A DYNAMIC PROGRAM FOR TWO-DIMENSIONAL FRACTURE MECHANICS PROBLEMS

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ABSTRACT

In this report, the theoretical and numerical fundamentals of boundary integral equations method techniques in real transformed dynamic are presented. Numerical results for impact problems with crack obtained by B.I.E. Method, through a dynamic program of general application, are shown. Interesting remarks on how the numerical integrations have been done, are pointed out in the method.

KEYWORDS

Boundary element method; dynamic fracture.

INTRODUCTION

The solution of problems in classical elastodynamics remains, for the most part, an extremely complicated and difficult task.
The basis for integral representations of the solution of general linear elasticity problems is Betti's work of 1872. One of the first attempts to obtain the solution of problems in elastokinetics was made by Cerruti |4| in 1879. Later Kirchhoff |14| obtained the solution of the scalar wave equations as a potential, which had been obtained before by Poisson |19|. In 1897, Teodone |23| obtained integral representations for the displacement vector in terms of the initial data and boundary displacement and traction. More recent work by Kupradze |15| outlines a method of solving the steady-state elastodynamic problem.
The solution of the transient problem in linear elastodynamics by integral equations is possible due to the existence of fundamental solutions of the Laplace and Fourier transformed equations of motion corresponding to dynamic concentrated forces in the infinite medium. These fundamental solutions appear in Sternberg and Eubanks |22| and Doyle |9|. These solutions, in conjunction with a reciprocal relation, yield a vector identity which corresponds to Somigliana's identity in elastostatics and Green's third identity in potential theory.
Taking the field point to lie on the physical boundary of the problem, constraint equations derived from transformed displacement and traction vectors are obtained. For a well-posed problem, the constraint equations become sets of simultaneous integral equations with the unknown transformed boundary data as explicit unknowns. These equations are functions of the Laplace

transform parameter. These parametric constraint equations have their equivalent in elastostatics, see Rizzo [20]. The solution of these parametric integral equations, for each parametric value, could be achieved numerically by discretizing the boundary and employing quadrature.
The field equations are derived on the basis of displacement theory for a linear, isotropic, homogeneous, elastic material. Boundary is defined in the sense given by Kellogg [13].

THE BOUNDARY INTEGRAL EQUATION METHOD IN TWO-DIMENSIONAL ELASTODYNAMIC CASE

Boundary integral equation

The Boundary equation, for the surface Γ of a system Ω will remain defined for the zero body force case as:

$$\bar{c}\,\bar{u}(x) + \int_\Gamma \bar{T}(x,y)\,\bar{u}(y)\,d\Gamma(y) = \int_\Gamma \bar{U}(x,y)\,\bar{t}(y)\,d\Gamma(y) \qquad (1)$$

$$x, y \in \Gamma$$

which connects the transformed displacement and stress fields in the surface of system $\Omega \cup \Gamma$.
The expressions for the transformed displacement and stress field for integral points can be obtained using equations

$$\bar{u}(x) = -\int_\Gamma \bar{T}(x,y)\,\bar{u}(y)\,d\Gamma(y) + \int_\Gamma \bar{U}(x,y)\,\bar{t}(y)\,d\Gamma(y) \qquad (2)$$

$$\bar{\sigma}(x) = -\int_\Gamma \bar{S}(x,y)\,\bar{u}(y)\,d\Gamma(y) + \int_\Gamma \bar{D}(x,y)\,\bar{t}(y)\,d\Gamma(y) \qquad (3)$$

$$x \in \Omega\ ;\quad y \in \Gamma$$

Discretization of the equations

The substitution of the boundary, Γ, of the system, by another discretized formed by "E" boundary elements with n (=3) nodes in each of them, causes equation (1) to transform itself into the discretized equation:

$$\bar{c}\,\bar{u}(x) + \sum_{e=1}^{E} \bar{H}_e\,\bar{u}_e = \sum_{e=1}^{E} \bar{G}_e\,\bar{t}_e \qquad (4)$$

where $\bar{H}_e$ and $\bar{G}_e$ are shape matrices:

$$\bar{B}_e = \int_{\Gamma e} \begin{bmatrix} \bar{B}_{11} & \bar{B}_{12} \\ \bar{B}_{21} & \bar{B}_{22} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} d\Gamma_e \qquad (5)$$

$$\forall\ \bar{B} = \bar{T}, \bar{U}$$

Establishing equation (4) for all the points, x, where the boundary is discretized, an algebraic system is obtained as follows:

$$\bar{c}\,\bar{u} + \hat{\bar{H}}\,\bar{u} = \bar{G}\,\bar{t} \qquad \text{or} \qquad \bar{H}\,\bar{u} = \bar{G}\,\bar{t} \qquad (6)$$

Similarly, equations (2) and (3) are discretized as:

$$\bar{u}(x) = -\sum_{e=1}^{E} \bar{H}_e\,\bar{u}_e + \sum_{e=1}^{E} \bar{G}_e\,\bar{t}_e \qquad \bar{\sigma}(x) = -\sum_{e=1}^{E} \bar{S}_e\,\bar{u}_e + \sum_{e=1}^{E} \bar{D}_e\,\bar{t}_e \qquad (7)$$

Calculation of integrations

The coefficients of equations (4) are defined in (5), and equations (7) can be expressed as:

$$\bar{H}_e = \int_{\Gamma e} \bar{T}(\zeta)N(\zeta)\,d\Gamma_e \qquad \bar{S}_e = \int_{\Gamma e} \bar{S}(\zeta)N(\zeta)\,d\Gamma_e$$

$$\bar{G}_e = \int_{\Gamma e} \bar{U}(\zeta)N(\zeta)\,d\Gamma_e \qquad \bar{D}_e = \int_{\Gamma e} \bar{D}(\zeta)N(\zeta)\,d\Gamma_e$$

or in compact form by:

$$I = \int_{\zeta=-1}^{\zeta=1} f(\zeta)\,d\zeta \qquad f=(\bar{T},\bar{U},\bar{S},\bar{D}).N.\text{Jacobian} \qquad (8)$$

The coefficient (8) represents the value provided by the integration of a boundary element from a point, which may belong to that element or not.
The numerical evaluation of (8) through a Gauss interpolation process, for example, of m points (m=6), gives:

$$I = \sum_{i=1}^{m} f(\alpha_i)\omega_i$$

where α_i, ω_i correspond to natural coordinates and weights.

The number of points m for which integral (8) is evaluated is related to the error made, and this error affects the results obtained at the boundary on solving the system of equations.
A way to decrease this error consists in maintaining the number of integration points m, a variable depending on an error function of the distance of the point from which it is integrated, x, and of the integrated element, e; as defined by Lachat [16]. Or, using the subelement technique [2,3], each element Γ_e is subdivided in subelements Γ_{se} with local natural coordinates (ζ_1, ζ_2). Equations (8) then becomes:

$$I = \sum_{se=1}^{SE} \int_{\zeta_1}^{\zeta_2} f(\zeta)\,d\zeta =$$

$$= \sum_{se=1}^{SE} \int_{\alpha=-1}^{\alpha=1} f[0.5\{\alpha(\zeta_2-\zeta_1) + (\zeta_2+\zeta_1)\}].(\zeta_2-\zeta_1)/2\,d\alpha =$$

$$= \sum_{se=1}^{SE} 0.5(\zeta_2-\zeta_1).\sum_{i=1}^{m} f[0.5\{\alpha(\zeta_2-\zeta_1) + (\zeta_2+\zeta_1)\}]\,\omega_i$$

where "SE" represents the number of subelements into which the original element "e" has been subdivided.
With this technique, all the integrations permormed are numerical, and the error made in the evaluation of the coefficients will depend on the subelements number into which the element been integrated at that moment is divided. The error also depends on some parameter as the minimum distance from the point of integration to the element the distance to the centroid, the minimum length required for the subelement obtained and the severity parameter needed in the process.

APPENDIX

FREE TERM CALCULATION

In this section we are going to calculate the expressions:

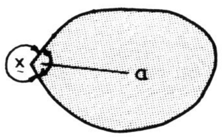
$$\lim_{\epsilon\to 0} \int_{\Gamma\epsilon} \bar{T}_{ji}(x,y)\,\bar{u}_i(y)\,d\Gamma(y) = \bar{c}_{ji}(x)\,\bar{u}_i(x) \qquad (D1)$$

In two-dimensional case:

$$d\Gamma_\epsilon = r\, d\alpha$$

$$r_{,i} = i' = \delta_{i1}\cos\alpha + \delta_{i2}\sin\alpha \qquad \text{(D2)}$$

$$n_i = r_{,i} = i'$$



By using the expressions (D2) on (D1), it will yield:

$$\bar{c}_{ji} = (1/2\pi G)\Big\langle \lambda \int i'j'\,d\alpha \,.\lim_{r\to 0}\{r\frac{\partial\bar{\psi}}{\partial r} - r\frac{\partial\bar{\chi}}{\partial r} - \bar{\chi}\}$$

$$+ \; G\int i'j'\,d\alpha \,.\lim_{r\to 0}\{r\frac{\partial\bar{\psi}}{\partial r} - 2r\frac{\partial\bar{\chi}}{\partial r} + \bar{\chi}\}$$

$$+ \; G\delta_{ij}\int d\alpha \,.\lim_{r\to 0}\{r\frac{\partial\bar{\psi}}{\partial r} - \bar{\chi}\}\Big\rangle$$

$$= (1/2\pi)\Big\langle (c_2^2/c_1^2 - 1)[\sin\alpha.\cos\alpha(\delta_{i1}\delta_{j1} - \delta_{i2}\delta_{j2}) + \sin^2\alpha(\delta_{i1}\delta_{j2} + \delta_{i2}\delta_{j1}) - \alpha\delta_{ij}\Big\rangle$$

where α represents the boundary external angle in the singular point x considered.

In the particular case of a smooth surface ($\alpha=\pi$) $\bar{c}_{ji} = -0.5\;\delta_{ij}$. The matrix c_{ji} which appears in (1) will be:

$$\bar{c}_{ji} = \delta_{ij} + \bar{c}_{ji}$$

And where:

$$\frac{\partial\bar{\psi}}{\partial r} = K_1(\xi_1 r)[2\xi_1/\xi_2^2 r^2] + K_1(\xi_2 r)[-\xi_2 - 2/\xi_2 r^2] + K_0(\xi_1 r)[\xi_1^2/\xi_2^2 r] + K_0(\xi_2 r)[-1/r]$$

$$\frac{\partial\bar{\chi}}{\partial r} = K_1(\xi_1 r)[\xi_1^3/\xi_2^2 + 4\xi_1/\xi_2^2 r^2] + K_1(\xi_2 r)[-\xi_2 - 4/\xi_2 r^2] + K_0(\xi_1 r)[2\xi_1^2/\xi_2^2 r] + K_0(\xi_2 r)[-2/r]$$

$$\frac{\partial^2\bar{\psi}}{\partial r^2} = K_1(\xi_1 r)[-6\xi_1/\xi_2^2 r^3 - \xi_1^3/\xi_2^2 r] + K_1(\xi_2 r)[6/\xi_2 r^3 + 2\xi_2/r] + K_0(\xi_1 r)[-3\xi_1^2/\xi_2^2 r^2] + K_0(\xi_2 r)[\xi_2^2 + 3/r^2]$$

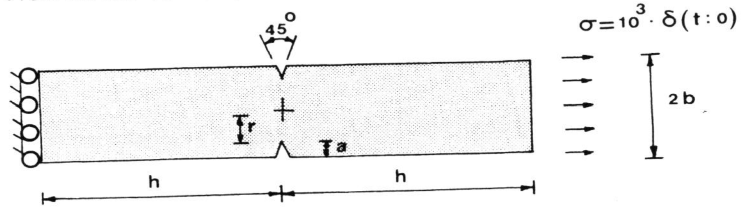
$$\frac{\partial^2\bar{\chi}}{\partial r^2} = K_1(\xi_1 r)[-12\xi_1/\xi_2^2 r^3 - 3\xi_1^3/\xi_2^2 r] + K_1(\xi_2 r)[12/\xi_2 r^3 + 3\xi_2/r] + K_0(\xi_1 r)[-\xi_1^4/\xi_2^2 - 6\xi_1^2/\xi_2^2 r^2] + K_0(\xi_2 r)[\xi_2^2 + 6/r^2]$$

EXAMPLE

Double-notched beam

This application consists in calculating the stress in the tip of the notch for impact loading versus Laplace parameter k.

The model, whose material properties and dimensions are shown in Figure 1, has been discretized in 104 parabolic elements, as Figure 2 shows.



ρ(specific weight) = 1.00
ν(Poisson modulus) = 0.36
Behavior of the problem = plane stress

Fig. 1. Dimensions and material properties.

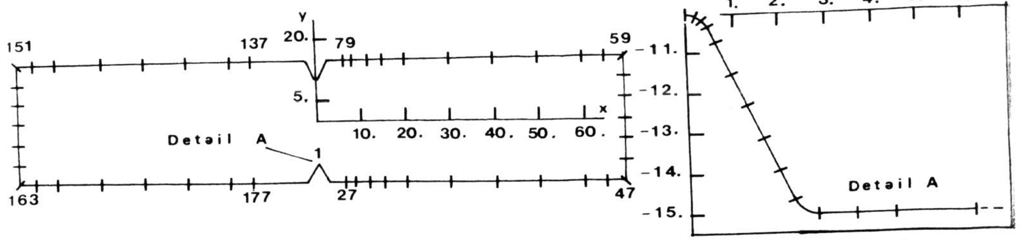


Fig. 2. Discretization analysed (208 nodes, 104 elements).

The b dimension has been considered variable in the range (b/h = 0.3-0.9), h = 69, and a = 5., the notch tip has a radius of r = 0.5.

The model is solved statically, giving a value for the displacement of the stressed side(nodes 47-59)of u = 0.145 10^6.

The stresses for internal points to a distance r away from the crack tip are represented in Figure 3 for b = 15. In the figure, it is shown how the stress varies parabolically with the distance to the crack tip for $r/a \geq 0.002$, and linearly for a distance $r/a \leq 0.002$.

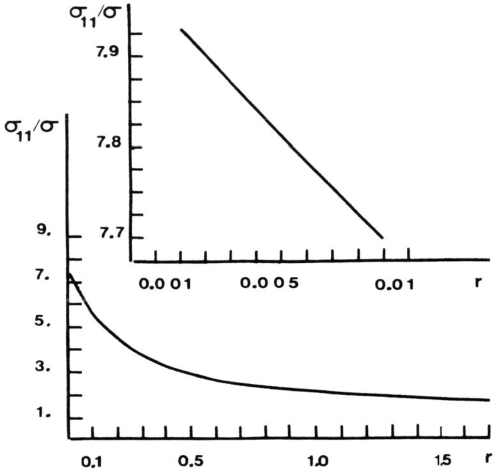
In Figure 4, stresses versus $1/\sqrt{r}$ for several models with different width are given. Only the region $1/\sqrt{r} \leq 3.16$ ($r \geq 0.1$) can be considered for calculating the stress intensity factor.

Figure 4 shows the normalized stresses for each case of variation of parameter b against the tip crack distance r.

Stress intensity factors, obtained from Figure 4, are printed in Figure 5 together with the theoretical values provided by Benthem and Koiter solutions |21|:

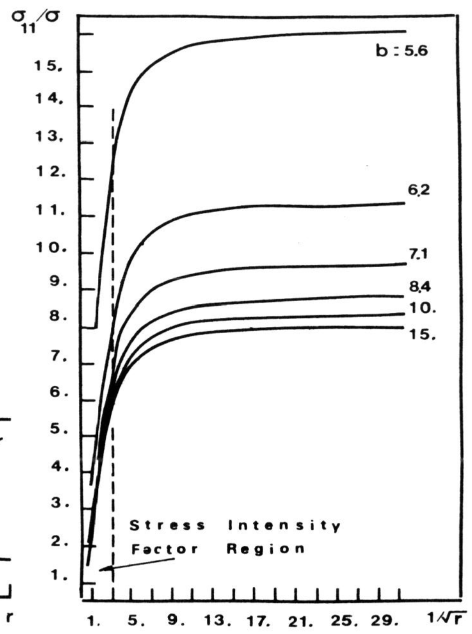
$$K_1/K_0 = (1-a/b)^{-0.5}\;[1.12(1-0.5a/b)-0.015(a/b)^2+0.091(a/b)^3]$$

The model has been analysed dynamically. Figure 6 shows the horizontal motion of points on the stressed side (nodes 47-59) versus Laplace parameter k. Figure 7 presents the stresses σ_{11} in direction x for points in the crack section far away from the crack tip a distance r.



a: 5. h: 69. b: 15.

Fig. 3. Normalized stress versus r.



a: 5. h: 69.

Fig. 4. Normalized stress versus $1/\sqrt{r}$

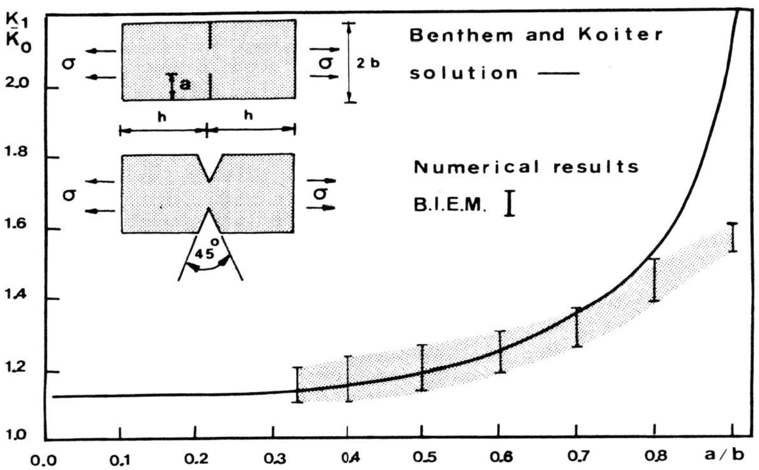


Fig. 5. Stress intensity factors against parameter a/b. Numerical BIEM and theoretical results.

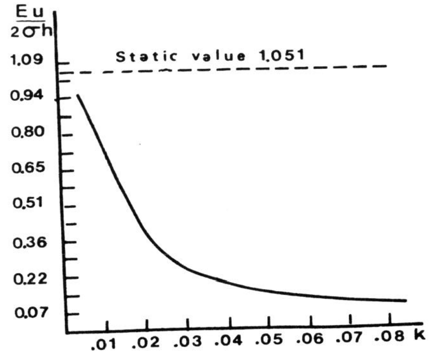


Fig. 6. Horizontal motion of free sides for double-notched specimen.

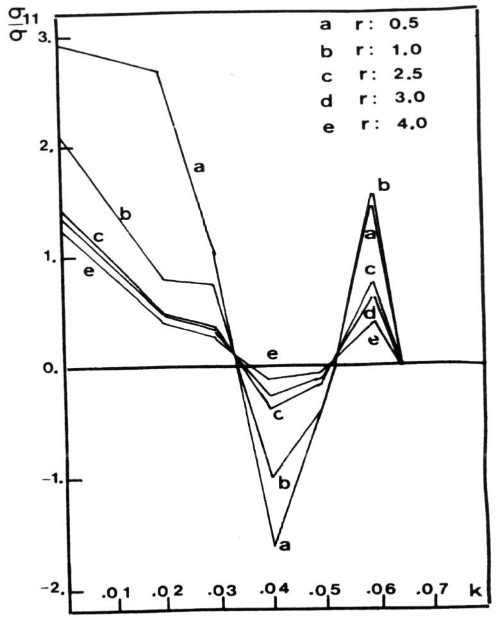


Fig. 7. Stresses σ_{11} against Laplace parameter k.

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