A THEORY FOR LAMINATED PLATES WITH A THROUGH-THE-THICKNESS CRACK

by

E. P. Chen
Sandia National Laboratories
Systems Safety Technology Division
Division 4442
Albuquerque, New Mexico, USA 87185

ABSTRACT

The flexural response of a laminated plate with a through-the-thickness crack was considered in this investigation. A laminated plate theory of the Reissner type has been developed to treat this problem. For demonstration purposes, the theory was applied to a balanced symmetric laminate which contains a through-the-thickness central crack. The stress intensity factor for the crack was determined and the influences of the various material and geometric parameters on the stress intensity factor are identified.

KEYWORDS

Laminated plate; crack; flexure; stress intensity factor.

INTRODUCTION

The response of laminated plates to bending action is considered in this investigation. This is an important problem area in composite material mechanics since the laminated plate is one of the most widely used structural elements. However, the analysis of laminated plates presents difficulties because of the added length dimension - namely, the plate thickness. Exact three-dimensional solution to the problem is very complicated, if at all possible. The approximate approach, then, is to assume stress or strain dependence on the thickness coordinate and reduce the problem to a two-dimensional one by considering the stress resultants.

The classical plate theory for isotropic elastic plates was developed by Kirchhoff [1]. The effect of transverse shear deformation was neglected and, consequently, only two out of three of the traction-free condition on a free surface can be satisfied. This deficiency was corrected by Reissner [2]. The extension of Reissner's theory to include material anisotropy was accomplished by Calcote [3], Smith [4], Reissner and Stavsky [5], Dong, et. al. [6], Whitney [7] and Pagano [8], among others. In the field of fracture mechanics, the bending of isotropic plates containing a through-the-thickness crack was treated by Hartranft and Sih [9] and Knowles and Wang [10] in the context of the Reissner theory. An improvement over the Reissner theory which relaxes the linear dependence of the stresses in the thickness coordinate was made by Hartranft and Sih [11]. A laminated theory along this line was developed by Sih, Badaliance and Chen [12]. However, the theory presented in [12] is very complicated and presents difficulties when extension to orthotropic laminates is attempted.

In this study, a simple laminated plate theory of the Reissner type is developed to deal with crack problems. The basic approach is to assume that the strains are continuous across the material interfaces. Consequently, the stresses are necessarily discontinuous by virtue of the distinct constitutive relationships from lamina to lamina. The theory is demonstrated through a specific example of a balanced symmetric laminate with a through-the-thickness crack. Fracture mechanics parameters are determined and the influences of various material and geometric properties on these parameters are discussed.

FORMULATION OF THE PROBLEM

Consider the laminated plate in Figure 1. The plate consists of four layers each of which is assumed to be isotropic and homogeneous. The two middle layers are made of the same material whose shear modulus of elasticity and Poisson's ratio are denoted as μ_1 and ν_1, respectively. The two outer layers are of another material whose material properties are characterized by μ_2 and ν_2. All four layers are of the same thickness h/4 such that the laminate is of thickness h. A through-the-thickness central crack of length 2a exists in the plate. The in-plane dimensions of the plate are assumed to be large in comparison to both h and a such that boundary effects may be neglected. A set of Cartesian coordinates x, y and z are attached to the center of the plate as shown in Figure 1.

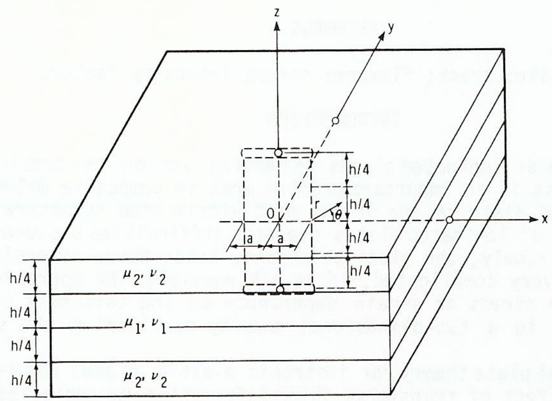

Fig. 1. A symmetrically layered plate with a through crack.

The approximation here is to treat the plate as a nonhomogeneous one whose properties are functions of the thickness coordinate z. Then, the derivation procedure for dynamic plate theory by Mindlin [13] will be followed. Although only a time-independent example will be worked out here, the general elastodynamic equation will be derived for completeness.

Let the properties of the laminated plate be

$$(\mu, \nu, \rho) = \begin{cases} (\mu_1, \nu_1, \rho_1) & 0 < |z| < \frac{h}{4} \quad (1) \\ (\mu_2, \nu_2, \rho_2) & \frac{h}{4} < |z| < \frac{h}{2} \quad (2) \end{cases}$$

in which the symbol ρ denotes the mass density of the material. Further, it is assumed that the faces of the plate are free from tangential tractions but under normal pressure q_1 and q_2, that is,

$$\tau_{xz} = \tau_{yz} = 0 \qquad z = \pm \frac{h}{2} \tag{3}$$

$$\sigma_z = -q_1(x, y, t) \qquad z = \frac{h}{2} \tag{4}$$

$$\sigma_z = -q_2(x, y, t) \qquad z = -\frac{h}{2} \tag{5}$$

The bending and twisting moments and transverse shearing forces, all per unit length, are defined by

$$(M_x, M_y, H_{xy}) = \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \tau_{xy})\, z\, dz \tag{6}$$

$$(Q_x, Q_y) = \int_{-h/2}^{h/2} (\tau_{xz}, \tau_{yz})\, dz \tag{7}$$

Making use of the Hooke's law relating stresses to strains, equations (6) and (7) render

$$M_x = D_1[(\Gamma_x)_1 + \nu_1(\Gamma_y)_1] + D_2[(\Gamma_x)_2 + \nu_2(\Gamma_y)_2] \tag{8}$$

$$M_y = D_1[(\Gamma_y)_1 + \nu_1(\Gamma_x)_1] + D_2[(\Gamma_y)_2 + \nu_2(\Gamma_x)_2] \tag{9}$$

$$2H_{xy} = (1-\nu_1)D_1(\Gamma_{xy})_1 + (1-\nu_2)D_2(\Gamma_{xy})_2 \tag{10}$$

$$2Q_x = K^2 h[\mu_1(\Gamma_{xz})_1 + \mu_2(\Gamma_{xz})_2] \tag{11}$$

$$Q_y = \frac{1}{2}K^2 h[\mu_1(\Gamma_{yz})_1 + \mu_2(\Gamma_{yz})_2] \tag{12}$$

where the strain resultants are

$$h^3[(\Gamma_x)_1, (\Gamma_y)_1, (\Gamma_{xy})_1] = 96\int_{-h/4}^{h/4}(\epsilon_x, \epsilon_y, \gamma_{xy})z\,dz \tag{13}$$

$$7h^3[(\Gamma_x)_2, (\Gamma_y)_2, (\Gamma_{xy})_2] = 96\left[\int_{-h/2}^{-h/4} + \int_{h/4}^{h/2}\right](\epsilon_x, \epsilon_y, \gamma_{xy})z\,dz \tag{14}$$

$$[(\Gamma_{xz})_1, (\Gamma_{yz})_1] = \frac{2}{h}\int_{-h/4}^{h/4}(\gamma_{xz}, \gamma_{yz})\,dz \tag{15}$$

$$[(\Gamma_{xz})_2, (\Gamma_{yz})_2] = \frac{2}{h}\left[\int_{-h/2}^{-h/4} + \int_{h/4}^{h/2}\right](\gamma_{xz}, \gamma_{yz})\,dz \tag{16}$$

and the flexure rigidities are

$$D_1 = \frac{\mu_1 h^3}{48(1-\nu_1)}, \quad D_2 = \frac{7\mu_2 h^3}{48(1-\nu_2)} \tag{17}$$

In equation (11) and (12), the term K^2 is inserted to account for the thickness-shear motion of the plate. The value for K^2 was found to be $\pi^2/12$ in [13].

Let the displacements be continuous through the interfaces by assuming

$$u_x = z\psi_x(x,y,t), \quad v_y = z\psi_y(x,y,t), \quad w_z = w(x,y,t) \tag{18}$$

Substituting equations (18) into equation (13) - (16) and making use of the strain-displacement relationship for an elastic body yield the expressions relating the strain resultants and the potential function ψ_x, ψ_y and w. Inserting these expressions into equations (8) - (12) renders

$$M_x = D_0\left[\frac{\partial\psi_x}{\partial x} + \nu_0\frac{\partial\psi_y}{\partial y}\right] \tag{19}$$

$$M_y = D_0\left[\frac{\partial\psi_y}{\partial y} + \nu_0\frac{\partial\psi_x}{\partial x}\right] \tag{20}$$

$$H_{xy} = D_0(1-\nu_0)\left(\frac{\partial\psi_y}{\partial x} + \frac{\partial\psi_x}{\partial y}\right)/2 \tag{21}$$

$$Q_x = \pi^2 h\mu_0\left(\psi_x + \frac{\partial w}{\partial x}\right)/12 \tag{22}$$

$$Q_y = \pi^2 h\mu_0\left(\psi_y + \frac{\partial w}{\partial y}\right)/12 \tag{23}$$

in which the contractions

$$D_0 = D_1 + D_2 \tag{24}$$

$$\nu_0 = (D_1\nu_1 + D_2\nu_2)/D_0 \tag{25}$$

$$\mu_0 = \frac{1}{2}(\mu_1 + \mu_2) \tag{26}$$

have been used. Note that equations (19) - (23) are similar to those derived in [13] except the material constants D, ν and μ are replaced by D_0, ν_0 and μ_0, respectively.

Making use of the elastodynamic equations of motion and equations (6), (7), (18) and after some manipulation yield

$$\frac{\partial M_x}{\partial x} + \frac{\partial H_{xy}}{\partial y} - Q_x = \frac{\rho_0}{12}h^3\frac{\partial^2\psi_x}{\partial t^2} \tag{27}$$

$$\frac{\partial H_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} - Q_y = \frac{\rho_0}{12}h^3\frac{\partial^2\psi_y}{\partial t^2} \tag{28}$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q_2 - q_1 = \bar{\rho}h\frac{\partial^2 w}{\partial t^2} \tag{29}$$

where

$$\rho_0 = (\rho_1 + 7\rho_2)/8, \quad \bar{\rho} = \frac{1}{2}(\rho_1 + \rho_2).$$

By substituting equations (19) - (23) into equations (27) - (29), the differential equations governing the function ψ_x, ψ_y and w are obtained as

$$D_0\left[(1-\nu_0)\nabla^2\psi_x + (1+\nu_0)\frac{\partial}{\partial x}\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\right)\right] - \frac{\pi^2}{6}h\mu_0\left(\psi_x + \frac{\partial w}{\partial x}\right) = \frac{\rho_0}{6}h^3\frac{\partial^2\psi_x}{\partial t^2} \tag{30}$$

$$D_0\left[(1-\nu_0)\nabla^2\psi_y + (1+\nu_0)\frac{\partial}{\partial y}\left(\frac{\partial\psi_x}{\partial x} + \frac{\partial\psi_y}{\partial y}\right)\right] - \frac{\pi^2}{6}h\mu_0\left(\psi_y + \frac{\partial w}{\partial y}\right) = \frac{\rho_0}{6}h^3\frac{\partial^2\psi_y}{\partial t^2} \tag{31}$$

$$\frac{\pi^2}{12} h\mu_o \left(\nabla^2 w + \frac{\partial \Psi_x}{\partial x} + \frac{\partial \Psi_y}{\partial y}\right) + q_2 - q_1 = \bar{\rho} h \frac{\partial^2 w}{\partial t^2} \qquad (32)$$

in which ∇^2 is the Laplacian operator in two-dimension, i. e. $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

BENDING OF A LAMINATED PLATE BY A STATICALLY APPLIED MOMENT

Consider for the moment that the plate in Figure 1 is initially at rest and that a constant moment $M_y = M_0$ is slowly applied to the plate such that it may be considered as time-independent. It is desired to determine the response of the plate under this loading condition.

Under these considerations, equations (30) - (32) reduce to

$$Q_x - k^2 \nabla^2 Q_x = -D_o \frac{\partial}{\partial x}(\nabla^2 w) \qquad (33)$$

$$Q_y - k^2 \nabla^2 Q_y = -D_o \frac{\partial}{\partial y}(\nabla^2 w) \qquad (34)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0 \qquad (35)$$

in which $k^2 = D_0 (1 - \nu_0)/[4K^2 h \mu_0]$. In the limit for an isotropic plate, $\mu_1 = \mu_2$, $\nu_1 = \nu_2$ and k^2 reduces to h^2/π^2. This is about the same value as $h^2/10$ obtained in [9]. Equations (33) - (35) are to be solved under the following boundary and symmetry conditions :

$$H_{xy}(x,0) = Q_y(x,0) = 0 \qquad 0 < x < \infty \qquad (36)$$

$$M_y(x,0) = -M_o \qquad x < a \qquad (37)$$

$$\Psi_y(x,0) = 0 \qquad x \geq a \qquad (38)$$

Taking the Fourier cosine transform on Equations (39) - (41) and enforcing the boundary conditions (36) - (38) render the following pair of dual integral equations for the solution of the unknown constant [illegible]):

$$\int_0^\infty A(s) \cos(sx)\, ds = 0 \qquad x \geq a \qquad (39)$$

$$\int_0^\infty s F(s) A(s) \cos(sx)\, ds = -\frac{\pi M_o}{2(1+\nu_o)} \qquad x < a \qquad (40)$$

where

$$F(s) = \frac{3+\nu_o + 4k^2 s^2 (1-m)}{1+\nu_o}, \qquad m = \left(1 + \frac{1}{k^2 s^2}\right)^{1/2} \qquad (41)$$

The solution to equation (39) and (40) has been obtained by a modified procedure of Copson's methods [14] as

$$A(s) = -\frac{\pi M_o a^2}{2(1+\nu_o)} \int_0^1 \sqrt{\xi}\, \Phi(\xi) J_o(sa\xi)\, d\xi \qquad (42)$$

Where J_o is zero order Bessel function of the first kind and the function $\Phi(\xi)$ is governed by a Fredholm integral equation of the second kind:

$$\Phi(\xi) + \int_0^1 L(\xi,\eta)\, \Phi(\eta)\, d\eta = \sqrt{\xi} \qquad (43)$$

The kernel $L(\xi,\eta)$ is defined as

$$L(\xi,\eta) = \sqrt{\xi\eta} \int_0^\infty s\left[F\left(\frac{s}{a}\right) - 1\right] J_o(s\xi) J_o(s\eta)\, ds \qquad (44)$$

This completes the solution to the problems. The task now is to determine the moment distribution near the crack.

Following the same procedure as [9], the crack tip moment distribution is found as

$$M_x(r,\theta) = \frac{k_1}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) + O(r^o) \qquad (45)$$

$$M_y = \frac{K_1}{\sqrt{2r}} \cos\frac{\theta}{2} \left(1 + \sin\frac{\theta}{2} \sin\frac{3\theta}{2}\right) + O(r^0) \tag{46}$$

$$H_{xy} = \frac{K_1}{\sqrt{2r}} \cos\frac{\theta}{2} \sin\frac{\theta}{2} \cos\frac{3\theta}{2} + O(r^0) \tag{47}$$

$$Q_x = Q_y = O(r^0) \tag{48}$$

with the moment intensity factor k_1 being given as

$$K_1 = M_0 \sqrt{a}\, \Phi(1) \tag{49}$$

Note that the shear resultants remain finite at the crack point while the moment resultants possess the familiar inverse square root singularity in r, where r and θ are the local polar coordinates to the crack. This qualitative feature is similar to the result obtained in [9] for isotropic plates.

RESULTS AND DISCUSSIONS

Equation (43) has been solved numerically on an electronic computer and the results are shown in Figures 2-5. For all these figures, the mass densities were taken as $\rho_1 = \rho_2$. In Figure 2 the normalized moment intensity factor $k_1/(M_0\sqrt{a})$ is plotted as a function of the normalized plate thickness parameter $h/(\pi a)$ for $\nu_1 = \nu_2 = 0.3$ and $\mu_2/\mu_1 = 10.0$, 1.0 and 0.1. It is observed that the local moments to the crack increases with increasing $h/(\pi a)$ ratio for all μ_2/μ_1 ratios. Also, the stress intensity factor is greater with increasing μ_2/μ_1 values. These features are not altered when the values of the Poisson's ratios are changed. These are shown in Figure 3 for $\nu_1 = 0.2$, $\nu_2 = 0.4$ and in Figure 4 for $\nu_1 = 0.4$, $\nu_2 = 0.2$. The effect of Poisson's ratio on the moment intensity factor is depicted in Figure 5 for γ_1/γ_2 ratio fixed at 10.0. It is seen that the set $\nu_1 = 0.2$, $\nu_2 = 0.4$ yields the largest $k_1/M_0\sqrt{a}$ value, while the set $\nu_1 = 0.4$, $\nu_2 = 0.2$ renders the smallest value with the set $\nu_1 = \nu_2 = 0.3$ yields values in between the two previous sets.

In summary, a laminated plate theory which is suitable for the treatment of crack problems was presented. An example problem was solved to demonstrate the applicability of this theory. The theory is sufficiently simple for extension into anisotropic laminated plate problems.

REFERENCES

[1] Kichhoff, G. (1850). Uber das Gleichgewicht and die Bewegung eimer elastischen Scheibe, Journal fur Reine and Angewandte Mathematik, 40, 51-88.

[2] Reissner, E. (1945). The Effect of Transverse Shear Deformation on the Bending of Elastic Plates, Journal of Applied Mechanics, Trans. ASME, 67, A69-A77.

[3] Calcote, L. R. (1969). The Analysis of Laminated Composite Structures, Van Nostrand Reinhold Company.

[4] Smith, C. B. (1953). Some New Types of Orthotropic Plates Laminated by Orthotropic Material, Journal of Applied Mechanics, 20, 286-288.

[5] Reissner, E. and Stavsky, Y. (1961). Bending and Stretching of Certain Types of Heterogeneous Aeolotropic Elastic Plates, Journal of Applied Mechanics, 28, 402-408.

[6] Dong, S. B., Pister, K. S., and Taylor, R. L. (1962). On The Theory of Laminated Anisotropic Shells and Plates, Journal of the Aerospace Sciences, 29, 969-975.

[7] Whitney, J. M. (1969). The Effect of Transverse Shear Deformation on the Bending of Laminated Plates, Journal of Composited Materials, 3, 534-547.

[8] Pagano, N. J. (1969). Exact Solutions for Composite Laminated in Cylindrical Bending, Journal of Composite Materials, 3, 398-411.

[9] Hartranft, R. J. and Sih, G. C. (1968). Effect of Plate Thickness on the Bending Stress Distribution Around Through Cracks, Journal of Mathematics and Physics, 47, 276-291.

[10] Knowles, J. K. and Wang, N. M. (1960). On the Bending of an Elastic Plate Containing a Crack, Journal of Mathematics and Physics, 39, 223-235.

[11] Hartranft, R. J. and Sih, G. C. (1970). An Approximate Three-Dimensional Theory of Plates with Application to Crack Problems, International Journal of Engineering Sciences, 8, 711-729.

[12] Sih, G. C., Badaliance, R. and Chen, E. P. (1977). Through Cracks in Multilayered Plates, Plate and Shells with Cracks, G. C. Sih, editor, Noordhoff International Publishing.

[13] Mindlin, R. D. (1951). Influence of Rotatory Inertia and Shear on Flexural Motion of Isotropic Elastic Plates, Journal of Applied Mechanics, 18, 31-38.

[14] Copson, E. T. (1961). On Certain Dual Integral Equations, Proceedings Glasgow Mathematical Association, 5, 19-24.

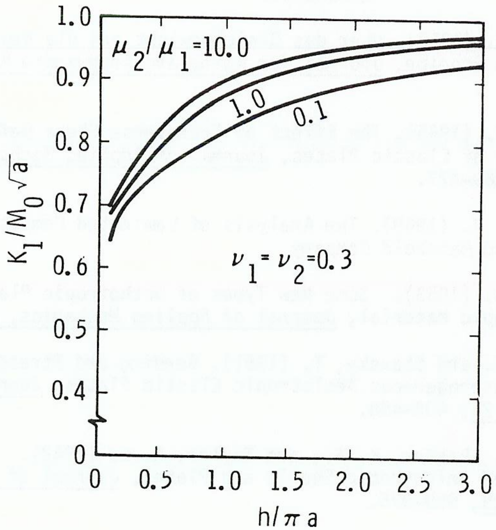

Fig. 2. Normalized moment intensity factor as a function of $h/\pi a$ for $\nu_1 = \nu_2 = 0.3$.

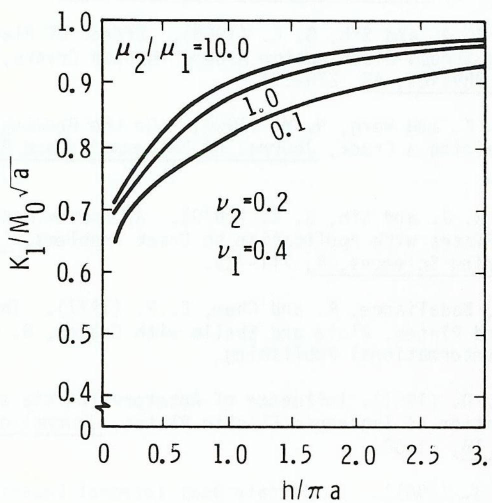

Fig. 3. Variations of moment intensity factor with laminate plate thickness for $\nu_1 = 0.2$ and $\nu_2 = 0.4$.

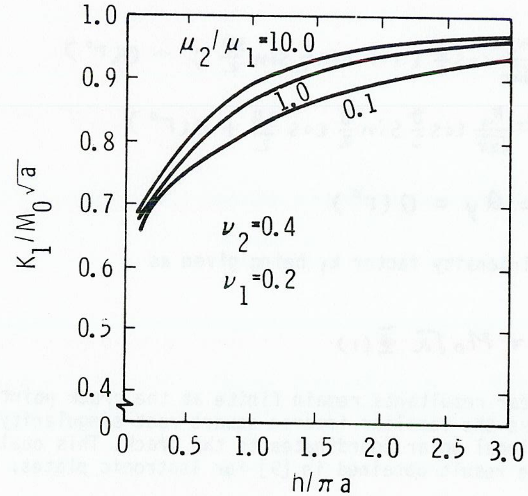

Fig. 4. Variations of moment intensity factor with laminate plate thickness for $\nu_1 = 0.4$ and $\nu_1 = 0.2$.

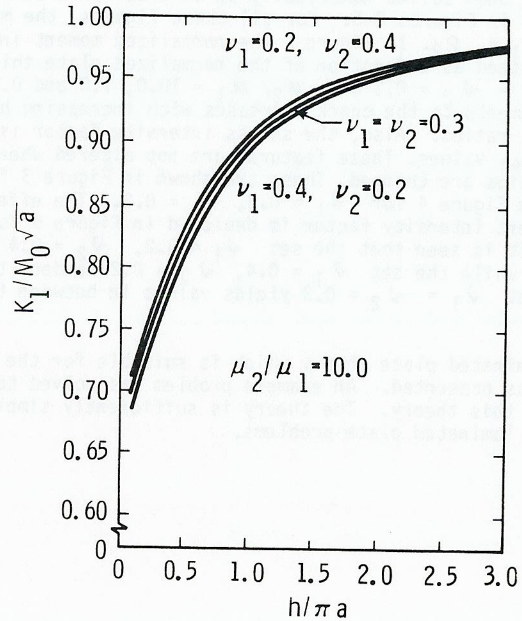

Fig. 5. Normalized moment intensity factor as a function of $h/\pi a$ for $\mu_2/\mu_1 = 10.0$.