CRACK-TIP PLASTICITY FOR RAPID CRACK PROPAGATION

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ABSTRACT

Dynamic effects have been investigated for the steady-state fields of stress and deformation in the immediate vicinity of a rapidly propagating crack tip in an elastic perfectly-plastic material. Both the cases of anti-plane strain and in-plane strain have been considered. The governing equations in the plastic regions are hyperbolic in nature. Simple wave solutions together with uniform fields provide explicit asymptotic expressions for the stresses and strains in the near-tip regions. Ahead of the crack tip, in the plane of the crack, scaled independent variables have been introduced to investigate the domain of validity of the dynamic solutions, and the transition to the quasi-static solution with decreasing crack-tip speed.

KEYWORDS

Dynamic effects; near-tip fields; fast fracture; elastic perfectly-plastic material; steady-state fields.

INTRODUCTION

The literature on dynamic effects near a rapidly propagating crack tip in the presence of elastic-plastic constitutive behavior is very limited. An investigation of the dynamic near-tip fields in an elastic perfectly-plastic material was presented by Slepyan (1976), who considered both the cases of anti-plane and in-plane strains. Dynamic effects near a propagating crack tip in a material with strain hardening were investigated by Achenbach and Kanninen (1978), and Achenbach, Kanninen and Popelar (1980), on the basis of J_2 flow theory, and a bilinear effective stress-strain curve. These authors found results which are very similar to the ones obtained by Amazigo and Hutchinson (1977) for the corresponding quasi-static problem. In the presence of strain hardening the governing equations are elliptic when the crack-tip speed is less than a certain critical value. The usual separation-of-variables asymptotic analysis can then be carried out, which yields singularities of the general type $r^p(-1 < p < 0)$ for the stresses and the strains. As the crack-tip speed increases (or alternatively as the strain-hardening curve becomes flatter) the nature of the governing equations becomes, however, hyperbolic, and the near-tip fields appear to change character. Indeed in the limit of elastic perfectly-plastic behavior the stresses become bounded and only some strains display singularities, as shown by Slepyan (1976), and

Achenbach and Dunayevsky (1980).

In this paper dynamic effects on near-tip fields are investigated for elastic perfectly-plastic constitutive behavior. The approach is different from the one employed by Slepyan (1976), and some different results have been obtained. The full expressions for the fields in the immediate vicinity of the propagating crack tip, which have been obtained elsewhere (Achenbach and Dunayevsky, 1980), are briefly reviewed. As the crack-tip speed decreases the expressions for the stresses reduce to the ones for the corresponding quasi-static problem, as might be expected on the basis of intuitive arguments. The strains, become, however, unbounded in the limit of vanishing crack-tip speed, which indicates that the transition from dynamic to quasi-static conditions is non-uniform. A detailed investigation of the fields ahead of the crack tip, in the plane of the crack, shows that the transition from the dynamic to the quasi-static solution with decreasing crack-tip speed is effected because the dynamic solution is asymptotically valid in a small zone, which shrinks on the crack tip in the limit of vanishing crack-tip speed.

GOVERNING EQUATIONS

Both a stationary coordinate system with axes denoted by x_i, and a moving coordinate system with axes denoted by (x,y,z) are considered. The moving coordinate system has its origin at the propagating crack tip. The geometry is shown in Fig. 1. In this section the equations governing the motions of an elastic perfectly-plastic material are stated in the stationary coordinate system. In the next sections these equations are simplified for anti-plane strain and plane strain, for the special case of "steady-state" fields of stress and deformation relative to the moving crack tip.

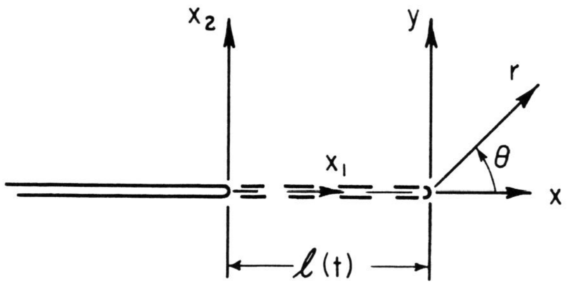

Fig. 1. Propagating crack tip with stationary and moving coordinate systems.

Relative to the stationary coordinate system the equations of motion are

$$\partial_j \sigma_{ij} = \rho\, \partial_t^2 u_i \tag{1}$$

In the zone of plastic deformation the stresses are assumed to satisfy the Tresca yield condition, which states that

$$|\tau|_{max} = k \tag{2}$$

where $|\tau|_{max}$ is the maximum shear stress, and k is the yield stress in pure shear. For an elastic perfectly-plastic solid the total strain rates are defined by

$$\partial_t \varepsilon_{ij} = \partial_t \varepsilon_{ij}^{el} + \partial_t \varepsilon_{ij}^{pl} \tag{3}$$

Here the elastic strain-rates are defined by

$$\partial_t \varepsilon_{ij}^{el} = \frac{1}{2\mu}\left(\partial_t \sigma_{ij} - \frac{\nu}{1-\nu}\, \partial_t \sigma_{kk} \delta_{ij}\right), \tag{4}$$

where μ and ν are the elastic shear modulus and Poisson's ratio, respectively. The plastic strain rates are

$$\partial_t \varepsilon_{ij}^{pl} = \lambda\, s_{ij} \tag{5}$$

where s_{ij} defines the stress deviator, and λ is a non-negative proportionality factor, which may vary in space and time. Equations (1) - (5) should be supplemented by appropriate boundary conditions.

For the case that the crack-tip speed approaches a constant value v_∞ as t increases, it may be assumed that steady-state fields of stress and deformation are established relative to the coordinate system moving with the crack tip. This assumption implies that the time derivatives may be expressed as

$$\partial_t(\ ) \sim -v_\infty \frac{\partial}{\partial x}\ ; \qquad \partial_t^2(\ ) \sim v_\infty^2 \frac{\partial^2}{\partial x^2} \tag{6a,b}$$

In the sequel the speed of the crack tip will appear in the "Mach number" M, which is defined as

$$M = v_\infty/(\mu/\rho)^{1/2} \tag{7}$$

where $(\mu/\rho)^{1/2}$ defines the velocity of shear waves in an elastic solid with shear modulus μ.

CRACK PROPAGATION IN MODE-III

In the moving coordinate system (x,y,z), steady-state motion in anti-plane strain is defined by a displacement w(x,y) in the z-direction. By using (6b), the equation of motion (1) then reduces to

$$\partial_x \sigma_{xz} + \partial_y \sigma_{yz} - \rho\, v_\infty^2\, \partial_{xx} w = 0 \tag{8}$$

The Tresca yield condition (2) may be expressed in the form

$$\sigma_{xz}^2 + \sigma_{yz}^2 \le k^2\ , \tag{9}$$

and the Prandtl-Reuss flow law (3)-(5) yields by using (6a)

$$\partial_{xx} w = \frac{1}{\mu}\, \partial_x \sigma_{xz} - 2\lambda\, \sigma_{xz} \tag{10}$$

$$\partial_{xy} w = \frac{1}{\mu}\, \partial_x \sigma_{yz} - 2\lambda\, \sigma_{yz} \tag{11}$$

It is sufficient to consider the solution in the half-plane $y \geq 0$. The boundary conditions at $y = 0$ are

$$x < 0 : \quad \sigma_{yz} = 0 \tag{12}$$

$$x \geq 0 : \quad \sigma_{yz} > 0 \quad , \; w = 0 \tag{13}$$

The yield condition (9) is identically satisfied by

$$\sigma_{xz} = -k \sin\omega, \quad \sigma_{yz} = k \cos\omega \tag{14a,b}$$

By introducing the strain component $w_x = \partial_x w$ together with (14a,b) in (8), and eliminating 2λ from (10) and (11), we obtain

$$\cos\omega\, \partial_x \omega + \sin\omega\, \partial_y \omega + M^2 \frac{\mu}{k}\, \partial_x w_x = 0 \tag{15}$$

and

$$\cos\omega\, \partial_x w_x + \sin\omega\, \partial_y w_x + \frac{k}{\mu}\, \partial_x \omega = 0 \tag{16}$$

where M is defined by (7).

Equations (15) and (16) constitute a hyperbolic system of equations. Appropriate solutions to this system of equations have been constructed by Achenbach and Dunayevsky (1980) as:

For $0 \leq \theta \leq \theta^*$:

$$\sigma_{xz} = -\, k\, [(1-M^2\sin^2\theta)^{1/2} - M\cos\theta]\, \sin\theta \tag{17}$$

$$\sigma_{yz} = k\, [(1-M^2\sin^2\theta)^{1/2} \cos\theta + M \sin^2\theta] \tag{18}$$

$$w_x = -\,(k/\mu M)\, \cos^{-1}[M \sin^2\theta + (1-M^2\sin^2\theta)^{1/2}\cos\theta] \tag{19}$$

$$w_y = \frac{k}{\mu}\Big\{\frac{1-M}{2M}\, \ell n[1-M\sin^2\theta - (1-M^2\sin^2\theta)^{1/2}\cos\theta] + \frac{1+M}{2M}\, \ell n[1+M\sin^2\theta + (1-M^2\sin^2\theta)^{1/2}\cos\theta] + \Psi(y) \tag{20}$$

where $w_y = \partial w/\partial y$ and $\Psi(y)$ is an as yet undetermined function. The angle θ^* is given by the relation

$$\theta^* = -\tan^{-1}(1/M) \tag{21}$$

For $\theta^* \leq \theta \leq \pi$:

$$\sigma_{xz} = -\,k\,, \qquad \sigma_{yz} = 0 \tag{22}$$

$$w_x = -\,(\pi/2)(k/\mu M), \qquad w_y = \Psi(y) \tag{23}$$

For various values of M, σ_{xz} and σ_{yz} have been plotted for the domain $0 \leq \theta < \theta^*$, in Figs. 2a and 2b.

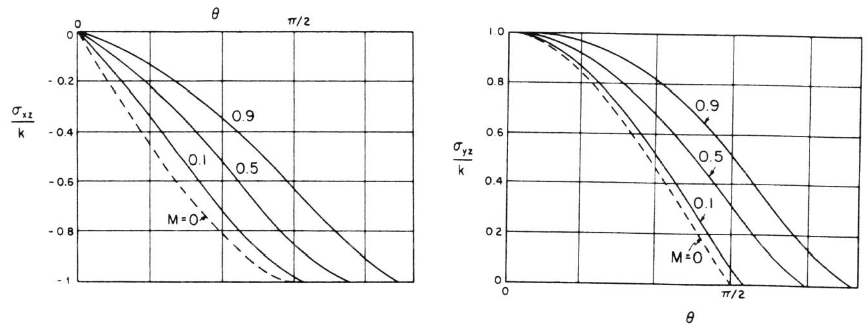

Fig. 2. Dimensionless shear stresses, σ_{xz}/k and σ_{yz}/k, versus θ, for various crack tip speeds; k = yield stress in pure shear; - - - quasi-static solution.

For $M = 0$, (17) and (18) reduce to $\sigma_{xz} = -k \sin\theta$ and $\sigma_{yz} = k\cos\theta$, respectively, which are the expressions for the quasi-static stresses derived by Chitaley and McClintock (1971). Thus, as one would perhaps expect intuitively, the dynamic stresses reduce to the quasi-static ones in the limit $M \to 0$.

An approximate expression for $\Psi(y)$ can be found from the condition that w_y should be bounded for $y = 0$, $x = r_p$ where r_p is the length of the plastic zone. For small y the expression for w_y given by (20) then implies that

$$\Psi(y) = \frac{k}{\mu}\Big[\frac{1-M}{M}\, \ell n(r_p/|y|) - \frac{1-M}{M}\, \ell n(1-M) - \ell n\, 2 + 1\Big] \tag{24}$$

It follows from (20) and (24) that w_y is singular at the crack tip. In the domain $0 \leq \theta \leq \theta^*$ we find

$$w_y \sim -\frac{k}{\mu}\, \frac{1-M}{M}\, \ell n(r/r_p) \tag{25}$$

This result shows that w_y not only becomes unbounded as $r \to 0$, but also as $M \to 0$. The strain component w_x is bounded as $r \to 0$, but it becomes unbounded in the limit $M \to 0$. Thus, for the strains there is no uniform transition from dynamic to quasi-static conditions. This transition will be discussed in more detail in the sequel.

A final result of interest is the crack-opening angle. This angle, which is defined by $\tan\frac{1}{2}\alpha = |w_x|$ at $\theta = \pi$, follows from (19) as

$$\alpha = 2 \tan^{-1} (\pi k/2\mu M) \tag{26}$$

As $M \to 0$ we find $\alpha = \pi$, which is in agreement with the quasi-static result.

MODE-I CRACK PROPAGATION IN PLANE STRAIN

In the moving coordinate system (x,y,z), steady-state motion in plane strain is defined by displacements u(x,y) and v(x,y) in the x- and y- directions, respectively. We assume that σ_z is the intermediate stress, i.e., $\sigma_x \le \sigma_z \le \sigma_y$. For the elastic case this is true, for the elastic-plastic problem it must be checked a posteriori. It was noted by Koiter (1953), see also Geiringer (1973, p. 514), that the Tresca yield condition then implies that $\partial_t \varepsilon^{pl} = \varepsilon^{pl}_z \equiv 0$. The condition of plane strain consequently reduces to $\varepsilon^{el}_z = 0$, which by virtue of (4) yields the result

$$\sigma_z = \nu(\sigma_x + \sigma_y) \tag{27}$$

We now introduce the following new variables

$$\sigma_- = \frac{1}{2} (\sigma_x - \sigma_y) \; ; \qquad \sigma = \frac{1}{3} (\sigma_x + \sigma_y + \sigma_z) \tag{28a,b}$$

$$u_x = \partial_x u \; ; \qquad v_x = \partial_x v \tag{29a,b}$$

By the use of Eq.(27) it is then easily checked that the equations of motion in the moving coordinate system may be expressed in the form

$$\partial_x \sigma_- + \frac{3}{2(1+\nu)} \partial_x \sigma + \partial_y \sigma_{xy} - \rho v_\infty^2 \partial_x u_x = 0 \tag{30}$$

$$-\partial_y \sigma_- + \frac{3}{2(1+\nu)} \partial_y \sigma + \partial_x \sigma_{xy} - \rho v_\infty^2 \partial_x v_x = 0 \tag{31}$$

Expressions for $\partial_x \varepsilon_x$ and $\partial_x \varepsilon_y$ follow from the Prandtl-Reuss flow equation (3)-(5). By considering $\partial_x(\varepsilon_x - \varepsilon_y)$ we find

$$\partial_y v_x - \partial_x u_x = (2\lambda/v_\infty) \sigma_- - (1/\mu) \partial_x \sigma_- \tag{32}$$

Similarly the relation for $\partial_x \varepsilon_{xy}$ yields

$$\partial_x v_x + \partial_y u_x = - (2\lambda/v_\infty) \sigma_{xy} + (1/\mu) \partial_x \sigma_{xy} \tag{33}$$

For plane strain the Tresca yield condition can be expressed in the form

$$\sigma_-^2 + \sigma_{xy}^2 = k^2 \tag{34}$$

Finally, the boundary conditions at y = 0 are

$$x < 0: \sigma_y = 0 \; , \; \sigma_{xy} = 0 \; ; \qquad x \ge 0 : v = 0 \; , \; \sigma_{xy} = 0 \; , \; \sigma_{yy} > 0 \tag{35a,b}$$

It is noted that the yield condition (34) is satisfied identically by

$$\sigma_- = - k \cos\omega \; ; \; \sigma_{xy} = - k \sin\omega \tag{36a,b}$$

By introducing these expressions in (30) - (33), and subsequently eliminating (λ/v_∞) we arrive at a system of equations of the general form

$$\sum_{j=1}^{4} L_{ij} \frac{\partial u_i}{\partial x} + \frac{\partial u_i}{\partial y} = 0 \; , \tag{37}$$

where

$$u_1 = \omega, \; u_2 = \sigma, \; u_3 = u_x \; , \; u_4 = v_x \; , \tag{38}$$

and the components of the matrix L_{ij} are given by Achenbach and Dunayevsky (1980).

Just as for the anti-plane case we seek simple wave solutions of Eq.(37). Unfortunately, for the plane strain case the system is too complicated to yield an analytical solution. It is, however, possible to obtain simple wave solutions for small values of M, where M is defined by (7), by using a perturbation method which is discussed in some detail by Achenbach and Dunayevsky (1980). It turns out that the stress fields depend on θ for $\theta_1^* \le \theta \le \theta_2^*$, while constant states prevail for $0 \le \theta \le \theta_1^*$ and $\theta_2^* \le \theta \le \pi$. Here we are particularly interested in the fields ahead of the crack tip. We find

$$\theta_1^* = \frac{\pi}{4} + \frac{1}{2}(1-\nu)^{1/2} M + O(M^2) \; , \tag{39}$$

and for $0 \le \theta \le \theta_1^*$

$$\sigma_{xy} = O(M^2) \; ; \qquad \sigma_y = k(2+\pi) + O(M^2) \tag{40a,b}$$

$$\sigma_x = k\pi + O(M^2) \; ; \qquad \sigma_z = 2\nu k(1+\pi) + O(M^2) \tag{41a,b}$$

$$\varepsilon_x = \frac{k}{\mu} [\frac{\pi}{2} - \nu(1+\pi)] + O(M^2) \; ; \; \varepsilon_y = \frac{k}{\mu}[- \frac{\pi}{2} + (1-\nu)(1+\pi)] + O(M^2) \tag{42a,b}$$

Thus, in this important domain all dynamic effects are higher order. It should be noted that the fields for $\theta > \theta_1^*$ do show a stronger dependence on M.

Another result of interest is the Mode-I crack-opening angle, which is defined by $\tan\frac{1}{2}\alpha = |v_x|$. We find

$$\tan\tfrac{1}{2}\alpha = M^{-1} (4k/\mu)(1-\nu)^{1/2} \tag{43}$$

UNIFORM ASYMPTOTIC RESULTS IN THE PLANE OF THE CRACK

The expressions that were stated in the previous sections may not be the most general solutions satisfying the boundary conditions and the governing equations. They are the so-called simple-wave solutions. In the plastic zone these solutions, which are the simplest ones that can be found, do not seem to have uniform validity for arbitrary values of M and the spatial coordinates. For the Mode-III case this is suggested by the observation, that the strain fields do not uniformly evolve into the corresponding quasi-static solutions as M decreases. It appears that the domain of validity of the simple wave solutions in the plastic zone shrinks on the crack tip as M decreases. To investigate this behavior in somewhat more detail for the Mode-III case, a separate asymptotic analysis has been carried out for a wedge-shaped region defined by small values of the angle θ.

As point of departure we rewrite (15) and (16) in polar coordinates (see Fig. 1), and we seek solutions of the form

$$\omega = \bar{\omega}(r)\ \theta + \omega_3(r)\ \theta^3 + - - - \tag{44}$$

$$w_x = \bar{w}_x(r)\ \theta + [w_x(r)]_3\ \theta^3 + - - - \tag{45}$$

Substitution of these expansions into the governing equations yields a system of ordinary differential equations for the coefficients. Here we restrict the attention to the first-order terms, i.e., to $\bar{\omega}(r)$ and $\bar{w}_x(r)$. After introducing $\rho = \ln(r/r_p)$ where r_p defines the boundary of the plastic zone in the plane of the crack, we find

$$(1-M^2)\ \frac{d\bar{\omega}}{d\rho} = (1-M^2)\bar{\omega} - \bar{\omega}^2 + \frac{\mu}{k} M^2\ \bar{\omega}\ \bar{w}_x = 0 \tag{46}$$

$$(1-M^2)\ \frac{d\bar{w}_x}{d\rho} = (1-M^2)\bar{w}_x + \frac{k}{\mu}\bar{\omega}^2 - \bar{\omega}\ \bar{w}_x = 0 \tag{47}$$

It was found necessary to analyze the solutions to (46) and (47) separately for $M \to 1$ and $M \to 0$.

For the case $M \to 0$ we introduce the following "boundary layer" variable

$$\tau = M\ \ln(r/r_p) \tag{48}$$

A length y asymptotic analysis, which includes matching of w_x to the elastic strain at $x = r_p$ (the elastic-plastic boundary), yields the following expressions, which are uniformly valid for $\theta = 0$, $0 \leq r \leq r_p$ as $M \to 0$

$$\bar{w}_x = M^{-1}(k/\mu)(r/r_p)^M - M^{-1}(k/\mu) + O(1) \tag{49}$$

$$\bar{\omega} = 1 + O(M) \tag{50}$$

For $r \to 0$ we find

$$w_y = -M^{-1}\ (k/\mu)\ \ln(r/r_p) + O(1) \tag{51}$$

while for $r/r_p \to 1$ we have

$$w_y = -M^{-1}\ (k/\mu)\ \ln(r/r_p) + B\ [\ln(r/r_p)]^2 + O(M) \tag{52}$$

where B is a constant.

Equation (49) should be compared with the expansion for small θ that can be obtained from (19) as

$$w_x = -M^{-1}\ (1-M)(k/\mu)\theta \tag{53}$$

Clearly (49) and (53) agree in the limit $r \to 0$. Similarly (25) and (51) agree as $r \to 0$. Thus the simple wave solutions (17)-(20) are valid for small values of r.

The extent of the domain of validity can be estimated by comparing the orders of magnitude of the two terms in Eq.(49). The first term is negligible as compared to the second one when

$$(r/r_p)^M = O(M)\ ,\ \text{i.e., when}\ r/r_p \sim M^{1/M} \tag{54}$$

Thus the boundary layer in which the dynamic solution is valid becomes extremely small as M decreases.

When $(r/r_p) \sim O(1)$, an expansion of $(r/r_p)^M$ with respect to M can be carried out [i.e., $(r/r_p) \sim 1 + M\ \ln(r/r_p)$], to yield

$$\bar{w}_x = (k/\mu)\ [\ln(r/r_p)-1] \tag{55}$$

$$\bar{\omega} = 1 \tag{56}$$

These expressions agree with the asymptotic approximations to the quasi-static solutions presented by Chitaley and McClintock (1971).

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