approximations for thin plates in bending.

THREE-DIMENSIONAL EIGENFUNCTION ANALYSIS

A brief summary of the three-dimensional eigenfunction analysis by Hartranft and Sih (1969) is included in this section. In this approach the displacement vector $\underline{u}$ is first expressed in a double series as

$$2\mu\underline{u} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} r^{\lambda_m+n}\, \underline{u}_n^{(m)}(\theta,z;\lambda_m) \quad , \tag{1}$$

where λ_m (m=0,1,2,...) are the eigenvalues, μ is Lame's constant, and the components of $\underline{u}_n^{(m)}$ $(u_n^{(m)}, v_n^{(m)}, w_n^{(m)})$ are functions of θ and z only. From Eq. (1) the corresponding stress components can be obtained using constitutive relations for a three-dimensional isotropic body. Applying the crack border boundary conditions (without introducing any approximations), the λ_m (m=0,1, 2,...) are found to be the roots of the characteristic-value equation

$$\sin 2\pi\lambda_m = 0 \quad , \tag{2}$$

which gives

$$\lambda_m = \frac{m}{2} \quad , \quad m=0,1,2,... \tag{3}$$

Since the displacements must be bounded as $r \to 0$, negative values of m have been excluded from Eq. (3). From Eq. (3), the double-series representation of each component of the displacement vector in Eq. (1) can be reduced to a single-power series in r, as

$$\begin{aligned} 2\mu u_r &= \sum_{n=0}^{\infty} r^{n/2}\, f_n(\theta,z) \\ 2\mu v_\theta &= \sum_{n=0}^{\infty} r^{n/2}\, g_n(\theta,z) \\ 2\mu w_z &= \sum_{n=0}^{\infty} r^{n/2}\, h_n(\theta,z) \end{aligned} \tag{4}$$

where f_n, g_n and h_n are unknown functions which will be evaluated using the equations of equilibrium. From Eqs. (4) the expressions for the stresses can be derived as

$$(1-2\nu)\sigma_{rr} = \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{[\tfrac{n}{2} - (\tfrac{n}{2}-1)\nu]f_n + \nu[\frac{\partial g_n}{\partial\theta} + \frac{\partial h_{n-2}}{\partial z}]\right\} \quad ,$$

$$\begin{aligned} (1-2\nu)\sigma_{\theta\theta} &= \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{[1+(\tfrac{n}{2}-1)\nu]f_n + (1-\nu)\frac{\partial g_n}{\partial\theta} + \nu\frac{\partial h_{n-2}}{\partial z}\right\} \quad , \\ (1-2\nu)\sigma_{zz} &= \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{[(\tfrac{n}{2}+1)f_n + \frac{\partial g_n}{\partial\theta}]\nu + (1-\nu)\frac{\partial h_{n-2}}{\partial z}\right\} \quad , \\ 2\tau_{r\theta} &= \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{\frac{\partial f_n}{\partial\theta} + (\tfrac{n}{2}-1)g_n\right\} \quad , \\ 2\tau_{\theta z} &= \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{\frac{\partial h_n}{\partial\theta} + \frac{\partial g_{n-2}}{\partial z}\right\} \quad , \quad \text{and} \\ 2\tau_{zr} &= \sum_{n=0}^{\infty} r^{\frac{n}{2}-1}\left\{\frac{n}{2}h_n + \frac{\partial f_{n-2}}{\partial z}\right\} \qquad \text{where } \nu = \text{Poisson's ratio.} \end{aligned} \tag{5}$$

In Eqs. (4) and (5) the unknown functions f_n, g_n and h_n can be evaluated using the equations of equilibrium in terms of displacements. Hartranft and Sih (1969) have evaluated these functions for three-dimensional crack problems and provided the following results, where terms containing $r^{-\frac{1}{2}}$, r^0 and $r^{\frac{1}{2}}$ have been retained to determine whether they result in singular contributions to $\tau_{\theta z}$ and τ_{rz} ($\tau_{\theta\tau}$ and τ_{rz} are evaluated using the equations of equilibrium for thin plates as explained in the following section)

$$\begin{aligned} \sigma_{rr} = {} & \frac{1}{2\sqrt{r}}\left\{{}^{(1)}B_1(\cos\tfrac{3\theta}{2} - 5\cos\tfrac{\theta}{2}) + {}^{(2)}B_1(\sin\tfrac{3\theta}{2} - \tfrac{5}{3}\sin\tfrac{\theta}{2})\right\} + {}^{(1)}B_2(1+\cos 2\theta) \\ & + \frac{3\sqrt{r}}{2}\left\{{}^{(1)}B_3(\cos\tfrac{5\theta}{2} + 3\cos\tfrac{\theta}{2}) + {}^{(2)}B_3(\sin\tfrac{5\theta}{2} + \tfrac{3}{5}\sin\tfrac{\theta}{2})\right\} - \frac{\sqrt{r}}{5}\,{}^{(2)}A_1'\sin\tfrac{\theta}{2} + O(r) \quad , \end{aligned}$$

$$\begin{aligned} \sigma_{\theta\theta} = {} & \frac{-1}{2\sqrt{r}}\left\{{}^{(1)}B_1(\cos\tfrac{3\theta}{2} + 3\cos\tfrac{\theta}{2}) + {}^{(2)}B_1(\sin\tfrac{3\theta}{2} + \sin\tfrac{\theta}{2})\right\} + {}^{(1)}B_2(1-\cos 2\theta) \\ & - \frac{3\sqrt{r}}{2}\left\{{}^{(1)}B_3(\cos\tfrac{5\theta}{2} - 5\cos\tfrac{\theta}{2}) + {}^{(2)}B_3(\sin\tfrac{5\theta}{2} - \sin\tfrac{\theta}{2})\right\} + O(r) \quad , \end{aligned}$$

$$\begin{aligned} \tau_{r\theta} = {} & \frac{-1}{2\sqrt{r}}\left\{{}^{(1)}B_1(\sin\tfrac{3\theta}{2} + \sin\tfrac{\theta}{2}) - {}^{(2)}B_1(\cos\tfrac{3\theta}{2} + \tfrac{1}{3}\cos\tfrac{\theta}{2})\right\} - {}^{(1)}B_2\sin 2\theta \\ & - \frac{3\sqrt{r}}{2}\left\{{}^{(1)}B_3(\sin\tfrac{5\theta}{2} - \sin\tfrac{\theta}{2}) - {}^{(2)}B_3(\cos\tfrac{5\theta}{2} - \tfrac{1}{5}\cos\tfrac{\theta}{2})\right\} - \frac{\sqrt{r}}{10}\,{}^{(2)}A_1'\cos\tfrac{\theta}{2} \quad , \end{aligned}$$

$$\begin{aligned} \sigma_{zz} = {} & \frac{-4\nu}{\sqrt{r}}\left\{{}^{(1)}B_1\cos\tfrac{\theta}{2} + \tfrac{1}{3}\,{}^{(2)}B_1\sin\tfrac{\theta}{2}\right\} + 2\nu\,{}^{(1)}B_2 + (1+\nu)\,{}^{(1)}A_0' \\ & - 4\nu\sqrt{r}\left\{{}^{(1)}B_1\cos\tfrac{\theta}{2} + \tfrac{1}{3}\,{}^{(2)}B_1\sin\tfrac{\theta}{2}\right\} + O(r) \quad , \end{aligned}$$

$$\tau_{\theta z} = \frac{1}{4\sqrt{r}} {}^{(2)}A_1 \cos\frac{\theta}{2} - \frac{1}{2}\left[{}^{(1)}A_2 + {}^{(1)}B_0'\right] \sin\theta$$
$$+ \left[\frac{3}{4} {}^{(2)}A_3 \cos\frac{3\theta}{2} + (3-4\nu){}^{(1)}B_1'(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2})\right] \sqrt{r}$$
$$+ \frac{1}{2} {}^{(2)}B_1' \left[\cos\frac{3\theta}{2} - \frac{2}{3}(3-4\nu)\cos\frac{\theta}{2}\right] \sqrt{r} + O(r) \quad , \tag{6}$$
$$\tau_{rz} = \frac{1}{4\sqrt{r}} {}^{(2)}A_1 \sin\frac{\theta}{2} + \frac{1}{2}\left[{}^{(1)}A_2 + {}^{(1)}B_0'\right] \cos\theta$$
$$- \left[(3-4\nu)\cos\frac{3\theta}{2} + (1-4\nu)\cos\frac{\theta}{2}\right] {}^{(1)}B_1'\sqrt{r} + \frac{3}{4}\sqrt{r}\, {}^{(2)}A_3\sin\frac{3\theta}{2}$$
$$+ \frac{1}{2} {}^{(2)}B_1'\left[\sin\frac{3\theta}{2} - \frac{2}{3}(1-4\nu) \sin\frac{\theta}{2}\right] \sqrt{r} + O(r)$$

In Eq. (6) the constants will be evaluated using the equilibrium equations for thin plates and appropriate boundary conditions, after imposing the assumptions for thin plates.

APPLICATION TO THIN PLATES SUBJECTED TO BENDING AND TWISTING

The equations of equilibrium for thin plates can be expressed in the form

$$\frac{\partial M_{rr}}{\partial r} + \frac{1}{r}\frac{\partial M_{r\theta}}{\partial \theta} + \frac{M_{rr} - M_{\theta\theta}}{r} = q_r \quad ,$$
$$\frac{\partial M_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial M_{\theta\theta}}{\partial \theta} + \frac{2}{r} M_{r\theta} = q_\theta \quad , \quad \text{and} \tag{7}$$
$$\frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r} = p$$

where p is the transverse load on the plate and q_r and q_θ are transverse shear forces. The following thin plate relationships

$$M_{rr} = \frac{h^2}{6}\sigma_{rr} \quad , \quad M_{r\theta} = \frac{h^2}{6}\tau_{r\theta} \quad , \quad M_{\theta\theta} = \frac{h^2}{6}\sigma_{\theta\theta} \tag{8}$$

can be employed, where h is the plate thickness, to apply the equilibrium conditions to the stress components σ_{rr}, $\tau_{r\theta}$, and $\sigma_{\theta\theta}$ given in Eqs. (6). From the first two equations of Eqs. (7), q_r and q_θ can be evaluated subject to the condition that the last equation there is also satisfied. Substituting for M_{rr}, $M_{r\theta}$ and $M_{\theta\theta}$ in Eqs. (7) allows q_r and q_θ to be evaluated as

$$q_r = -\frac{1}{4\sqrt{r}} {}^{(2)}A_1' \sin\frac{\theta}{2} + O(r^0)$$
$$q_\theta = -\frac{1}{4\sqrt{r}} {}^{(2)}A_1' \cos\frac{\theta}{2} + O(r^0) \tag{9}$$

Substituting for q_r and q_θ in the last of Eqs. (7) requires that ${}^{(2)}A_1'=0$ when p=0, or when the transverse load in the plate is zero. Hence for the case of pure bending moments or twisting moments applied at infinity (transverse load p=0), q_r and q_θ do not contain singular terms. For the case of symmetric bending moments at infinity this conclusion agrees with the results of Hartranft and Sih's (1968) analysis for bending. For the case of twisting moments at infinity, the present results agree with the conclusions of Subramonian (1980). Thus, for the case of thin plates without transverse load (p=0), it is shown that τ_{rz} and $\tau_{\theta z}$ do not have singular terms, since for thin plates

$$\tau_{rz} = \frac{3}{2h}(1 - \frac{4z^2}{h^2})\, q_r$$
$$\tau_{\theta z} = \frac{3}{2h}(1 - \frac{4z^2}{h^2})\, q_\theta \quad . \tag{10}$$

where h is the plate thickness and z is the distance measured from the middle plane of the plate. For the case of thin plates without transverse loads, Hartranft and Sih's (1969) results given in Eqs. (6) are applicable, except for the out-of-plane stresses which are modified according to the conventional approximation for thin plates.

$$\sigma_{zz} \simeq 0$$
$$\tau_{rz} = \tau_{\theta z} = O(r^0) \tag{11}$$

The results of this analysis can be summarized as follows

$$\sigma_{rr} = \frac{1}{2\sqrt{r}}\left[{}^{(1)}B_1(\cos\frac{3\theta}{2} - 5\cos\frac{\theta}{2}) + {}^{(2)}B_1(\sin\frac{3\theta}{2} - \frac{5}{3}\sin\frac{\theta}{2})\right] + O(r^0)$$
$$\sigma_{\theta\theta} = \frac{-1}{2\sqrt{r}}\left[{}^{(1)}B_1(\cos\frac{3\theta}{2} + 3\cos\frac{\theta}{2}) + {}^{(2)}B_1(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2})\right] + O(r^0)$$
$$\tau_{r\theta} = \frac{-1}{2\sqrt{r}}\left[{}^{(1)}B_1(\sin\frac{3\theta}{2} + \sin\frac{\theta}{2}) - {}^{(2)}B_1(\cos\frac{3\theta}{2} + \frac{1}{3}\cos\frac{\theta}{2})\right] + O(r^0)$$
$$\tau_{rz} = \tau_{\theta z} \simeq O(r^0) \quad \text{and} \quad \sigma_{zz} \simeq 0 \quad . \tag{12}$$

From a comparison of Eqs. (12) with the results of Hartranft and Sih (1968) and Wang (1970), it can be seen that

$${}^{(1)}B_1 = \frac{-K_1}{\sqrt{2}} \cdot (\frac{2z}{h}) \cdot \Phi(1)$$

$$^{(2)}B_1 = \frac{3K_2}{\sqrt{2}} \cdot \left(\frac{2z}{h}\right) \cdot \Psi(1) \quad , \tag{13}$$

where $K_1 = \frac{6}{h^2} M^\infty \sqrt{a}$ and $K_2 = \frac{6}{h^2} N^\infty \sqrt{a}$.

M^∞ and N^∞ are the uniform bending and twisting moments applied at infinity; h is the plate thickness, a is the half crack length and z is the distance from the mid-plane of the plate. $\Phi(1)$ and $\Psi(1)$ are the thickness correction factors given by Hartranft and Sih (1968) and Wang (1970) respectively.

Comparison of the results of this analysis and the results provided by Williams (1961) can be best illustrated using the trajectories of maximum shear stress given as

$$\tau_{max}^2 = \left(\frac{\sigma_{rr} - \sigma_{\theta\theta}}{2}\right)^2 + \tau_{r\theta}^2 \quad . \tag{14}$$

Substituting for σ_{rr}, $\sigma_{\theta\theta}$ and $\tau_{r\theta}$ from Eqs. (12) and (13), for the case of skew symmetric bending moments at infinity ($M^\infty=0, N^\infty=N_0$), gives

$$\tau_{max}^2 = \frac{2}{r}\left(\frac{z}{h}\right)^2 K_2^2 \left(1 - \frac{1}{2}\sin^2\theta\right) \quad . \tag{15}$$

Williams' (1961) analysis gives

$$\tau_{max}^2 = \frac{K_2^2 \left(\frac{z}{h}\right)^2}{8r(3+\nu)^2}\left[30\nu\cos\theta - 10(1-\nu)\cos 2\theta + (1-\nu)^2 - 6(1-\nu)\nu\cos 3\theta + 25 + 9\nu^2\right] \quad . \tag{16}$$

Trajectories of maximum shear stress obtained from Eq. (15) are shown in Fig. 1, while those obtained from Eq. (16) are given in Fig. 2. Photoelastic experiments reported by Jones and Subramonian (1983) indicate that the present solutions are more accurate near the crack-tip than Williams' solutions (1961).

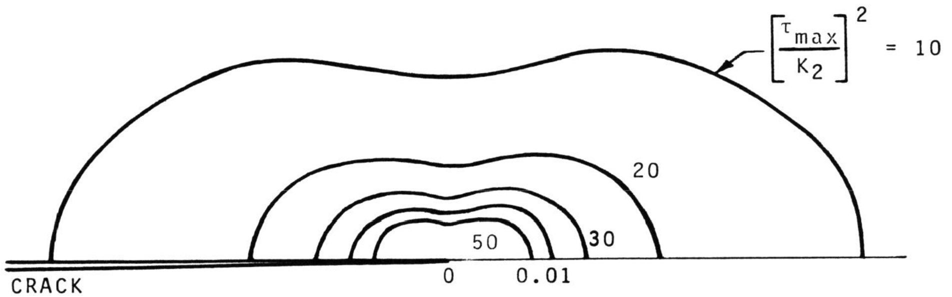

Fig. 1. Contours of the maximum crack-tip shear stress in the plate tearing mode using present solutions.

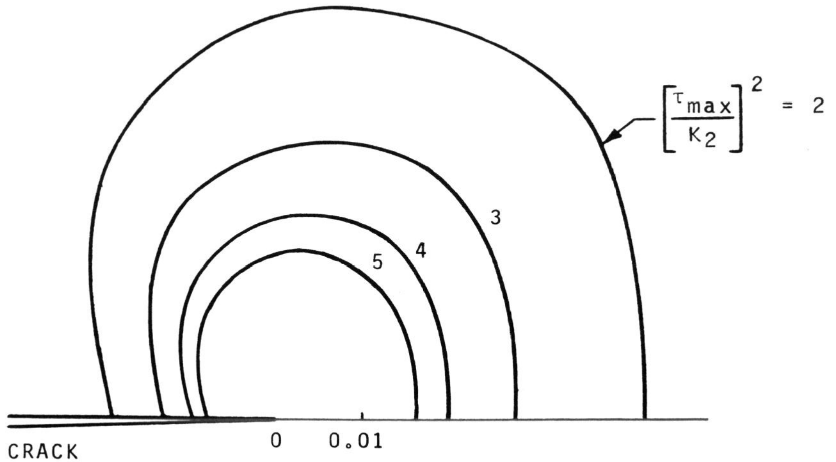

Fig. 2. Contours of the maximum crack-tip shear stress in the plate tearing mode using Williams' solutions.

CONCLUSIONS

An approach is proposed in this paper for obtaining the crack-tip stress field in thin plates subjected to bending moments. This approach employs the three-dimensional eigenfunction expansion procedure proposed by Hartranft and Sih (1969). The form of the crack-tip stress field is in agreement with the results of Hartranft and Sih (1968) and Wang (1970) for symmetric and skew-symmetric bending respectively. The difference between the results of Wang (1970) and the present study is that the transverse shear stresses τ_{rz} and $\tau_{\theta z}$ do not have singular terms in the present study. Maximum shear stress trajectories are compared using the results of the present study and the results of Williams (1961) for the case of skew-symmetric bending at infinity. The trajectories obtained using the present results are in better agreement with the photoelastic test results reported by Jones and Subramonian (1983).

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