AN APPROACH TO A CRACK-INCLUSION INTERACTION PROBLEM

I. Kunin* and B. Gommerstadt**

**Department of Mechanical Engineering, University of Houston, USA*
***Department of Civil Engineering, Case Western Reserve University, Cleveland, USA*

ABSTRACT

The problem of a crack near an inclusion is treated using the restriction integral equation method. An approximation for elastic fields and the energy of interaction is obtained. Energy release rates relations are established.

KEYWORDS

Crack-inclusion interaction, energy release rates.

INTRODUCTION

The interaction of cracks with elastic inclusions is of importance in mechanics and micromechanics of fracture, the theory of dislocations, geomechanics, etc. Composite materials, such as ceramics and fiber reinforced composites, are heterogeneous solids consisting of homogeneous phases. The stress field around the crack tip at a bimaterial interface is drastically different than that for a homogeneous medium. A number of analyses attempted to predict the stress fields and stress intensity factors (Atkinson, 1972; Erdogan, 1975). Microscopic observations clearly indicate that in many cases cracks are quite different from an ideal cut. Typically, a zone of "damage" develops in the vicinity of the crack tip and accompanies the crack propagation. This zone can be identified and is visible as an entity adjacent to the crack. Interaction between such damage and the main crack strongly effects the stress-strain field and related phenomena. Modeling of this interaction is presented in the concept of the "crack-layer" (Chudnovsky, 1984). The attention in this case should be concentrated on energy release consideration of the crack-inclusion system as a whole.

In this paper, the problem of a crack with an inclusion is treated using the restriction integral equations method (Kunin, 1983), which permits one to obtain approximate analytical or efficient numerical solutions for the inclusion problem. An approximation for the elastic fields and the energy of interaction between the crack and the small inclusion near the tip of the crack is obtained. Energy release rates are considered in detail and new relations for a crack-inclusion interaction are established.

GENERAL MATHEMATICAL MODEL

The equation for the displacement $u(x)$ in an unbounded elastic space with an elastic moduli tensor $C(x)$ is

$$-\partial_\lambda C^{\lambda\alpha\mu\beta}(x)\ \partial_\mu u_\beta(x) = q^\alpha(x) \quad , \tag{1}$$

where $q(x)$ is body force. The equation is to be understood in the sense of the generalized functions. In compact form we write

$$Lu = q \quad , \quad L = -\partial C \partial \quad . \tag{2}$$

The solution of this equation which tends to zero at infinity is expressed through a Green's tensor for the displacement

$$u_\alpha(x) = \int G_{\alpha\beta}(x,x')q_\beta(x')dx' \tag{3}$$

or, in compact form

$$u = Gq \quad , \quad GL = I \quad , \tag{4}$$

where I is the identity operator.

We resolve the operator L into two components

$$L = L_* + L_1 \quad , \quad L_1 = -\partial C_1(x)\partial \quad , \tag{5}$$

where L_* describes a medium with a crack and $C_1(x)$ is a perturbation in the elastic moduli due to an inclusion. The case $C_1 \to \infty$ corresponds to a rigid inclusion, and $C_1 = -C_o$ corresponds to a void. Let G_* be a Green's operator corresponding to L_*. The Green's function $G_*(x,x')$ is known for certain two-dimensional crack problems.

Applying G_* to (2) we obtain the integral equation

$$u + G_*L_1u = u_* \quad , \tag{6}$$

where u_* is the displacement field in the medium with a crack in the absence of the inclusion. Applying the symmetrized gradient operator def to (6) we obtain the integral equation for the strain ε = def u

$$\varepsilon + K_*C_1\varepsilon = \varepsilon_* \quad , \tag{7}$$

where ε_* = def u_* and

$$K_* = -\text{def } G_* \text{ def} \tag{8}$$

is a Green's operator for the strain in the medium with the crack. Its kernel $K_*(x,x')$ is a generalized function defined by the corresponding regularization. Properties of Green's operator K = -def G def are discussed in detail in (Kunin, 1983).

Consider the case when the inclusion is localized in a finite domain V^+ with the characteristic function $V^+(x)$ (which is a support of the function $C_1(x)$). Let V^- be the compliment to V^+ and $V^-(x)$ be the corresponding characteristic function. Note that $V^+(x)C_1(x) = C_1(x)$, $V^-(x)C_1(x) = 0$, $V^+(x) + V^-(x) = I$. Denoting the multiplication with the functions $V^+(x)$, $V^-(x)$ by the operator V^+, V^-, we have $V^+ + V^- = I$. Let us introduce the operators

$$K_*^+ = V^+K_*V^+ \quad , \quad K_*^- = V^-K_*V^+ \tag{9}$$

and let $\varepsilon_*^+ = V^+\varepsilon_*$, and so on, be the corresponding restrictions. Then the equation (7) is equivalent to the pair of equations

$$\varepsilon^+ + K_*^+C_1\varepsilon^+ = \varepsilon_*^+ \quad , \tag{10}$$

$$\varepsilon^- = \varepsilon_*^- - K_*^-C_1\varepsilon^+ \quad . \tag{11}$$

The first equation determines the solution ε^+ inside V^+ and the second determines its continuation on V^-.

Thus, for the inclusion localized in the domain V^+, the problem is reduced to the integral equation (10) inside V^+. The essence of the restriction integral equations method is based on a correct definition of the singular operators K_*^+, K_*^-, in the sense of the generalized functions (Kunin, 1983).

The solution of the problem is equivalent to representing the Green's function G for a medium with both crack and inclusion in the form

$$G = G_* - G_*\partial P_*\partial G_* \quad , \tag{12}$$

where the operator P_* satisfies an integral equation in the domain V^+ and admits a representation

$$P_* = -(C_1^{-1} + K_*^+)^{-1} \quad . \tag{13}$$

The solution of (10) is now given by

$$\varepsilon^+ = \varepsilon_*^+ + K_*^+P_*\varepsilon_*^+ \quad . \tag{14}$$

The operator P_* can be interpreted as the interaction energy operator. Indeed, the energy of interaction, Φ_{int}^*, between the inclusion and field ε_* is expressed as

$$\Phi_{int}^* = \frac{1}{2}\iint_{VV'} \varepsilon_*^+(x)P_*(x,x')\varepsilon_*^+(x')dxdx' \quad , \tag{15}$$

where $P_*(x,x')$ is the kernel of the operator P_*.

The total energy of deformation Φ is equal to the sum

$$\Phi = \Phi^* + \Phi_{int}^* \quad , \tag{16}$$

where Φ^* is the energy of the field ε_* itself, i.e. in the medium with the crack in the absence of the inclusion. The total energy can also be written in the form

$$\Phi = \Phi^o + \Phi_{int} \quad , \tag{17}$$

where Φ^o is the energy of the external field ε_o (without the crack and inclusion), and Φ_{int} is the total energy of interaction of the field ε_o with the crack-inclusion system. The interaction between the crack and the inclusion also contributes to Φ_{int}. Analogously to (15) we have

$$\Phi_{int} = \frac{1}{2}\iint_{VV'} \varepsilon_o^+(x)\; P(x,x')\,\varepsilon_o^+(x')\,dx\,dx' \quad , \tag{18}$$

where P is the corresponding interaction energy operator. To obtain an expression for P, let us first represent the Green's operator G_* in a form that is analogous to (12)

$$G_* = G_o - G_o \partial P_o \partial G_o \quad , \tag{19}$$

where G_o is a Green's operator for a homogeneous medium with elastic constants C_o and P_o is the operator considered to be given. This operator determines the energy of interaction of the crack and the external field ε_o. Employing (19) we have (cf. (14))

$$\varepsilon_*^+ = \varepsilon_o^+ + K_o^+ P_o \varepsilon_o^+ \quad . \tag{20}$$

Substituting this into (10), we finally obtain

$$P = P_o + (I + P_o K_o^+)\; P_* \;(I + K_o^+ P_o) \quad . \tag{21}$$

Thus, the total interaction energy Φ_{int} is expressed as a sum of two terms. The first term is the usual interaction energy of the external field ε_o and the crack that is analyzed in fracture mechanics. The second term reflects an additional contribution due to the inclusion. As was indicated above, the computation of this term is equivalent to solving the integral equation (10) in the domain of the inclusion. The problem is essentially simplified when the domain can be considered as a small one.

INCLUSION IN THE ASYMPTOTIC CRACK TIP FIELD

Let a be a characteristic size of the inclusion and r be the distance from the center of the inclusion to the crack tip. The essential simplification of (21) is achieved when the inclusion is located in a vicinity of the crack tip, i.e. a << ℓ and r << ℓ where 2ℓ is the crack length. Then the asymptotic stress field σ_* near the crack tip is given by

$$\sigma_* = k(2\pi r)^{-1/2} \quad , \tag{22}$$

where k is a tensor stress intensity factor. In the absence of the inclusion and for σ_o = const

$$k \equiv k_o = \sigma_o(\pi \ell)^{1/2} \quad . \tag{23}$$

Assume first that a single inclusion is localized in a small domain in a neighborhood of the point x_o. Then the operator $C_1(x)$ for the elastic moduli of the inclusion can be approximated appropriately by a δ-functional model

$$C_1(x) = vC_1\delta(x - x_o) \quad , \tag{24}$$

where v is the volume of the domain, and C_1 is an effective elastic constant. Note that a quasicontinuum with a characteristic parameter of the order of the inclusion size should be incorporated to make the δ-functional model mathematically correct (Kunin, 1983). The δ-functional model (24) reduces (21) to

$$P = P_o + (I + P_o g_o)\; P_* \;(I + g_o P_o) \quad , \tag{25}$$

where

$$P_* = -v(C_1^{-1} + v g_*)^{-1} \tag{26}$$

and g_o, g_* are known constants, v is the volume of the inclusion.

Distinct from (21), the operator P is defined explicitly if the Green's tensor G_* and thus the operator P_* are known. The strain field ε and the interaction energy Φ_{int} may be obtained using (14) and (18). The kernel of the operator P_o is approximated by the first term of its suitable multipole expansion in a neighborhood of the x_o. The validity of such an approximation is considered in detail in (Kunin, 1983).

In the particular case of an elliptical crack in an anisotropic elastic space, the first approximation to P_o gives rise to an exact expression for the energy of interaction Φ°_{int} of the crack and an external field σ_o = const

$$\Phi^\circ_{int} = \frac{1 - \nu_o}{2\mu_o}\, k_o^2 \ell \quad , \tag{27}$$

where μ_o, ν_o are shear elastic module and Poisson's ratio of a medium, respectively. Let μ_1, ν_1 be elastic constants of the inclusion and $\bar\mu = \mu_1/\mu_o$, $\bar\nu = \nu_1/\nu_o$. Then using (22), (23), (25) and putting $g_* = g_o$ in (26) we obtain from (18)

$$\Phi_{int} = \Phi^\circ_{int}\,[1 + \eta\xi^2\psi_o] \quad , \tag{28}$$

where η = r/ℓ, ξ = a/r and

$$\psi_o = \frac{\bar\mu(b_1 + b_2\bar\mu)}{(1 + b_3\bar\mu)(1 + b_4\bar\mu)} \quad , \tag{29}$$

where $b_i = b_i(\nu_o, \nu_1)$ are constants of the order of unity which are calculated explicitly for a given model.

Returning to the case of an arbitrary inclusion when ξ is not assumed to be small we obtain an expression for Φ_{int} which is a generalization of (28)

$$\Phi_{int} = \Phi^\circ_{int}\,[1 + \eta\xi^2\psi_o(\xi)] \quad , \tag{30}$$

where $\psi(\xi) = \psi(\xi, \bar\mu, \bar\nu)$ depends on the shape of the inclusion and $\psi(o) = \psi_o$ given by (29).

There are certain cases when ψ(ξ) can be computed explicitly using results obtained for a crack interacting with another single crack (Isida, 1970; Savin, 1970), with several cracks (Chudnovsky, 1984), and with an elastic cylindrical inclusion (Atkinson, 1972; Erdogan, 1975).

For the model under consideration, $\Phi_{int} = \Phi_{int}(\ell, r, a)$. Respectively, three cases of the crack-inclusion motion can be distinguished: 1) the crack length 2ℓ is varying while r,a = const (translation of the system as a whole); 2) the distance r is varying while ℓ,a = const (relative translation of the inclusion); 3) the inclusion size a is varying, while ℓ,r = const (inclusion is swelling). Let us calculate the corresponding energy release rates for all three cases.

As is well known, the energy release rate for a single crack propagating in a uniform stress field σ_o = const is described by the J-integral. In notation corresponding to the treated model

$$J_o = \frac{1}{2}\frac{\partial\Phi^\circ_{int}}{\partial\ell} = \ell^{-1}\Phi^\circ_{int} . \qquad (31)$$

Analogously we define for the first case

$$J = \frac{1}{2}\frac{\partial\Phi_{int}}{\partial\ell} , \qquad (32)$$

for the second case

$$J^* = \frac{\partial\Phi_{int}}{\partial r} , \qquad (33)$$

and for the third case

$$M^* = a\frac{\partial\Phi_{int}}{\partial a} . \qquad (34)$$

It can be proved that J, J^* and M^* are path-independent integrals corresponding to conservation laws with respect to the translation and dilatation defined above.

Using (30) we find

$$J = J^\circ[1 + \frac{1}{2}\eta\xi^2\psi(\xi)] , \qquad (35)$$

$$J^* = J^\circ[\xi^2\psi(\xi) + \xi^3\psi'(\xi)] , \qquad (36)$$

$$M^* = J^\circ r[2\xi^2\psi(\xi) + \xi^3\psi'(\xi)] . \qquad (37)$$

Expressions (30) and (35)-(37) permit one to establish certain important relations between the interaction energy and all path-independent integrals. They are:

$$\Phi^*_{int} = M^* - rJ^* , \qquad (38)$$

$$J = \ell^{-1}(\Phi^\circ_{int} + \frac{1}{2}\Phi^*_{int}) = J^\circ + \frac{1}{2\ell}M^* - \frac{r}{2\ell}J^* . \qquad (39)$$

One may try to interpret (38) as

$$M = M^* - rJ^* , \qquad (40)$$

where M should stand for a new dilatation energy release rate corresponding to a new coordinate system origin. However, M would not satisfy the relation corresponding to (34), i.e., M is not a true dilatation energy release rate but rather a combination of dilatational modes.

It is clearly seen from (39) that J is a superposition of energy release rates due to the absolute and relative translations and the dilatation.

Note that the contribution of the last two terms to J depends on the shape, location and rigidity of the inclusion. As a rule, for soft (rigid) inclusions the contribution will be positive (negative). Under some conditions an essential screening effect is possible (Chudnovsky, 1984).

In conclusion note that the results obtained can be extended to more general models. Localized defects can be not only inhomogeneities, but in addition can also include a source of internal (residual) stress. To extend the presented approach to this general case an appropriate renormalization of the effective characteristics of the inclusion is necessary.

An extension of the proposed approach to a system of defects is also possible. In this case a solution is not a superposition of perturbations caused by individual defects because of interactions between defects. The tensor describing corresponding interactions can be calculated explicitly for δ-functional models.

In the general case, the kernel of the operator P might be found by numerical methods. Use of the latter is facilitated by the fact that, unlike Green's functions, the operator P is localized in defect domains. Numerical methods are applied directly to integral equations similar to equation (10).

ACKNOWLEDGEMENT

The authors thank A. Chudnovsky for stimulating discussions.

REFERENCES

Atkinson, C. (1972). The Interaction Between Crack and an Inclusion. Int. J. Engng. Sci., Vol. 10, 127-136.
Chudnovsky, A., Dolgopolsky, A., and Kachanov, M. (1984). Elastic Interaction of a Crack with a System of Microcracks. Proceedings of this Conference.
Erdogan, F., and Gupta, G. D. (1975). The Inclusion Problem with a Crack Crossing the Boundary, Int. J. Fract., Vol. 11, 13-27.
Isida, M. (1970). Bull. ISME, Vol. 13, 635.
Kunin, I. A. (1983). Elastic Media with Microstructure. Vol. 1,2. Springer-Verlag, Berlin.
Savin, G. N. (1970). Stress Distribution around Holes. NASA TT, F-607.