

## A THERMODYNAMIC DESCRIPTION OF INELASTIC FRACTURE

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### ABSTRACT

The problem of line crack extension in dissipative continua is studied within the theory of classical thermodynamics. By a global dissipation analysis, the expression of intrinsic dissipation is given and the crack tip force  $G$  associated with the crack extension velocity is introduced. If  $G \neq 0$ , it is shown that the crack tip behaves like a moving heat source and  $T \sim -\log r$ . The obtained results are new and give a general framework for the study of coupled thermomechanical behaviour near the crack tip. Classical notions such as Rice-Eshelby integral, energy flux of Freund, energy release rate are extended for arbitrary continua. Usual models of elastic, viscoelastic and plastic materials are considered.

### KEYWORDS

Thermodynamics ; dissipation ; singularity transport condition ; crack tip force ; heat source ; thermal equation ; dissipative continua.

### INTRODUCTION

The objective of this paper is to present a thermodynamic description of the running crack problem in arbitrary dissipative continua [12, 13, 4]. This description illustrates the development of energy methods in the context of fracture. In many points, the proposed theory is different from Cherepanov's [5], or Rice's [15] or Gurtin's [9] descriptions, the two last ones are devoted to the Griffith model of brittle fracture.

The basic problem associated with the formulation of a criterion for crack extension is the characterization of the local state of stress and strain near the crack tip. This problem has been successfully studied in elasticity. Significant results have been obtained such as the notions of stress intensity factors, path independent integrals, energy release rate, etc... [3], [14].

From a macroscopic point of view, if we consider that crack extension is an irreversible thermodynamic process in a similar way as friction or viscosity or plasticity, then classical methods of continua mechanics can be applied and a satisfactory description can be derived in plasticity, viscoelasticity as in elasticity. Here it is shown that classical notions are extended in a consistent manner and that basic equations governing the coupled thermomechanical behaviour are established.

For this, we present in the first section the thermodynamical analysis of a plane body with a running line crack, undergoing small transformation. The material is not necessarily elastic. The crack tip force  $G$  associated with the crack extension velocity is introduced. The expression of the intrinsic dissipation is given. In particular, it is shown that if  $G \neq 0$ , the crack tip behaves like a moving heat source and the temperature is singular as  $-\text{Log } r$ . The general theory is then illustrated by usual models of elastic, visco-elastic and elastic plastic materials. Significant results given in the literature are reviewed in the context of the present description. New results will be presented, such as the coupled thermomechanical singularity in thermoelasticity. The estimation of the crack tip force is discussed in detail. Our attention will be focussed on Rice's estimation  $G = 0$  in the elastic perfectly plastic case. Although our definition of  $G$  seems to be different, the estimation  $G = 0$  results effectively from the stress and strain singularity analysis [12], not only for the elastic perfectly plastic model, but also for certain models in viscoelasticity, when the reversible surface energy is absent. The estimation of  $G$  in the presence of reversible surface energy is a rather complex problem, the reader may refer to [13] for a short discussion in the case of mechanical reversible surface energy. Here, we assume that reversible surface energy is neglected, all surface energy will be considered as dissipative.

THERMODYNAMIC DESCRIPTION

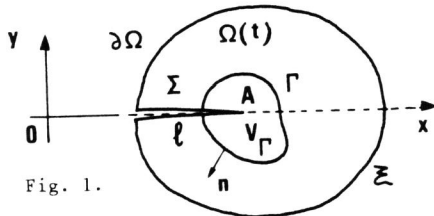


Fig. 1.

Let us consider the dynamic response of a simple body with a line crack which corresponds to the two dimensional problem of the fig. 1. The material, not necessarily elastic, is defined by a free energy density per unit mass  $W$ , function of the state variables  $(\epsilon, \alpha, T)$  where  $\epsilon$  denotes the strain tensor,  $\alpha$  a family of internal parameters and  $T$  the temperature. The internal energy is  $e(\epsilon, \alpha, S) = TS + W$ , the entropy density  $S$

and the temperature verifies :

$$S = - \frac{\partial W}{\partial T}, \quad T = \frac{\partial e}{\partial S} \quad (1)$$

The first principle of thermodynamics gives the energy balance :

$$\dot{U} + \dot{C} = P_e + P_{cal} \quad (2)$$

for an arbitrary system of material points occupying a volume  $V$ .

The quantities  $U, C, P_e, P_{cal}$  denote respectively the global internal energy, the kinetic energy, the external power and the calorific power :

$$U = \int_V \rho e dV, \quad C = \int_V \rho \dot{u}^2 / 2 dV \quad (3)$$

$$P_e = \int_{\partial V} n \cdot \sigma \cdot \dot{u} ds, \quad P_{cal} = - \int_{\partial V} q \cdot n ds$$

The second principle inequality leads to the notion of intrinsic dissipation and thermal dissipation. We recall that for regular system, the intrinsic dissipation is :

$$D = \int_V \rho T \dot{S} dV - P_{cal} \geq 0 \quad (4)$$

where  $D = \int_V D dV$  and  $D = \sigma \dot{\epsilon} - \rho(\dot{e} - T \dot{S}) = \sigma \dot{\epsilon} - \rho(\dot{W} + S \dot{T}) \geq 0$  is the volumic dissipation.

At the crack tip  $A$ , the thermomechanical fields may be singular. If we take for  $V$  the whole body  $\Omega(t)$ , the relation (4) is still available under the following assumption :

Assumption H1

The functions  $D, T \dot{S}$  and  $S \dot{T}$  are Lebesgue-integrable functions in  $\Omega(t)$ . So we rule out the presence of shock waves where both  $T$  and  $S$  may have finite discontinuities and  $T \dot{S}, S \dot{T}$  are not integrable functions. The integrability of  $D$  is a quite physical assumption since  $D$  corresponds to various volumic dissipations. For further purpose, we introduce also the following basic assumption :

Assumption H2

The transport condition of the singularity for a physical quantity  $g$  is :  $\dot{g} = - \dot{\ell} g_{,1} + (\text{more regular terms integrable})$  (5)

i.e. the dominant term of  $\dot{g}$  is exactly  $-\dot{\ell} g_{,1}$ .

This assumption, which is identically verified for steady motions of crack, means that the singularity of the quantity  $g$  must have the same nature. One can remark that  $\dot{g} + \dot{\ell} g_{,1}$  is the derivative of  $g$  with respect to time in the moving reference with the crack tip.

For the whole body, the computation of  $\dot{U}$  and  $\dot{C}$  is not straightforward since  $\dot{e}$  and  $\dot{u}$  are not necessarily integrable. Let us isolate the crack tip by a closed curve  $\Gamma$ , delimiting a volume  $V_\Gamma(t)$  in translation with the crack. If  $V_\Gamma$  and  $C_\Gamma$  denote respectively :

$$U_\Gamma(t) = \int_{\Omega(t) - V_\Gamma(t)} \rho e dV, \quad C_\Gamma(t) = \int_{\Omega(t) - V_\Gamma(t)} \rho \dot{u}^2 / 2 dV \quad (6)$$

then we obtain with the assumption H2 for  $g = \rho(e + \dot{u}^2/2)$  :

$$\lim_{\Gamma \rightarrow 0} (\dot{U}_\Gamma + \dot{C}_\Gamma) = \dot{U} + \dot{C} \quad (7)$$

$$\frac{d}{dt} [(U + C) - (U_\Gamma + C_\Gamma)] = \frac{d}{dt} \int_{V_\Gamma} \rho(e + \dot{u}^2/2)(x + \ell(t) - \ell(t_0), y, t) dV_0$$

$$\dot{U} + \dot{C} - \lim_{\Gamma \rightarrow 0} (\dot{U}_\Gamma + \dot{C}_\Gamma) = \lim_{\Gamma \rightarrow 0} \int_{V_\Gamma(t_0)} \{\rho(e + \dot{u}^2/2)_{,t} + \dot{\ell} \rho(e + \dot{u}^2/2)_{,1}\} dV_0 = 0$$

because the integrant is an integrable function.

The first principle gives for the system of material points occupying the volume  $\Omega(t) - V_\Gamma(t)$  at time  $t$  :

$$\int_{\Omega - V_\Gamma} \rho(\dot{e} + \dot{u} \cdot \dot{u}) dV = \int_{\partial \Omega} n \cdot \sigma \cdot \dot{u} ds - \int_\Gamma n \cdot \sigma \cdot \dot{u} ds + \int_\Gamma q \cdot n ds - \int_{\partial \Omega} q \cdot n ds - \int_{\Sigma_\Gamma} q \cdot n ds \quad (8)$$

The left hand side can also be expressed as :

$$\dot{U}_\Gamma + \dot{C}_\Gamma + \int_\Gamma \dot{\ell} \rho(e + \dot{u}^2/2) n_1 ds \quad (9)$$

From (2), (8), (9), we obtain simply :

$$\lim_{\Gamma \rightarrow 0} \int_\Gamma q \cdot n ds = \lim_{\Gamma \rightarrow 0} \int_\Gamma \{\dot{\ell} \rho(e + \dot{u}^2/2) n_1 + n \cdot \sigma \cdot \dot{u}\} ds \quad (10)$$

The second principle gives for the same system :

$$\int_{\Omega - V_\Gamma} D dV = \int_{\Omega - V_\Gamma} \rho T \dot{S} dV + \int_{\partial \Omega} q \cdot n ds + \int_{\Sigma_\Gamma} q \cdot n ds - \int_\Gamma q \cdot n ds \quad (11)$$

From (4), (11), we obtain simply :

$$D = \int_\Omega D d\Omega + \lim_{\Gamma \rightarrow 0} \int_\Gamma \{\dot{\ell} \rho(e + \dot{u}^2/2) n_1 + n \cdot \sigma \cdot \dot{u}\} ds \quad (12)$$

If we introduce the crack tip force  $G$  :

$$G = \lim_{\Gamma \rightarrow 0} \int_\Gamma \{\rho(e + \dot{u}^2/2) n_1 + n \cdot \sigma \cdot \dot{u} / \dot{\ell}\} ds \quad (13)$$

then the relation (10) shows that the crack tip behaves like a heat source of intensity  $G \dot{\ell}$ . The relation (12) shows that the global dissipation consists of two terms : volumic dissipation  $D$  and local dissipation  $G \dot{\ell}$  at the crack tip.

The relation (13) gives the expression of G in terms of the singularity of the thermomechanical fields. We remark from the two assumptions H 1, H 2 that (TS)<sub>1</sub> is an integrable function, thus :

$$\lim_{\Gamma \rightarrow 0} \int_{\Gamma} \dot{\ell} TS n_1 ds = \lim_{\Gamma \rightarrow 0} \int_{V_{\Gamma}} \dot{\ell} (TS)_{,1} dV = - \lim_{\Gamma \rightarrow 0} \int_{V_{\Gamma}} (TS) \dot{\ell} dV = 0 \quad (14)$$

This relation and the transport condition of u show that the crack tip force can also be written as :

$$G = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \{ \rho(W + \dot{u}^2/2) n_1 - n \cdot \sigma \cdot u_{,1} \} ds \quad (15)$$

The thermal evolution of the crack body follows from (10). We obtain indeed :

$$\text{Div } q + \rho \dot{T} S = D + G \dot{\ell} \delta(A) \quad (16)$$

where  $\delta(A)$  denotes the Dirac measure at the moving crack tip. For example, if the Fourier law  $q = -k \text{ grad } T$  is assumed, we obtain the thermal equation :

$$-k \Delta T + \rho \dot{T} S = D + G \dot{\ell} \delta(A) \quad (17)$$

Let us prove that the temperature is singular ( $T \sim \text{Log } r$ ) if and only if  $G \neq 0$ , otherwise T is finite at the crack tip.

Indeed, if the temperature can be expanded as  $T = H(t) r^{\alpha}$ .  $(\text{Log } r)^{\beta} \hat{T}(\theta, \ell)$ , we obtain necessarily  $\alpha = 0$ ,  $\beta = 1$ ,  $\hat{T} = \hat{T}(\ell)$  since the heat source must be finite,  $q \sim -k \text{ grad } T \sim r^{-1}$ . The temperature field is then  $T = H(t) \text{Log } r \hat{T}(\ell) + \text{more regular terms } r^m (\text{Log } r)^n$ ,  $m \geq 0$ . But from the equation (17), it follows that :

$$T = - \frac{G \dot{\ell}}{2k\pi} \text{Log } r + \text{regular terms } r^m (\text{Log } r)^n, \quad m \geq 0 \quad (18)$$

This result is new and has been given recently [4], [13]. We underline the fact that the temperature singularity is here obtained for a large class of materials. Our description gives a general framework for the determination of coupled thermo-mechanical behaviour.

Remark

In quasi-static transformation, the crack tip force G can also be characterized as an energy release rate.

Indeed, we have in this case the possibility to define a global parametric description [12], [13]. The system is completely defined by the state variables  $(\xi, \alpha, \ell, T, \sigma_{IR})$  which are respectively the displacement over  $\partial\Omega$ , the internal parameter field, the crack length, the temperature field and the irreversible stress field  $\sigma_{IR} = \sigma - \rho \partial W / \partial \epsilon$ . Let us consider the functional :

$$\phi(\xi, \alpha, \ell, T, \sigma_{IR}) = \int_{\Omega(t)} (\rho W + \sigma_{IR} \epsilon) dV \quad (19)$$

The partial derivative  $\partial\phi/\partial\xi$  represents the surface force F acting on  $\partial\Omega$ , the partial derivative  $\partial\phi/\partial\alpha$  represents the generalized force field associated to  $\alpha$ , the partial derivative  $\partial\phi/\partial T$  the field  $-\rho S$  and the derivative  $\partial\phi/\partial\sigma_{IR}$  the field  $\epsilon$ .

In the references [12], [13] we established that :

$$G = - \frac{\partial\phi}{\partial\ell} (\xi, \alpha, \ell, T, \sigma_{IR}) \quad (20)$$

ILLUSTRATIONS IN ELASTICITY, VISCOELASTICITY & PLASTICITY

The theory is here illustrated by usual models of material in elasticity, viscoelasticity and plasticity. We wish to verify by reviewing the obtained results in the literature that the two assumptions H1 and H2 are fulfilled and the proposed theory is consistent. Almost classical results are related only to the pure mechanical problem. In this case, we recall that the basic equation is the energy balance  $P = E + C + D$  where E is the reversible, elastic energy of the whole system. The assumption H1

corresponds then to the integrability of  $D = \sigma \dot{\epsilon} - \rho \dot{W}$ , W denotes here the elastic energy density.

Elasticity

For elastic materials, the free energy density is a function of the state variables  $(\epsilon, T)$  and  $\sigma = \rho \partial W / \partial \epsilon$ . There is no volumic dissipation i.e.  $D = 0$ .

In quasi-static transformation, Rice [14] and Cherepanov [5] have discussed the stress and strain singularities for the mechanical problem, when  $W(\epsilon)$  is a homogeneous polynomial function of  $\epsilon$ . It has been shown that  $W = K(t) r^{-1} \hat{W}(\theta)$  and thus the assumption H2 is verified.

The formula (15) reduces to the classical Rice-Eshelby integral :

$$G = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \{ \rho W n_1 - n \cdot \sigma \cdot u_{,1} \} ds = \int_{\Gamma} \{ \rho W n_1 - n \cdot \sigma \cdot u_{,1} \} ds$$

Remark that  $\sigma \dot{\epsilon} \sim r^{-2}$  so the function  $\dot{W}$  is not integrable although its Cauchy principal value exists.

If the transformation is dynamic, smooth solutions have been studied in linear elasticity [1]. We verify without difficulty the two assumptions H1, H2. The quantity  $G \dot{\ell}$  where G is given by (13) has been introduced in this case by Freund [7] as the energy flux into the crack tip.

The general case of dynamic thermoelasticity has been recently discussed in the context of our theory [4]. The asymptotic solution of the thermomechanical fields is given up to the second terms for both displacement and temperature. It has been established that the temperature singularity (18) does not change the dominant singularity of the mechanical fields which are the same as that obtained in isotherm elasticity. This new result illustrates for the first time the coupled behaviour near the crack tip.

The intrinsic dissipation  $D = G \dot{\ell}$  is a finite measure at the crack tip for elastic materials. This expression enables us to construct systematically "standard" laws of crack propagation if we assume normal dissipation in a similar way as standard laws of plasticity and viscoplasticity [8], [10]. These laws correspond to the existence of a dissipation potential  $\Omega^*(\ell)$  such that :

$$G = \frac{\partial \Omega^*}{\partial \dot{\ell}}(\ell) \quad (21)$$

For example, the Griffith law corresponds to  $\Omega^* = G_c \langle \dot{\ell} \rangle$ , and the crack extension law (21) may be written as :

$$\begin{aligned} \text{If } G < G_c & \text{ then } \dot{\ell} = 0 \\ \text{If } G = G_c & \text{ then } \dot{\ell} \geq 0 \end{aligned} \quad (22)$$

where  $G_c$  denotes a critical dissipative surface energy.

Viscoelasticity

Only the purely mechanical problem has been discussed. Let us consider firstly Kelvin models of viscoelasticity. In perfect viscoelasticity i.e. without internal parameters the free energy is  $W(\epsilon, T)$ . The quantity  $\sigma_R = \rho \partial W / \partial \epsilon$  is the reversible stress,  $\sigma = \sigma_R + \sigma_{IR}$  and we obtain  $D = \sigma_{IR} \dot{\epsilon}$ . In the linear case, the constitutive equations are  $\sigma_R = L \cdot \epsilon$  and  $\sigma_{IR} = M \dot{\epsilon}$ . These equations show that if the tensors L and M are positive definite,  $\sigma_{IR}$  is more singular than  $\sigma_R$ . A singularity analysis shows that  $\sigma_{IR} \sim r^{-1/2}$  and  $\epsilon \sim \sigma_R \sim r^{1/2}$ . The formula (13) gives  $G = 0$ . Since  $D \sim r^{-1}$  and  $W \sim r$  the two assumptions H1 and H2 are well verified. Perfect Maxwell models are discussed in a similar way. The free energy is

$W(\epsilon - \epsilon^P, T)$  with  $\sigma = \rho \partial W / \partial \epsilon$ . The volumic dissipation is  $D = \sigma \dot{\epsilon}^P$ . In the linear case the constitutive equations are  $\sigma = L(\epsilon - \epsilon^P)$  and  $\sigma = M \dot{\epsilon}^P$ . These equations show that  $\sigma \sim L\epsilon$  and thus a singularity gives simply  $\sigma \sim \epsilon \sim r^{-1/2}$ , the singularities of stress and strain are those of linear elasticity for a moving crack. Non-linear Maxwell models have been discussed recently [11]. If we assume that  $\dot{\epsilon}_{ij}^P = C(|\sigma|/\sigma_0)^{m-1} \sigma_{ij}/\sigma_0$ ,  $|\sigma|^2 = \sigma_{ij} \cdot \sigma_{ij}$ ; it has been shown that if  $m < 3$  the stress singularity is that of linear elasticity, if  $m > 3$  the stress singularity is  $r^{-1/m-1}$ . In the later case  $G = 0$ . In both situations, the two assumptions H1, H2 are verified.

### Plasticity

In perfect plasticity, the free energy is  $W(\epsilon - \epsilon^P, T)$  with  $\sigma = \rho \partial W / \partial \epsilon$  and  $D = \sigma \dot{\epsilon}^P$ . Let us consider the case of linear elastic strain stress relation and quasi-static, isothermal transformation. The constitutive equations are  $\sigma = L(\epsilon - \epsilon^P)$  and  $\dot{\epsilon}^P = \lambda \partial f / \partial \sigma$  in the plastic range. Significant results are due to Chitaley, Mc Clintock [6], Slepian [17], Rice and coworkers [16]. Plane strain analysis in mode I shows that the stress distribution may be compared to the Prandtl's distribution, but an unloading region near the free boundary is observed. The stress is bounded and the velocity  $\dot{u} \sim \text{Log } r$  gives an opening displacement  $[u] \sim r \text{Log } r$ . The assumption H1 is fulfilled since  $D = \sigma \dot{\epsilon}^P \sim r^{-1}$ . The basic assumption H2 is naturally verified. The formula (13) gives simply  $G = 0$ .

For hardening materials, there exists a discussion from Amazigo and Hutchinson [2] in the case of linear isotropic hardening. It has been shown that the stress is singular as  $r^s$  with  $-1/2 \leq s \leq 0$ ,  $s \rightarrow 0$  if  $E_L/E \rightarrow 0$  and  $s \rightarrow -1/2$  if  $E_L/E \rightarrow 1$ . The formula (13) gives always  $G = 0$  from this singularity analysis. The estimation  $G = 0$  does not result from bounded stress distribution near the crack tip. In incremental plasticity, it is expected that  $\sigma \dot{\epsilon}$  has the same singularity as  $\sigma \dot{\epsilon}^P$ , thus it is integrable and gives  $G = 0$ . This estimation has been suggested by Rice in many published papers. But the definition of  $G$  used by Rice is not the same, we denote it here by  $G^*$ :

$$G^* = \lim_{\Gamma \rightarrow 0} \int_{\Gamma} \{ W^* n_1 - n \cdot \sigma \cdot u_1 \} ds$$

where  $W^* = \int_{\sigma} \sigma d\epsilon$  is the deformation energy which is not a state function in incremental plasticity.

### CONCLUSION

Our discussion shows that for several models of material the energy method does not provide a local fracture parameter, except in elasticity. The global intrinsic dissipation consists of volumic term  $D$  and eventually a local term  $G^*$ , the estimation  $G = 0$  means simply that there is no crack tip energetic parameter. In that case, we underline the fact that the temperature is finite.

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