PATH INDEPENDENT INTEGRAL FOR THE CALCULATION OF THE ENERGY RELEASE RATE IN ELASTODYNAMICS

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ABSTRACT

The in-plane problem of a cracked elastic body is considered and, using a global energetic point of view, the existence of Bui's J-integral (1977) for the calculation of the energy release rate in the dynamic case is proved for general loading.

KEYWORDS

Elastodynamics; crack growing; energy release rate; path-independent integral; stress-intensity factors; kinematic intensity factors.

NOTATIONS

In this paper we shall use the following notations :

$U = (U_i)$	displacement vector
$\dot{U} = (\dot{U}_i)$	velocity
$\varepsilon = (\varepsilon_{ij})$	strain tensor $2\varepsilon_{ij} = U_{i,j} + U_{j,i}$
$\sigma = (\sigma_{ij})$	stress tensor
W	elastic energy density $W = 1/2\ \sigma_{ij}\ \varepsilon_{ij}$
$W_{e\ell}$	the strain energy of the solid $\int_\Omega W dv$
ρ	specific density
C	kinetic energy density $C = 1/2\ \rho \dot{U}_i \dot{U}_i$
W_{kin}	the kinetic energy of the solid $\int_\Omega C dv$
λ and μ	Lamé coefficients
C_1	compressive wave velocity $C_1 = [(\lambda+2\mu)/\rho]^{1/2}$
C_2	transverse wave velocity $C_2 = (\mu/\rho)^{1/2}$
E	Young's modulus
ν	Poisson's ratio

INTRODUCTION

In a paper devoted to the analysis of stress and strain concentration near the tips of cracks and notches, Rice (1968, a) introduced for homogeneous, but not necessarily isotropic solids, in two dimensional elastostatics, with infinitesimal deformations, with the possibility of non-linear stress-strain relations, a path-independent integral

(1) $$J = \int_\Gamma (W n_1 - \sigma_{kj}\, n_j\, U_{k,1})\, ds$$

where Γ is an oriented curve in the solid, $n = (n_i)$ is the outward normal unit vector on Γ . If Γ is a closed path not surrounding inhomogeneity nor singularity, then J=0. In the contrary, J remains constant when Γ is deformed without surrounding a new inhomogeneity or new singularity.

That integral is of great importance in fracture mechanics, if Γ is chosen begining on one face of a crack, surrounding the crack tip, and ending on the other face. Then one has (Rice, 1968,a,b)

(2) $$J = G$$

where G is the energy release rate by unit length of crack created in the x_1 direction (fig. 1).

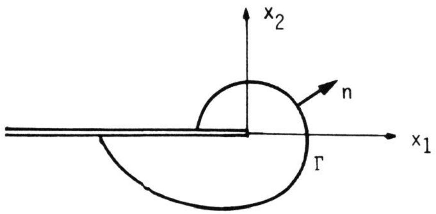

Fig. 1. Path Γ surrounding the crack tip.

These results which have an easy generalization in three dimensional elastostatics must be traced to the works of Eshelby (1956, 1970, 1975), where the integral value is interpreted as a force on the defect lying in the path of integration. Knowles and Sternberg (1972) have shown that these results as well as other conservation laws follow from an application of Noether's theorem (1918) on invariant variational principles to the principle of minimum potential energy in elastostatics. The interpretation of such conservation laws from an energetic point of view was given by Budiansky and Rice (1973). Some applications to stress-intensity factor calculations problems are given by Freund (1978).

These results valid for elastostatic cannot be easily obtained in elastodynamics. For example, it is impossible to calculate the energy release rate for a crack growing in its own plane by means of a line independent integral. It is easy to show this if, following Atkinson and Eshelby (1968), we consider two paths Γ_1 and Γ_2 so that Γ_2 is surrounded by Γ_1. Let us suppose that a wave, coming from out of Γ_1 has not reached Γ_2, a line integral along Γ_1 will be changed while the same integral along Γ_2 remains unchanged. The line integral, whatever it may be, will vary with the line of integration.

Atkinson and Eshelby (1968), and Freund (1972) (see also Kostrov and Nikitin, 1970) have then proposed, for the calculation of the energy release rate during the growing of a crack in elastodynamics, to consider a line integral, but the line is shrunk onto the crack tip. For a crack growing in its own plane at a velocity v in the x_1 direction, they have shown that

(3) $$G = \lim_{\Gamma\to 0} \int_\Gamma [(W+C)n_1 + n.\sigma.\dot{U}/v\ ]\, ds$$

where Γ is an oriented line surrounding the crack tip .

In a different way, Bui (1977) proposed to calculate the energy release rate by means of a truly path independent integral, with line terms and also surface term

(4) $$J = \int_\Gamma \{W n_1 - \sigma_{jk} n_k U_{j,1} - \tfrac{1}{2}\rho \dot{U}_h \dot{U}_h n_1 - \rho \dot{U}_j U_{j,1} v n_1\}\, ds + \frac{D}{Dt}\int_A \rho \dot{U}_j U_{j,1}\, dv$$

where Γ is a rigid path surrounding the crack tip and in translation at the same velocity v and A is the surface surrounded by Γ.

Bui refered to a conservation law given by Eshelby (1956, 1970) and also by Fletcher (1976) who applied Noether's theorem (1918) to the elastodynamic variational principle

(5) $$\frac{\partial}{\partial t}\{\rho \dot{U}_j\, U_{j,1}\} + \frac{\partial}{\partial x_k}\{(W - \tfrac{1}{2}\rho \dot{U}_h \dot{U}_h)\,\delta_{1k} - U_{j,1}\,\sigma_{jk}\} = 0$$

Bui (1977) has shown that J=G in the particular case of mode I loading. We prove here, by energetic considerations, that the result J=G remains true for general loading. We prove successively :

1- The existence of the J-integral, eq. (4).
2- The path independency property.
3- The identity $J \equiv G$.

EXISTENCE OF J

In the definition of J, eq. (4), there are two terms, a line integral and a surface integral. The existence of the line integral is quite obvious, since the fields have no singularity on Γ. To know whether the second term exists, we look at the singularities of the fields $\dot{U}$ and $U_{,1}$ near the crack tip. The singularity of $U_{,1}$ is known being $O(r^{-1/2})$ a long time ago from the work of Yoffé (1951) whose results were confirmed in a more general feature more recently by Achenbach and Bažant (1975). For the singularity of $\dot{U}$, the easiest way for its analysis is to refer to the "transport condition of the singularity" assumption introduced by Nguyen (1979) which is : " If the crack is growing in the x_1 direction at the velocity v,for each field f in the solid, we have near the crack tip

(6) $$\dot{f} = -v\, f_{,1} + (\text{more regular terms})\ "\quad .$$

From (6) we deduce that $\dot{U}$ will be an $O(r^{-1/2})$ term. The second term of the right member of (4) is then well defined and the existence of J is proved.

PATH INDEPENDENCY PROPERTY

Let us recall first the expression of the time derivative of an integral on a rigid domain D moving in the solid with the velocity v_i (Germain, 1973)

(7) $$\frac{D}{Dt}\int_D f\,dv = \int_D \frac{\partial f}{\partial t}\,dv + \int_{\partial D} f\,v_i n_i\,ds$$

where ∂D is the boundary of D, n is the outward unit normal vector of ∂D, f and $\partial f/\partial t$ are integrable in D.

Following Bui (1977), let us integrate the relation (5) in a rigid domain D including neither singularity nor inhomogeneity and moving in the solid at the velocity v in the x_1 direction. Using (7) we find

(8) $$\frac{D}{Dt}\rho\int_D \dot{U}_j U_{j,1}\,dv + \int_{\partial D} \{W n_1 - \sigma_{jk} n_k U_{j,1} - \frac{1}{2}\rho \dot{U}_h \dot{U}_h n_1 - \rho \dot{U}_j U_{j,1} v n_1\}\,ds = 0 \quad .$$

We notice that (8) is a little different from the integral expression of the conservation law given in Fletcher (1976), because Bui made use of a moving domain of integration, while Fletcher considered a fixed domain D_o. Let us consider now two curves Γ_1 and Γ_2 surrounding the crack tip, enclosing respectively the surfaces A_1 and A_2 (in translation at the velocity v), fig. 2. Apply the relation (8)

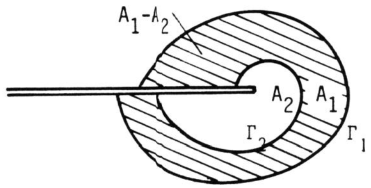

Fig. 2. Domain without singularity.

for the domain $D = A_1 - A_2$. It is possible to do this because $A_1 - A_2$ incloses no singularity. Separating the integrals over (A_1, Γ_1) and (A_2, Γ_2) we see that the values of the J-integral, eq. (4), corresponding to the paths Γ_1, Γ_2 are equal. It follows that J is independent of the path surrounding the crack tip.

IDENTITY J ≡ G

Bui (1977) has shown that J = G for the particular case of mode I loading, using a rectangular path for Γ, the corners of which are $(-a, -\eta)$, $(a, -\eta)$, (a, η), $(a, -\eta)$ fig. 3.

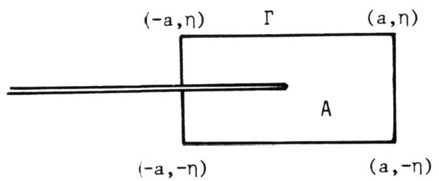

Fig. 3. Rectangular path for mode I loading.

Putting η equal to zero and using the asymptotic expression of the fields in mode I, he found J as a function of the intensity factors

(9) $$J = \frac{1-\nu^2}{E} K_I^\sigma \; K_I^u$$

where the stress-intensity factor K_I^σ and the kinematic intensity factor K_I^u are

(10) $$K_I^\sigma = \lim_{r\to 0} (2\pi r)^{1/2}\,\sigma_{22}(r, 0, t) \quad ,$$

(11) $$K_I^u = \lim_{r\to 0} \frac{\mu}{2(1-\nu)} (2\pi/r)^{1/2}\,\frac{U_2(r, \pi, t) - U_2(r, -\pi, t)}{2} \quad .$$

Bui has shown that

(12) $$K_I^\sigma = \frac{D(1-\nu)}{\beta_1(1-\beta_2^2)} K_I^u$$

where :
$$\beta_1 = \sqrt{(1 - v^2/C_1^2)} \quad , \quad \beta_2 = \sqrt{(1 - v^2/C_2^2)} \quad , \quad D = 4\beta_1\beta_2 - (1+\beta_2^2)^2$$

Freund (1972) has shown that in elastodynamics the energy release rate is

(13) $$G = \frac{1+\nu}{DE}\,\beta_1(1-\beta_2^2)\,(K_I^\sigma)^2 \quad .$$

From (9), (12) and (13) Bui deduced that the relation J=G remains valid in elastodynamics in mode I loading where J is given by (4). Let us prove now that the relation J=G remains true for a more general loading. We shall not use the same method as that of Bui, but we shall make use of an energetic point of view. Let us denote by F the energy flux per unit time into the crack tip, by v the crack extension velocity in the solid. The energy release rate is $G = F / v$. Since there is no volumic dissipation in an elastic solid, we have

(14) $$F = P_e - \frac{D}{Dt} W_{e\ell} - \frac{D}{Dt} W_{kin}$$

where P_e is the external power. If there are no volumic forces, then

(15) $$P_e = \int_{\partial\Omega} \sigma_{ik} n_k \dot{U}_i\,ds \quad .$$

We can rewrite F as follows :

(16) $$F = \int_{\partial\Omega} \sigma_{ik} n_k \dot{U}_i\,ds - \frac{D}{Dt}\int_\Omega (W + \frac{1}{2}\rho \dot{U}_h \dot{U}_h)\,dv \quad .$$

Using (7) it can be shown (A and Γ being defined in (4)) that :

(17) $$\frac{D}{Dt}\int_{\Omega-A} (W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h)\,dv = \int_{\Omega-A} \frac{\partial}{\partial t}(W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h)\,dv - v\int_\Gamma (W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h) n_1\,ds \quad .$$

From the equation of motion and Green's formula we have :

(18) $$\int_{\Omega-A} \frac{\partial}{\partial t}(W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h)\,dv = \int_{\partial\Omega-\Gamma} \sigma_{ik} n_k \dot{U}_i\,ds \quad .$$

The equations (16), (17), (18) show that :

(19) $$F = \int_\Gamma \{v(W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h) n_1 + \sigma_{ik} n_k \dot{U}_i\}\,ds - \frac{D}{Dt}\int_A (W + \frac{1}{2}\rho\dot{U}_h\dot{U}_h)\,dv \quad .$$

The domain of integration is now changed from a fixed domain Ω, with the boundary $\partial\Omega$, to a moving one A, with boundary Γ. Now let us study the expression F-vJ :

$$F - v J = \int_{\Gamma} (\rho v \dot{U}_h n_1 - \sigma_{hk} n_k)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}\right) U_h \, ds - \frac{D}{Dt}\int_A \left(W + \frac{1}{2}\rho \dot{U}_h \dot{U}_h + v\rho \dot{U}_h U_{h,1}\right) dv \quad . \tag{20}$$

F and Jv are independent of the path Γ. From the transport assumption (6) :

$$(\rho v \dot{U}_h n_1 - \sigma_{hk} n_k)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x_1}\right) U_h = o(r^{-1}) \quad . \tag{21}$$

If the path Γ is shrunk onto the crack tip, the first term of the right member of (20) tends to zero, as does the second term. Since the right member of (20) is independent of Γ we have proved that it is zero, so we have :

$$F = v J \quad . \tag{22}$$

From G=F/v and (22) we see that Bui's definition of the J-integral leads to the identity J≡G which is valid in elastodynamics, in general loading conditions without volumic forces. The J-integral of Rice (1) valid for the static and quasi-static cases is re-obtained from (4) by putting $\dot{U} = 0$ (static case) or $\rho = 0$ (neglecting the inertial forces, quasi-static case). Note also that if in (4) we shrunk the path onto the crack tip, using (6) we recover the relation (3). This result is in itself another proof of the relation J=G in elastodynamics.

DISCUSSION

We shall discuss some of the possible applications of Bui's integral. The real practical interest of the J-integral is in the energy release rate calculation for dynamic problems solved by the finite element method. We must calculate a surface integral at each step in time, and for that reason it seems to be a difficulty, but the precision in the result of the surface integral calculation would be quite good, since it is not useful to know exactly the fields near the crack tip. The singularity of the term we must integrate is like $O(r^{-1})$, and for that reason, the contribution to the surface integral of a small zone around the crack tip is of order O(r), and we should not need a lot of small elements.

But Bui's J-integral can also be useful in some analytical problems. For example, the result we obtain here for cracks can be easily generalised to damaged zones in the sense of Bui and Ehrlacher (1980), where the J-integral is used to deduce the dissipation rate of energy during the damaged zone propagation. The damaged zone is here in fact a destroyed zone and can be interpreted as a crack with a non null thickness.

We can give another simple analytical application. We use the J-integral to show why the energy release rate decreases from a stationary crack to a propagating crack. Let us recall first some important result. Consider the simple case of an infinite medium with a half infinite crack. It is now well known (see Kostrov (1975), Rose (1976)) that the energy release rate for a crack growing at the velocity v, for each pure mode of loading, is

$$G = g(v) \, G_{stat} \quad , \tag{23}$$

where g(v) is an universal function decreasing from one to zero when the velocity increases from zero to the Rayleigh wave velocity, in the case of mode I loading for example, and G_{stat} is, for the same geometry and loading, the energy release rate of a stationary crack. It is worth noting that eq. (23) is strictly proved for a half infinite crack (see Rose (1976)). Let us consider a static loading on our half infinite crack. The crack starts propagating with a constant velocity v at time t=0. The static energy release rate, for negative times, is given by

$$G_o = \int_{\Gamma_o} (W n_1 - \sigma_{kj} n_j U_{k,1}) \, ds \equiv G_{stat} \tag{24}$$

for any Γ_o surrounding the crack tip. Let us choose a large path Γ_o and denote by Γ_t the translated path from Γ_o with the length vt the x_1 direction, by A_t the surface surrounded by Γ_t. At a small time t>0, the waves generated by the moving crack tip will not have reached either the path Γ_o nor the path Γ_t (fig. 4.). Thus

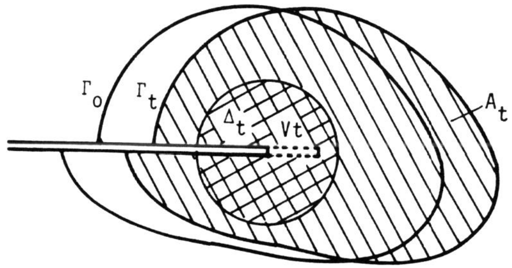

Fig. 4. Waves generated by the moving crack tip.

on Γ_t we have $\dot{U}=0$ and we find from (4) with the path Γ_t

$$G_t = \int_{\Gamma_t} (W n_1 - \sigma_{jk} n_k U_{j,1}) \, ds + \frac{D}{Dt}\int_{A_t} \rho \dot{U}_j U_{j,1} \, dv \quad . \tag{25}$$

As Γ_t is still a path surrounding the crack tip at time zero and since the fields have not yet been changed on Γ_t the first term of (25) is G_o. From (6) it is easy to see that the second term of (25) is negative and we have the result $G_t < G_o$. In fact, the moving domain A_t is restricted to a disc of radius $C_1 t$ and with a fixed center at the position of the crack tip at time t=0. It is possible to come back to a fixed domain of integration Δ' by changing the integration variables homothetically in the ratio 1/t. In Δ' we have a fixed singularity at the point (v,0). Moreover, we have

$$-\frac{D}{Dt}\int_{\Delta_t} \rho \dot{U}_j U_{j,1} \, dv = \{1 - g(v)\} \, G_o \quad , \tag{26}$$

and it can be interpreted as the energy given back to the solid with the help of inertial forces in spite of the crack tip progression.

CONCLUSION

Bui's J-integral in elastodynamics can be a useful technique for the energy release rate numerical calculation in fracture mechanics, but can also be the simple means to obtain some interesting analytical results about the dissipation rate in elastodynamic problems without volumic forces and with a flux of energy into new created boundary surfaces.

ACKNOWLEGMENT

The author wishes to thank Professor Bui for helpful discussions.

REFERENCES

Achenbach, J. P., and Z. P. Bažant (1975). Elastodynamic near tip stress and displacement fields for rapidly propagating cracks in orthotropic materials. J. Appl. Mech., 97, 183.

Atkinson, C., and J. D. Eshelby (1968). The flow of energy into the tip of a moving crack. J. Fract. Mech., 4, 3.

Budiansky, B., and J. R. Rice (1973). Conservation laws and energy release rates. J. Appl. Mech., 40, 201.

Bui, H. D. (1977). Stress and crack-displacement intensity factors in elastodynamics. Fracture. Vol. 3, ICF4, Waterloo, p. 91.

Bui, H. D., and A. Ehrlacher (1980). Propagation dynamique d'une zone endommagée dans un solide élastique-fragile en mode III et en régime permanent. C. R. Acad. Sc.Paris, Série B, 290, p. 273.

Eshelby, J. D. (1956). The continuum theory of lattice defects. Solid State Physics Vol. 3, Acad. Press, 79.

Eshelby, J. D. (1970). The energy-momentum tensor in continuum mechanics. Inelastic behavior of Solids. Mc Graw Hill, New York, p. 77.

Eshelby, J. D. (1975). The elastic energy-momentum tensor. J. Elasticity, 5, 321.

Fletcher, P. C. (1976). Conservation laws in linear elastodynamics. Arch. Rat. Mech. and Anal., 60, 329.

Freund, L. B. (1972). Energy flux into the tip of an extending crack in an elastic solid. J. Elasticity, 2, 342.

Freund, L. B. (1978). Stress intensity factors calculations based on a conservation integral. Int. J. Solids & Struct., 14, p. 241.

Germain, P. (1973). Cours de Mécanique des milieux continus. Masson, Paris.

Knowles, J. K., and E. Sternberg (1972). On a class of conservation laws in linearized and finite elastostatics. Arch. Rat. Mech. and Anal., 44, 187.

Kostrov, B. V. and L. V. Nikitin (1970). Arch. Mech. Stos., 22, 749.

Kostrov, B. V. (1975). On the crack propagation with variable velocity. Int. J. Fract., 11, p. 47.

Nguyen, Q. S. (1979). Une description thermodynamique du problème de fissure mobile C. R. Acad. Sc. Paris., Série B, 288, 201.

Noether, E. (1918). Invariante Variationsprobleme. Göttinger Nachrichten, Mathematisch-Physikalish Klasse 2, p. 235.

Rice, J. R. (1968a). A path independent integral and the approximate analysis of strain concentration by notches and cracks. J. Appl. Mech., 35, 379.

Rice, J. R. (1968b). Mathematical analysis in the mechanics of fracture. Fracture, Vol. 2., Acad. Press, p. 101.

Rose, L. R. (1976). Recent theoretical and experimental results on fast brittle fracture. Int. J. Fract., 12, 799-813.

Yoffe, E. H. (1951). The moving Griffith crack. Phil. Mag., 12, p. 739.