INELASTIC ANALYSIS OF SURFACE FLAWS
USING THE LINE-SPRING MODEL

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ABSTRACT

A formulation of the elastic-perfectly plastic analysis of part-through surface cracks in plate or shell structures is presented which is based on the Rice line-spring model. By using a simple elastic plate or shallow shell theory embodying the Kirchhoff hypothesis, the line-spring model is reduced to the numerical solution of a pair of coupled singular integral equations. In the linear elastic regime, stress intensity factor distributions for axial surface cracks in moderately thin-walled pressurized cylinders (R/t=10) are found to be five to twelve per cent greater than those obtained from more refined boundary integral equation solutions. For higher pressure levels, the inferred elastic-plastic distribution of crack front deformation, as parameterized by J, is consistent with a strong tendency for stable ductile tearing and tunneling; commencing at the deepest penetration of the surface crack.

KEY WORDS

Surface cracks; plates and shells; line-spring model; elastic-plastic fracture; computational mechanics; singular integral equations.

INTRODUCTION

Although considerable progress has been made in the development of computational techniques for problems in linear elastic and elastic-plastic fracture mechanics, it is fair to say that the computational costs associated with detailed three-dimensional finite element or boundary integral formulations of many crack configurations encountered in engineering practice render these problems economically intractable. Of course, in the case of linear elasticity, stress intensity solutions can be readily normalized and catalogued so that one can easily justify the expenditure of a fairly large amount of effort in obtaining accurate solutions to certain generic problems such as the surface flaw in a plate or cylindrical shell, (Raju and Newman, 1979) (Heliot, Labbens and Pellisier-Tanon, 1979). On the other hand, similarly detailed elastic-plastic analyses would require excessive commitments of resources in order to obtain a reasonable parametric study of flaw and structure geometry and material hardening behaviors.

It is the purpose of this paper to present some further results in the continuing development of an approximate tool for the analysis of part-through surface cracks in plates and shells, namely the line-spring model introduced by Rice and Levy (1972) and further developed by Levy and Rice (1972), Rice (1972), and more recently by Parks (1980). The essential computational economy afforded by the line-spring model is that these crack configurations can be reduced to formulations in shell or plate theory, rather than in three-dimensional continua.

Before proceeding with the current development, it is worthwhile recalling some of the history of the line-spring model. At the time of its introduction in 1972, there was considerable disagreement in the literature as to stress intensity calibrations for surface cracks in plates. It is likely, however, that this uncertainty contributed to the general lack of further development of the line-spring model as a potentially useful tool for approximate engineering of such cracks. Recently, Parks (1980) re-examined the model and found very good agreement, of order two to five percent difference in stress intensity factor distribution along a semi-elliptical surface crack front in a plate subject to far-field tension, as compared with the very detailed three-dimensional finite element solutions of Raju and Newman (1979), which are representative of a growing computational consensus regarding this problem. Furthermore, Parks (1980) extended some of the concepts presented by Rice (1972) in the development of inelastic constitutive behavior of the line-spring, and was able to obtain approximate evaluations of the evolution of the intensity of crack tip deformation along the surface crack in the elastic-plastic regime for the case when the bulk of the plate deformed elastically, and all inelastic response could be lumped into the line-spring itself. The intensity of the crack front deformation was parameterized in terms of a variation of a local crack front J variation, but it could just as well have been given in terms of a variation of δ_t, the crack tip opening displacement.

In the present work, we extend the line-spring model to axial surface cracks in cylindrical shells, using the simple shallow shell theory presented by Copley and Sanders (1969), which retains the Kirchhoff hypothesis, to represent bulk behavior of the shell. It is noted that in the case of through cracks, the use of such classical plate or shallow shell theories leads to certain discrepancies in the singular asymptotic fields at the ends of the through cracks. These difficulties can be obviated by the use of higher order plate and shell theories, such as Reissner's, which admit the possibility of transverse shear, e.g., Bergez and Radenkovic (1973). Recently Krenk (1978) has examined the effects of such formulations on through-crack bending and membrane stress intensity factors in cylinders. Although at a later time we shall investigate the consequences of higher order shallow shell theories on line-spring results, we may expect that the differences may be small, especially for surface cracks whose total length is substantially greater than shell thickness. For similarly long through-cracks in shells, the differences in crack face opening and relative rotation are slight except in the regions close to the ends of the through-crack. In the line-spring formulation, the regions near the end of the model through-crack correspond to the intersection of the part-through surface crack and the shell free surface, where in the limit, the line-spring model becomes inapplicable. On the other hand, significant differences may be expected when comparisons are made between line-spring solutions based on a first-order shallow shell theory and those which would be obtained from a line-spring formulation which was embedded in a more general finite element or finite difference shell theory, especially when the projected surface crack length is of the same order as a typical shell radius of curvature, or shell thickness.

FORMULATION

We present the formulation of the line-spring model of an axial surface crack in a cylinder in two stages. In the first part, we introduce the behavior of the shell structure of mean radius R and thickness t and containing a symmetrical part-through surface crack of total length 2c. In the second part we couple the line-spring model to the structural model.

First, introduce a local cartesian reference frame at the center of the crack, with coordinate x along the cylindrical generator which contains the discontinuity, coordinate z is positive outwards, and (x, y, z) form a right-handed system. Let the depth, a, of the interior surface crack vary according to $a=a(x)$ for $|x|<c$. The thin shallow shell is then idealized as a two-dimensional continuum, and the part-through surface crack is accordingly idealized as a one-dimensional discontinuity in the shell of length 2c. The case of arbitrarily large R/t, namely a flat plate, is indicated schematically in Fig. 1.

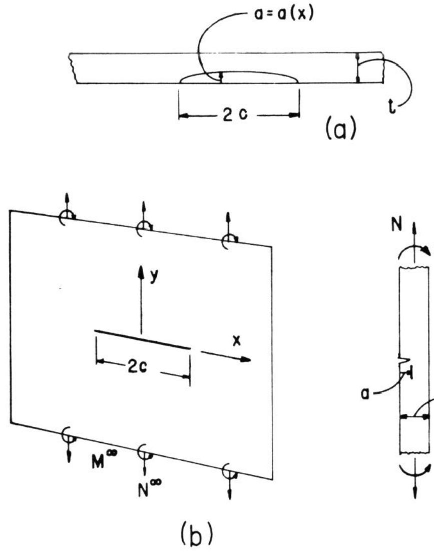

Fig. 1 Schematic illustration of basic features of the line-spring model, after Rice (1972)

In contrast to the through crack case, the discontinuity cut is not free of membrane force and bending moment because of the presence of the remaining ligament of size t-a(x). Let the local membrane force N_{yy} and bending moment M_{yy} per unit length transmitted across the discontinuity be denoted N(x) and M(x), respectively. Furthermore, introduce the fields which represent the jump in shell kinematical quantities which are work-conjugate to N and M. The first of these is $\delta(x)$, the discontinuity in shell midsurface displacement normal to the plane of the surface crack: $\delta(x) = [u_y(x,0)]$. The second, $\theta(x)$, is the jump in midsurface rotation, and, in view of the adopted Kirchhoff condition $\theta(x) = [w,_y(x,0)]$, where w is the outward displacement of the shell and the comma indicates partial differentiation. In each case, the square brackets denote the discontinuity across the cut for the function indicated; for example,

$$[u_y(x,0)] = u_y(x, y=0^+) - u_y(x, y=0^-). \quad (1)$$

We have introduced four functions, M, N, θ and δ, and now must introduce connections between them. The first set of connections comes from consideration of the elastic shell structure. We will follow the formulation of Copley and Sanders (1969), based on a complex function representation of the shell behavior. Introduce the dimensionless shell parameter λ given by

$$\lambda^2 = (.75(1-\nu^2))^{1/2} \; c^2/(Rt) \quad (2)$$

where ν is the Poisson's ratio. The case of $\lambda \to 0$ can be interpreted as the flat plate limit of shallow shell theory as $R \to \infty$. Introduce dimensionless coordinates

$$(X,Y,Z) = (x/c,\ y/c,\ z/c). \quad (3)$$

Let a dimensionless normal displacement W (positive outwards) be given by

$$Et^2\, w = p_0 R c^2\, (12(1-\nu^2))^{1/2}\, W \quad (4)$$

in terms of a reference pressure p_0, and let the membrane stress resultants be given in terms of a dimensionless Airy function ϕ by

$$N_{xx} = p_0 R \phi,_{YY}, \quad N_{yy} = p_0 R \phi,_{XX}, \quad N_{xy} = -p_0 R \phi,_{XY}. \quad (5a\text{-}c)$$

If a reference moment resultant is introduced as $M_0 = p_0 R t\, (12(1-\nu^2))^{-1/2}$, the remaining physical resultants are:

$$M_{xx} = M_0\, (W,_{XX} + \nu W,_{YY}) \quad (6)$$

$$M_{yy} = M_0\, (W,_{YY} + \nu W,_{XX}) \quad (7)$$

$$M_{xy} = (1-\nu)\, M_0 W,_{XY} \quad (8)$$

$$Q_x = (M_0/c)\, (\hat{\nabla}^2 W),_X \quad (9)$$

$$Q_y = (M_0/c)\, (\hat{\nabla}^2 W),_Y \quad (10)$$

where $\hat{\nabla}^2$ is the two-dimensional Laplacian operator in the dimensionless coordinates (X,Y). Note that the sign convention for (6-10) is the negative of that often used.

A solution of the uncracked shell under internal pressure p is given by $\phi,_{XX}^{(0)} = p$ = constant and W = constant. The complete solution for the cracked structure is obtained by superposing this homogeneous solution and that of a residual problem in which the loads which are applied to the surfaces of the cut are the actual loads transmitted by the line-springs minus the loads of the homogeneous solution, with the stresses of this residual problem decaying to zero far from the cut.

With these introductions, and the combination of W and ϕ into the dimensionless complex function $\psi=\phi+iW$, the field equations of shallow shell theory for the residual problem are:

$$\hat{\nabla}^4\psi - 4i\lambda^2\psi,_{XX} = 0 \quad (11)$$

with following boundary conditions on the cut $|X| < 1$.

$$\phi,_{XX}(X,0) = (N(X)/R-p)/p_0 \quad (12)$$

$$W,_{YY}(X,0) + \nu W,_{XX}(X,0) = M(X)/M_0 \quad (13)$$

$$\phi,_{XY}(X,0) = 0 \quad (14)$$

$$W,_{YYY}(X,0) + (2-\nu)\, W,_{XXY}(X,0) = 0 \quad (15)$$

Equation (15) is the effective Kirchhoff shear condition. We may note that, if the uncracked solution had contained a uniform value of $M_{yy} = M_\infty$, then the right hand side of equation (13) would have been replaced by $(M(X)-M_\infty)/M_0$.

Copley and Sanders (1969) were able to use a Fourier transform representation of the solution to equations (11-15) to reduce the problem to two coupled singular integral equations in two dimensionless real density functions f(x) and g(x) defined on $|x| < c$. Straightforward algebraic manipulation of their equations leads to the relationships between the functions f and δ and g and θ as the dislocation densities

$$\mu_\theta \equiv \frac{d\theta}{dX} = \frac{p_0 R c \alpha^{1/2}}{Et^2}\, g \quad (16)$$

$$\mu_\delta \equiv \frac{d\delta}{dX} = -\frac{p_0 R c}{Et}\, f \quad (17)$$

where $\alpha = 12(1-\nu^2)$. We can directly substitute eqns. (16,17) plus the boundary conditions into eq. (43) of Copley and Sanders (1969) to obtain

$$pR - N(X) = \frac{Et}{c} \int_{-1}^{1} \{-t\alpha^{-1/2} H_1(X-X')\mu_\theta(X') + H_2(X-X')\mu_\delta(X')\}dX' \quad (18a)$$

$$-M(X) = \frac{Et^2}{\alpha c} \int_{-1}^{1} \{-tH_3(X-X')\mu_\theta(X') + \alpha^{1/2} H_4(X-X')\mu_\delta(X')\}dX' \quad (18b)$$

where the integrals are taken in Cauchy principal value sense and the kernels $H_i(X-X')$ are given in eq. (44) of Copley and Sanders (1969).

Equations (18a,b) represent the behavior of the shell, and constitute one of the couplings noted above. The second coupling comes from the line-spring and relates the local forces (N,M) to the local displacement discontinuities (δ,θ) through stiffness coefficients S_{ij} as

$$N(X) = S_{11}(X)\delta(X) + S_{12}(X)\theta(X) \tag{19a}$$

$$M(X) = S_{21}(X)\delta(X) + S_{22}(X)\theta(X) \tag{19b}$$

where the local stiffnesses S_{ij} depend on local relative crack depth $a(X)/t$. The rationale for the choice of the local stiffnesses S_{ij} is motivated in terms of the additional "cracked" compliances P_{ij} caused by an edge crack of depth a in a long plane strain strip of width t subject to combined tension and bending, as in Fig. 1(c). The local line-spring stiffness matrix S_{ij} is taken as the inverse of the "additional compliance" matrix P_{ij}. For further details, see Parks (1980) or Rice (1972).

We may note that $\delta(\pm 1) = \theta(\pm 1) = 0$ with consequent closure conditions

$$\int_{-1}^{1} \mu_\delta(X)dX = 0, \quad \int_{-1}^{1} \mu_\theta(X)dX = 0 \tag{20a,b}$$

Furthermore, we can use the boundary conditions on δ and the definition of μ_δ to obtain

$$\delta(X) = -\int_{-1}^{1} H(X'-X)\mu_\delta(X')dX' \tag{21}$$

where H is the Heaviside step function of its argument, with an identical relation derivable on replacing δ with θ. Equation (21) and its corresponding expression in θ can be inserted into eqs. (19a,b), and the resulting expressions for N and M can then be inserted into eqs. (18a,b). The final result, together with the closure eqs. (20a,b) is a system of coupled singular integral equations of the second kind, which can be readily solved numerically using the procedures developed by Erdogan and Gupta (1972).

In the range of loadings for which the bulk of the structure remains elastic, eqs. (18a,b) continue to hold, but if loads are sufficiently large, inelastic response of the line-spring can be accommodated through an incremental form of eqs. (19a,b), with S_{ij} replaced by tangent stiffnesses as discussed by Parks (1980). In such cases, incremental forms of equations (18a,b) and (20a,b) are solved.

In the elastic regime, the solution for the dislocation densities yield the displacement discontinuities from integrals of the form of eq. (21), and the crack tractions are recovered from the line-spring constitutive eqs. (19a,b). Finally, the stress intensity factor at a given location X is taken to be that which would obtain in a long single edge notched (SEN) specimen of width t and crack length a(X) subjected to an axial force and bending moment per unit thickness equal to N(X) and M(X), respectively. In the elastic-plastic regime, Parks (1980) noted that the plastic part of a crack tip opening increment, $\dot{\delta}_t^p$, could be related to the plastic parts of the displacement discontinuity increments through the kinematics of a nonhardening slipline analysis of the SEN specimen subject to tension and bending. The intensity of crack tip deformation could then be assessed by a local J value along the crack front, taking $J = J_{elastic} + J_{plastic}$, with $J_{elastic} = K_I^2(1-\nu^2)/E$. The plastic part of the J value was taken as the product of local plastic crack tip opening, the tensile flow stress of the material σ_0, and a pure number m which is of order one: $J_{plastic} = m\sigma_0\delta_t^p$. Thus the local J is given by

$$J = K_I^2(1-\nu^2)/E + m\sigma_0\delta_t^p \tag{22}$$

and an essentially equivalent description, in terms of crack tip opening displacement would be

$$\delta_t = .6\, K_I^2/E\sigma_0 + \delta_t^p \tag{23}$$

where the first term on the right hand side of eq. (23) represents the crack tip opening displacement in small scale yielding.

RESULTS

We first compare the stress intensity factor distribution for internal pressure of an axial interior surface crack in a cylinder of inner radius $R_i = 10t$. The crack shape is semi-elliptical, as indicated in Fig. 2, with deepest penetration a and of aspect ratio a/c = 1/3. More detailed results for this geometry are given by Heliot, Labbens, and Pellisier-Tanon (1979). Normalized stress intensity factors are given in terms of a function $H(\phi)$ given by

$$H(\phi) = \frac{K_I(\phi)E(k)}{\sigma_\theta(\pi a)^{1/2}}\left(\sin^2\phi + a^2\cos^2\phi/c^2\right)^{-1/4} \tag{24}$$

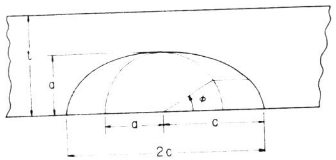

Fig. 2 Schematic illustration of semi-elliptical surface crack

where E(k) is the complete elliptic integral of the second kind, $k^2 = 1 - a^2/c^2$, σ_θ is a reference stress equal to the hoop stress at the inner radius and ϕ is a coordinate locating points on the ellipse $(x/c)^2 + (y/a)^2 = 1$ such that $y = a\sin\phi$. Thus ϕ ranges from 0 at the free surface to $\pi/2$ at the deepest penetration, a convention opposite to that used by Heliot, Labbens, and Pellisier-Tanon (1979).

Fig. 3 shows the line-spring and boundary integral calculations for a/t = .25 and .8. The line-spring K_I results are a few percent greater, but the maximum differences are roughly twelve percent. This level of agreement seems satisfactory in view of the simple shell theory used. Indeed this shell is not all that thin, $(t/R_i = .1)$ and the error inherent in the use of such a shell theory can be expected to be of order t/R. If the reference membrane stress σ_θ used in the line-spring shell calculations had been taken as related to the pressure p through a mean R/t value instead of through the Lamé solution, then both of the dotted line curves in Fig. 2 would be lowered by approximately five percent, resulting in somewhat better agreement for the deep crack and somewhat less agreement for the shallow crack. As in the plate results of Parks (1980), the line-spring model somewhat unexpectedly seems to capture the sense of the K_I variation near the free surface even though the basic premises of the model are not obtained.

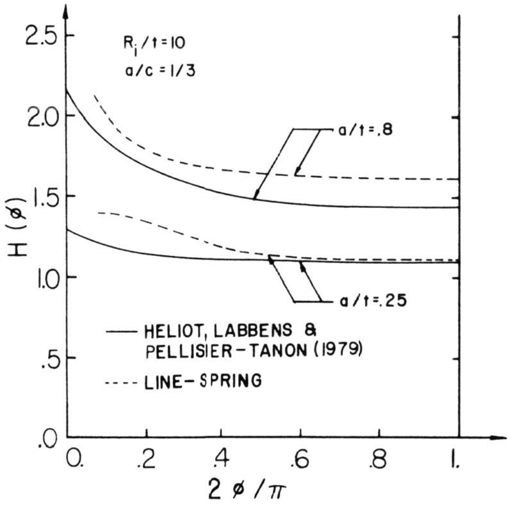

Fig. 3 Comparison of the variation of normalized stress-intensity factor for a pressurized cylinder containing an interior surface crack

Fig. 4 shows the effect of surface crack length on stress intensity factor distribution for a semi-elliptical internal crack of maximum relative depth a/t=.25. Here σ_θ is a reference hoop stress and for conservatism should probably be taken as the inner radius hoop stress of the Lamé solution.

Each of the line-spring solutions shown in Fig. 3 and 4 required approximately six (6) seconds of computer time on a Digital VAX computer. It hardly needs to be stated that such a combination of accuracy and economy makes the model uniquely promising for further development and application.

Fig. 5 shows the elastic-plastic concentration of deformation along the crack front for an axial semi-elliptical interior surface crack of maximum relative depth a/t=.5 with a/c = 1/3 in a pressurized cylinder with $R_i/t = 10$. As can be seen, the plasticity concentrates crack front deformation near the deepest penetration of the surface crack for more than near the intersection with the vessel

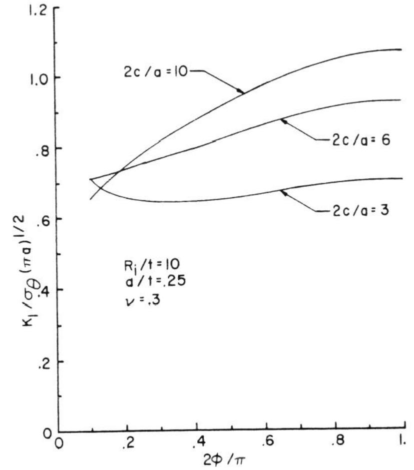

Fig. 4 Effect of projected surface crack length 2c on stress intensity factors for an axial semi-elliptical surface crack with a/t=.25 in a pressurized cylindrical vessel with $R_i/t = 10$.

inner wall. If material ductile tearing crack growth laws of the R-curve form $J(\Delta a)$ were available, graphs such as Fig. 5 would permit, by cross-plotting, an inferred variation of crack growth, $\Delta a(\phi)$.

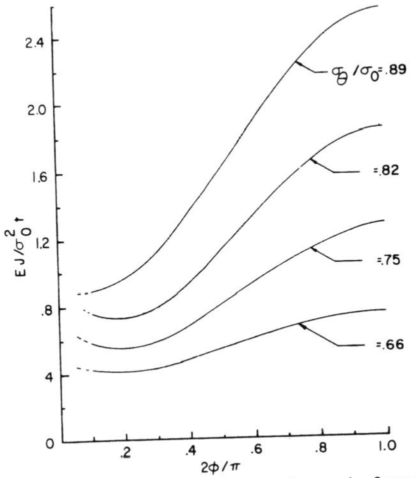

Fig. 5: Elastic-plastic concentrations of crack front deformation at deepest points of an axial interior surface crack with a/t=.5, a/c = 1/3 in a cylindrical vessel with $R_i/t = 10$.

Encouraging results of the sort shown here would seem to provide motivation for continued development of the line-spring model as a means of obtaining acceptably accurate characterizations of surface cracks in shells at a similarly acceptable cost.

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