PROPAGATION OF DAMAGE IN ELASTIC AND PLASTIC SOLIDS

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ABSTRACT

A simple model is proposed to analyse the dynamic steady state propagation of a damaged zone in elastic and plastic solids. Exact solutions are worked out for the small scale damage model in elastic material and for the strip problem, in mode III loading. The small scale model of mode I is numerically solved with the finite element method. Mathematically, one has an unknown-boundary value problem, the solution of which provides the shape of the damaged zone. The influence of parameters such as applied loads, fracture stress, and velocity is studied in connection with the J-integral in elastodynamics. The relationship with the classical theory of cracks is established. The implications for ductile fracture are discussed using the model of damage in elastic perfectly plastic material. The new result $G > 0$ is obtained for plastic damage models which imply a characteristic dimension related to the material constants, the applied loads and the velocity.

KEYWORDS

Damage ; elastic-damage ; elastic-plastic-damage ; dynamic propagation of damaged zone ; damage front ; path-dependent integral ; path-independent integral ; energy release rate ; fracture energy rate ; plastic rate ; ductile fracture ; characteristic dimension ; antiplane solution.

1. INTRODUCTION

Crack problems are extensively studied in the fracture mechanics. A crack in solid is geometrically idealized by a smooth surface of discontinuity. The crack front, or the crack-tip in 2-D problems, is necessarily a singularity line (or point) of the mechanical fields. The nature of such singularities depends on the physical laws of the material. Generally, it is assumed that the same mechanical behaviour, elastic or plastic etc..., applies to both continuum and crack tip region. The homogeneity assumption requires that the process zone at the tip is small compared to the crack length. Crack theories do not take into account the process zone, because at the scale of some grains, geometrical and physical idealizations of the crack are not adequate. The material is indeed damaged and so the continuum concept may be questioned. As far as the continuum approach is a reasonable assumption, any analyses of the process zone by continuous models should be very helpful in fracture mechanics.
There are continuous models of internal damage parameters, which can be incorpora-

ted into crack analysis. For instance, damage parameters have been related to the effective stress, Kachanov (1958), Lemaître and Chaboche (1978), Westlund (1979), etc..., to specific mass density, Rousselier (1977, 1978), and recently to microcracks density, Dragon and Mroz (1979). To our knowledge, the most elaborated models of damage are these of Dragon and Mroz, and Rousselier, who have introduced the thermodynamical state parameters such as strain ε, plastic strain ε^p, internal parameters associated with the hardening and the softening processes. Damage models have been especially developed for fatigue, creep, ductile fracture, etc..., see Dang Van and Cordier (1980), Kachanov (1978), Kubo and al. (1979).

In this paper, we make use of a rather elementary model of damage in order to focus our attention on the problem of fracture by dynamic propagation of a damaged zone.

Analytical solution will be worked out for dynamic mode III. It will be shown that the model of "small scale damage" in linear elastic solid is identical to the classical linear fracture model, provided certain limit procedure is made. A "large scale damage" model will be illustrated by the problem of an infinite strip. Approximate analysis of the quasi-static growth damage in mode I will be also given.

Mathematically, the analysis of fracture by damage propagation is an unknown-boundary value problem, which has some similar feature with cavitation in fluids dynamics. Moreover, the problem is non-linear. The aim of this paper is to determine how far the damaged zone may be influenced by different parameters such as loading conditions, fracture stress, velocity V of the damage front.

Energy considerations will be done in connection with a path-independent integral given for moving cracks in elasto-dynamics by Bui (1977, 1978). The extension of the theory to ductile fracture will also be discussed.

2. A SIMPLE MODEL OF DAMAGE

2.1 Qualitative considerations

In most polycrystalline metals under simple loading, the quasi-static stress-strain curve has schematically the shape presented on Fig. 1-a, with three characteristic ranges : elastic (E), plastic (P), and failure range (F). The last range after reaching the fracture stress σ_R or the limit strain ε_R is characterized by the decrease of the stress which may be explained by various factors, geometrical instabilities, such as necking, narrow slip bands through the sample, and structural instabilities occurring at two different levels. Firstly, on a microscopic level, there are nucleation of voids, cleavage, intense shear bands between voids (Grant 1971, etc...). Secondly, on a macroscopic scale, there are cavities and microcracks.

The plastic range depends on the materials and the loading condition. The effect of a very high strain rate is to sharpen the stress-strain curve (dotted line of fig. 1-a).

Let us consider some qualitative aspects on micro-mechanisms of plasticity and damage. Plastic strain is known to be the result of dislocations glides, twinning, etc... The effect of increasing strain rate is to increase both density of defects, dislocation loops and cells, and inertial effects, which constrain the motions of dislocations. These mechanisms are responsible for the workhardening of metals. Plasticity by slip is generally independent of the hydrostatic stress. Experimental results on steel, see survey of works given in Bement & al. (1971), show a sudden increase of the dynamic yield stress at $\dot{\varepsilon} \simeq 10^4\ s^{-1}$, at room temperature, more than 2 to 3 times the static value. In fact, below the transition temperature,

time effects and diffusion processes are often not very important to have influence on the work-hardening process. At high stress levels, and elevated densities of line defects, there are new mechanisms, such as relaxation of dislocations at microcracks along the boundary of grains, annihilation of dislocations between themselves, voids nucleation by jog mechanism, etc... These softening mechanisms are associated with an overall volumic dilatation, so that damage and fracture are expected to depend on hydrostatic pressure, François (1977), Auger & al. (1977). The effect of increasing strain rate is to reduce the ductility of metals, not the damage process and so the stress-strain curve has a sharp form. It is expected that such a stress-strain curve prevails in the process zone of the running crack, even at low speed. For instance, a small strain variation $\Delta\varepsilon \simeq 10^{-2}$ over a grain size distance $\Delta x \simeq 10^{-5}$ m results in a strain rate of order $\dot{\varepsilon} \simeq 10^3$ V, i.e. $10^4\ s^{-1}$ for V = 10 ms^{-1}. This very qualitative discussion indicates that dynamic behaviour must be considered in the process zone. Unfortunately, we have only a few experimental data during very large strain rate, $\dot{\varepsilon} \geqslant 10^5\ s^{-1}$ or higher value.

2.2 A model of damage

We first idealize a sharp $\sigma(\varepsilon)$ curve by the model of Fig. 1 (b), which is considered in many works, see Dragon and Mroz (1979), Rousselier (1978). We assume that the material is elastic below some fracture stress σ_R. Damage fracture occurs when some relation is satisfied :

(1) $\qquad f(I_1, J_2) = 0$

where $I_1 = \sigma_{ii}$ and $J_2 = 1/2\ s_{ij}s_{ij}$ are respectively the invariants of the stress tensor and the deviatoric stress tensor. At the critical state (1), according to the path-strain, there are two possible states, elastic unloading (path 1) and sudden damage (path 2). We can interpret the model in the framework of a damage theory, by putting the Kachanov's parameter D equal to zero for elastic behaviour, to one for full samage state. Since the effective stress $\sigma_e = \sigma/(1-D)$ must be a finite quantity, the value D = 1 implies $\sigma = 0$.

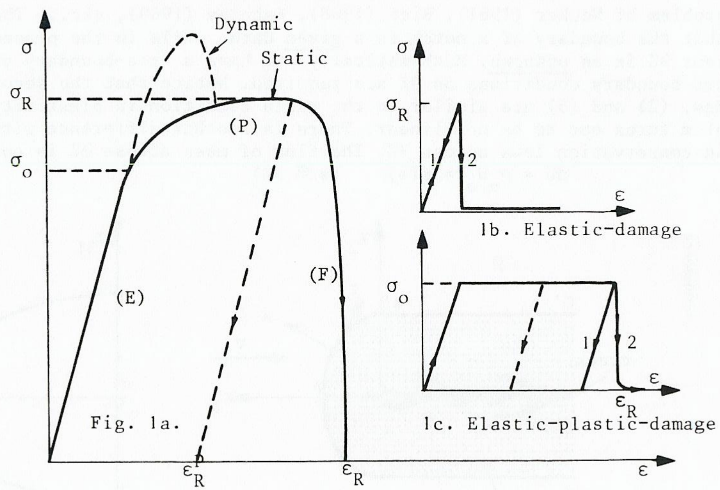

Fig.1. (a) Static and dynamic stress-strain curves.
(b) Idealization of elastic-damage law, σ_R fracture stress.
(c) Elastic-plastic-damage, yield stress σ_o, limit strain ε_R.

In section 7, we shall consider a more complex model including plasticity too, Fig. 1 (c). In what follows, we make use of the elastic-damage model to analyse the propagation of a damaged zone Z in solid. We assume that the material is damaged around the crack tip and along the crack surfaces, Fig. 2 (a). Neither the physical crack, nor the damaged and discontinuous material in Z may be geometrically, kinematically described.

Let us consider the steady state propagation of the damaged zone Z, parallel to Ox_1 with the velocity V. We assume that the boundary ∂Z consists of two straight lines CB, C'B' which are the stream-lines of particles being elastically unloaded and the arc BB', i.e. the damage front of the process zone. In the wake region Z, due to full damage, the stress tensor is null. Hence the stress-tensor is discontinuous across ∂Z, but the continuity of the stress vector $T = \sigma.n$ implies that $T_i = 0$ along ∂Z. In the moving axes $Ox_1x_2x_3$, the boundary conditions for the stress-tensor in the undamaged zone Ω are :

$$(2)\qquad \begin{cases} \sigma_{21} = \sigma_{22} = \sigma_{23} = 0 & \\ f(I_1, J_2) < 0 & \text{on BC, B'C'} \end{cases}$$

$$(3)\qquad \begin{cases} \sigma_{nn} = \sigma_{nt} = \sigma_{nz} = 0 & \\ f(I_1, J_2) = 0 & \text{on BB'.} \end{cases}$$

In a two-dimensional problem, the damage criterion takes a simple form. In mode III loading, the stress components are σ_{31}, σ_{32}. Therefore the condition $f \leq 0$ is equivalent to :

$$(4)\qquad \sigma_{31}^2 + \sigma_{32}^2 \leq 0 \quad , \quad (\sigma_{3n} = 0)$$

In mode I or II, the inequality takes the form :

$$(5)\qquad \sigma_{tt}^2 \leq \sigma_R^2 \quad , \quad (\sigma_{nn} = \sigma_{nt} = 0).$$

One notices that the first conditions of Eqs. (2) and (3) are the same for notches problem of Neuber (1961), Rice (1968), Eshelby (1969), etc... The difference is that the boundary of a notch is a given data, while in the present work, the contour ∂Z is an unknown. Mathematically, we have a free-boundary value problem, so two boundary conditions on ∂Z are required. Notice that the second conditions of Eqs. (2) and (3) are similar to the yield condition in plasticity, i.e. the problem turns out to be non-linear. There is another difference with notches problem in conservation laws across ∂Z. The flux of mass across ∂Z is conserved, i.e. :

$$\rho U = \rho_o U_o = m(s) \qquad (s \in \partial Z)$$

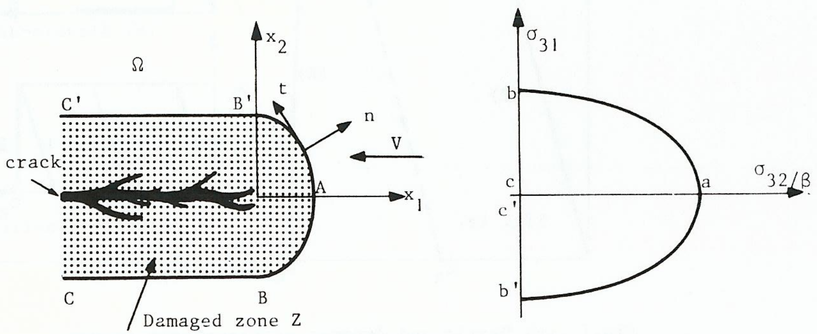

Fig. 2. (a) Steady-state moving damaged zone Z with the velocity V. The damage front is BB'.
(b) Complex τ-plane, $\tau = \sigma_{32}/\beta + i\sigma_{31}$.

where ρ and ρ_o are respectively the specific mass density of Ω and Z, U and U_o are the normal relative velocities of their material points with respect to ∂Z. The flux of mass m(s) vanishes on the stream-lines BC, B'C'. But along BB', we have $m(s) > 0$ and, according to the second principle of thermodynamics, we also have $m[S] \geq 0$, where [S] is the jump in specific entropy S . Consequently, $[S] \geq 0$. This means that some energy, from the continuous media Ω , is dissipated through BB' with non negative rate. The analysis of fracture energy rate through BB' will be discussed in section 7. We remark that the motion of a notch in a solid violates both Eq.(6) and the second principle.

3. DYNAMIC PROPAGATION OF A SMALL SCALE DAMAGE ZONE.

3.1 Equations

Small scale damage must be understood in the mathematical sense that both stress and strain decrease at infinity as $O(r^{-1/2})$. Let us recall the basic equation in mode III. The displacement w satisfies the wave equation in fixed axes Oxyz :

$$(6)\qquad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)w - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}w = 0$$

where $c = (\mu/\rho)^{1/2}$ is the shear wave speed, μ is the shear modulus. Introduce the moving axes $Ox_1x_2x_3$ for the steady state case, together with the well-known complex variables of Radok (1958) :

$$z = x_1 + i\beta x_2$$

$$\tau = \frac{1}{\beta}\sigma_{32} + i\sigma_{31}$$

where $\beta = (1 - v^2/c^2)^{1/2}$, $\sigma_{32} = \mu w_{,2}$, $\sigma_{31} = \mu w_{,1}$.

The wave equation (6) is equivalent to $\partial\tau/\partial\bar{z} = 0$. Hence the problem is readily solved by finding the analytic function $\tau(z)$ or $z(\tau)$ which satisfies appropriate boundary conditions.

In our problem, these conditions are the damage criterion Eq.(4), the stress-free condition on ∂Z and the small scale damage assumption taken in the following form:

$$(7)\qquad \tau \simeq \frac{K}{\beta\sqrt{2\pi z}} \qquad |z| \to \infty$$

The parameter K is similar to the dynamic stress-intensity factor K_{III}, while noting that it is not associated with stress singularity. Further discussion is needed to make clear its actual significance.

3.2 Solution

The analytical solution of the problem stated above was obtained by Bui and Ehrlacher (1980 a). Let us present the solution in details.

We let the damage criterion (4) satisfied by mapping z into the interior of the semi-ellipse of the τ-plane (Fig. 2 (b)), with semi-axes τ_R/β and τ_R. The front BAB' is mapped into the arc bab', while the lines BC, B'C' are mapped into bc, b'c'. The crucial step of the method of solution is to express the stress-free condition $\sigma_{31}n_1 + \sigma_{32}n_2 = 0$ in complex form. Let us consider the arc bab', in which : $\tau = \frac{1}{\beta}\tau_R\cos\theta + i\tau_R\sin\theta$, $-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$. If we denote by ds the arc element on BAB', then with the stress-free condition, we obtain $dz = (\sin\theta + i\beta\cos\theta)\,ds$. It is easy to verify that the stress-free condition is simply :

$$(8)\qquad \mathrm{Im}\left\{(\tau^2 - \gamma^2)\frac{dz}{d\tau}\right\} = 0 \qquad \text{along bab',}$$

where $\gamma^2 = (1 - \beta^2)\tau_R^2/\beta^2$. Now, on the straight lines BC, B'C', because of $\sigma_{32} = 0$ we see that $d\tau$ is purely imaginary, while dz and $(\tau^2 - \gamma^2)$ are real. Hence, the bracket in (8) is imaginary along bc and b'c' :

$$\text{(9)} \qquad \mathrm{Re}\,\{(\tau^2-\gamma^2)\frac{dz}{d\tau}\} = 0 \qquad \text{along bc and b'c'.}$$

The function $(\tau^2-\gamma^2)z'(\tau)$ has a pole of order 3 at $\tau = 0$, because of Eq.(7), and a simple zero at $\tau = \gamma$ since the function $z'(\tau)$ is regular in the semi-ellipse. Let us consider an auxiliary ω-plane and the mapping transformation $\tau \to \omega(\tau)$ which maps the interior of the ellipse $(\tau_R/\beta\ ,\ \tau_R)$ onto the unit circle $|\omega| \leq 1$. Then, the solution of Eqs. (7), (8), (9), is given by :

$$\text{(10)} \qquad z'(\tau) = \frac{C}{(\tau^2-\gamma^2)}\left[\frac{1}{\omega(\tau)} + \omega(\tau)\right]\left[\frac{1}{\omega^2(\tau)} - \omega^2(\gamma)\right]\left[\omega^2(\tau) - \omega^2(\gamma)\right]$$

where C is the real constant determined by Eq.(7) as :

$$\text{(11)} \qquad C = -\frac{K^2\gamma^2[\omega'(0)]^3}{\pi\omega^2(\gamma)\beta^2}$$

Eqs. (10), (11), completely solve the non-linear problem. The mapping function is explicitly known (Mikhlin, 1957) :

$$\omega(\tau) = \frac{2\tau}{(\alpha+1)\tau_R}\exp\left(Q(\tau)\right) \qquad (\alpha = 1/\beta)$$

$$\text{(12)} \qquad \begin{cases} Q(\tau) = \displaystyle\sum_{k\geq 1} (-1)^k \frac{1}{k}\,\frac{\gamma^{2k}\,\tau_R^{-4k}}{[(\alpha+1)^{2k}+(\alpha-1)^{2k}]}\,P_{2k}(\tau)\,(\alpha+1)^{-2k} \\ P_{2k}(\tau) = [\tau+\sqrt{\tau^2-\gamma^2}]^{2k} + [\tau-\sqrt{\tau^2-\gamma^2}]^{2k} \end{cases}$$

$P_{2k}(\tau)$ are polynomial of degree 2k.

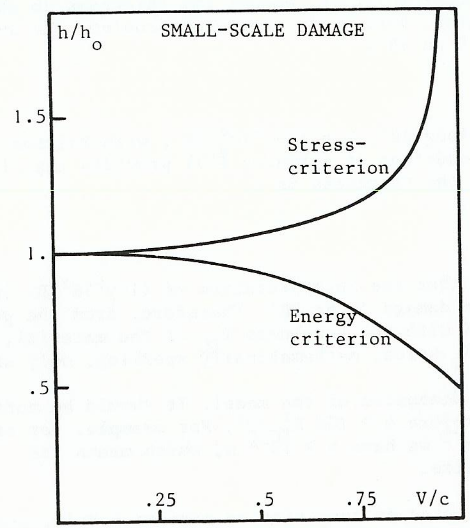

Fig. 3 Variation of h with the speed V. Stress-criterion (Constant K/τ_R), Energy-criterion (Const.$K^2/\beta\tau_R^2$).

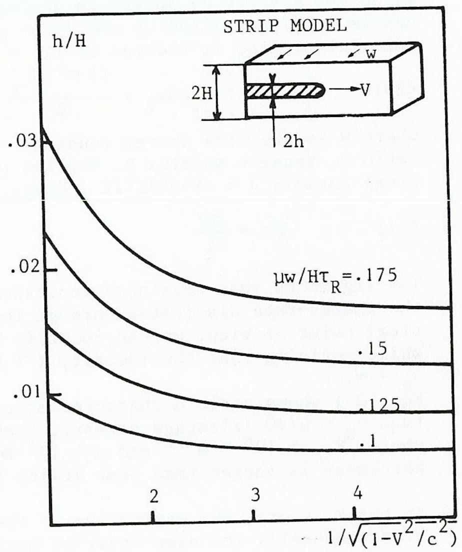

Fig. 4 Variation of the thickness h/H with the velocity V in the large scale damage model.

An interesting parameter may be the thickness 2h of the damaged zone. Taking one half of the residue of $z'(\tau)$ at $\tau = 0$, we obtain the result :

$$\text{(13)} \qquad 2h = K^2\frac{[\omega'(0)]^2}{\beta^3}\left\{1 - \omega^2(\gamma) + \frac{1}{\gamma^2[\omega'(0)]^2} - \frac{1}{\omega^2(\gamma)} - \frac{\omega'''(0)}{2[\omega'(0)]^3}\right\}$$

Figure 3 shows two plots of h versus V/c, corresponding firstly to constant K/τ_R (Stress-criterion) and secondly to constant $K^2/\beta\tau_R^2$. We remark that h decreases from $h_o = K^2/2\tau_R^2$ at V = 0 to 1/2 h_o at V = c, only in the latter case. We shall prove later that the case $K^2/\beta\tau_R^2$ = constant is closely related to the energy criterion. In the last case, we see that the damage models provide naturally a characteristic dimension, say h_o. In Fig. 5, we have plotted the damage fronts BB' for several velocities corresponding to energy criterion. They are getting thinner and thinner with increasing velocity.

Quasi-static loading.
We have $\beta = 1$, $\omega(\tau) = \tau/\tau_R$. The solution is very simple :

$$\text{(14)} \qquad z'(\tau) = -\frac{K^2}{\pi\tau^3} - \frac{K^2}{\pi\tau_R^2}\frac{1}{\tau}$$

or :

$$z(\tau) = \frac{K^2}{2\pi\tau^2} - \frac{K^2}{\pi\tau_R^2}\log\frac{\tau}{\tau_R} + \text{Real constant}.$$

We can arbitrarily fix the constant, for example to get $x_1(B) = 0$. Then, the damage front BB' has the shape of a cusped cycloid :

$$\text{(15)} \qquad \begin{cases} X_1(\theta) = \dfrac{K^2}{2\pi\tau_R^2}(\cos 2\theta + 1) & -\dfrac{\pi}{2} \leq \theta \leq \dfrac{\pi}{2} \\ X_2(\theta) = -\dfrac{K^2}{2\pi\tau_R^2}(\sin 2\theta + 2\theta) \end{cases}$$

The thickness $2h_o = K^2/\tau_R^2$ decreases to zero when $\tau_R \to \infty$ and the limiting case

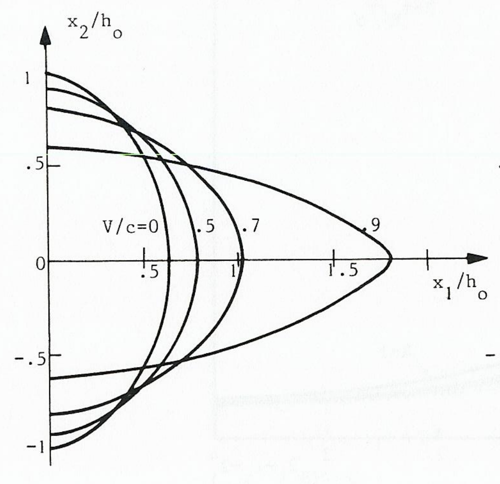

Fig. 5 Damage fronts in the small-scale damage model, with energy criterion for several velocities V/c.

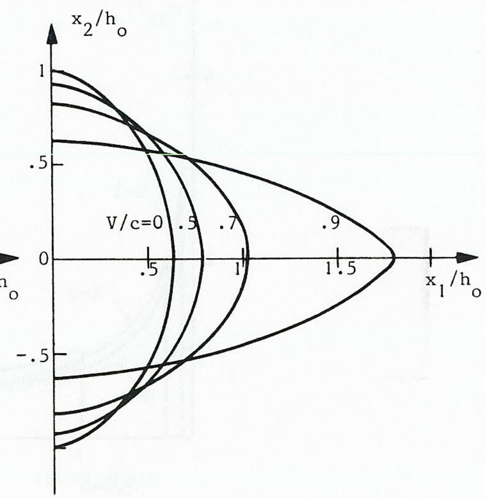

Fig. 6 Damage fronts in the large scale damage model of the infinite strip of height 2H.

$h = 0$ corresponds exactly to crack problems. The same conclusion is also true for dynamic loading, Eq. (13), when $\tau_R \to \infty$ (Perfect elasticity). The quasi-static stress σ_{32} along Ox_1 is plotted in Fig. 7, for several ratios $R = \tau_R/\tau_o$, where τ_o is the yield stress. The stress approaches pratically the value predicted by crack theory (dotted line of Fig.7) for $R \geqslant 3$. For comparison, the cleavage stress $\mu/10$ corresponds, in steel, to $R = \mu/10\tau_o \simeq 10$.

Remarks :
Stationary cracks in linear elastic body have been classically obtained by various limiting approaches, such as oblong elliptic shape hole, smooth root radius at flat notch, small-scale yielding model, (Rice, 1968, Neuber, 1961, ...) .
Our model $h \to 0$ is different in the sense that the propagation of the front BB' does not mean a removal of material. There is only an irreversible change of state, just like the cavitation in fluids dynamics, in which particles are changed into bubbles when the critical pressure of vapor is reached.

4. THE INFINITE STRIP.

The method of solution developed above is now applied to the problem of an infinite strip. Constant displacements $\pm w$ are applied to the lines $x_2 = \pm H$. Thus, the stresses at infinity are $\sigma_{32}(-\infty) = 0$ and $\sigma_{32}(+\infty) = \mu w/H$. The function $z'(\tau)$ has two distinct poles at $\tau = 0$ and $\tau = \mu w/\beta H \equiv \tau_\infty$. Taking into account the same conditions (8), (9) and the same mapping function $\omega(\tau)$, we find that :

$$(16) \qquad z'(\tau) = \frac{C}{(\tau^2-\gamma^2)}\left[\frac{1}{\omega}+\omega\right]\left[\frac{1}{\omega^2}-\omega^2(\gamma)\right]\left[\omega^2-\omega^2(\gamma)\right] \times \left[\omega^2-\omega^2(\tau_\infty)\right]^{-1}\left[\frac{1}{\omega^2}-\omega^2(\tau_\infty)\right]^{-1}$$

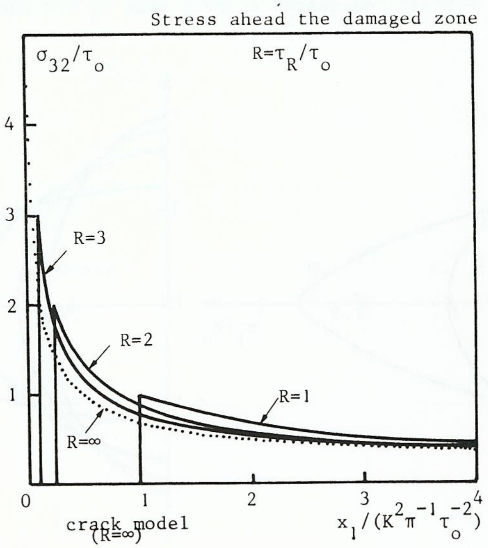

Fig. 7. Stress σ_{32} ahead the damage front. Singularity in the linear fracture Mechanics (dotted line, $R = \infty$).

In order to determine the real constant C, we consider the width 2H, i.e. the residue of $z'(\tau)$ at $\tau = \tau_\infty$. Accordingly we obtain :

$$(17) \qquad 2H = -\frac{C\,\pi}{\beta(\tau_\infty^2-\gamma^2)}\,\frac{[\omega^2(\tau_\infty)-\omega^2(\gamma)][1-\omega^2(\tau_\infty)\omega^2(\gamma)]}{\omega^2(\tau_\infty)[1-\omega^2(\tau_\infty)]\;\omega'(\tau_\infty)}$$

Finally, we integrate Eq. (16) along the semi-ellipse arc to get the damage front which is shown in Fig. 6. It is worthwhile to notice that the "large scale" model gives nearly the same picture as that obtained in the "small scale" model, Fig. 4, with the energy-criterion. The thickness 2h is determined by one half of the residue of $z'(\tau)$ at $\tau = 0$:

$$(18) \qquad \frac{h}{H} = 1 - \frac{\omega^2(\gamma)(\tau_\infty^2-\gamma^2)(1-\omega^2(\tau_\infty))\,\omega'(\tau_\infty)}{\gamma^2[\omega^2(\tau_\infty)-\omega^2(\gamma)][1-\omega^2(\tau_\infty)\omega^2(\gamma)]\,\omega'(0)}$$

The quasi-static value is $h_o = H\tau_\infty^2/\tau_R^2$, while the limit value at $V = c$ is $h(c) = \frac{1}{2}h_o$, i.e. the same formula as in section 3. Fig. 4 shows the variation of h/H with V/c for several ratios $\sigma_{32}(\infty)/\tau_R$.

5. QUASI-STATIC MODE I.

We look at a small scale model, in plane strain, in which stress and strain decrease at infinity as $O(r^{-1/2})$. In terms of the complex potentials of Muskhelishvili, Φ , Ψ , it is required that Φ' behaves, at infinity, as follows :

$$(19) \qquad \Phi'(z) \simeq \frac{K}{2\sqrt{2\pi z}} \qquad |z| \to \infty$$

The determination of the unknown boundary ∂Z is a rather difficult problem because of non-linearity of equations.
We do not attempt to derive an analytical solution, except the thickness 2h which can be easily obtained by the J-integral. Just as in the notch problem, the J-integral of Rice is reduced to ($T_i = 0$ on BB') :

$$(20) \qquad J = \int_{BB'} W dx_2 = \frac{(1-\nu^2)h\sigma_R^2}{E}$$

where W is the free energy constant along BB', $W = (1-\nu^2)\sigma_R^2/2E$, with Poisson's ratio ν, Young's modulus E. Now the condition at infinity (19) provides the classical formula $J = (1-\nu^2)K^2/E$. Hence, the thickness is :

$$(21) \qquad 2h = \frac{2K^2}{\sigma_R^2}$$

The arguments just developed indicate that the interpretation of $(1-\nu^2)K^2/E$ is the energy-rate dissipated through the damage front BB'. Therefore, from the physical point of view, we can identify K with the toughness K_{IC} of the material, while noticing that the identity $K \equiv K_I$ holds, mathematically speaking, only when $\sigma_R \to \infty$.
Eq. (21) shows again a characteristic dimension of the model. It should be noticed that $\sigma_R < \mu/10$ (cleavage stress), therefore $h > 100\ K_{IC}^2/\mu^2$. For example, for steel, where $K_{IC} \simeq 10^8\ N\ m^{3/2}$ and $\mu \simeq 10^{11} Nm^{-2}$ we have $h > 10^{-4}$ m, which means the thickness is larger than some grains size.

In crack theory, the estimation of the process zone size by various models, provides generally the same order of magnitude $\simeq K_I^2/\sigma^2$, etc... See surveys on elasto-plastic models by Mc Clintock (1971), Rice (1968). Recently, Rousselier (1978) has reached a result similar to (21), by numerical simulation of the damaged zone having a strip shaped form.
In this work, we may determine the unknown front BB' by the use of a finite element programming, combined with an iterative scheme since two conditions have to

be satisfied on BB', on the one hand, damage criterion $k = \sigma_{tt}/\sigma_R = 1$, and stress-free condition $T_i = 0$, on the other hand. The latter condition is fulfilled by the program at each step of the calculation. By modifying the curvature η along BB', increasing η if $k < 1$ and conversely, we satisfy the former condition. It is found that the iterative scheme converges very quickly. Fig. 8 shows the approximate front and the final mesh used near ∂Z. We see that the damage front is more blunted than the cycloid shaped front of mode III.

6. A PATH INDEPENDENT INTEGRAL IN ELASTODYNAMICS.

It is well known that Atkinson and Eshelby (1968), and Freund (1972) have given an integral for cracks in elastodynamics which requires a contour Γ having a zero radius around the crack tip. Another integral has been proposed by Bui (1977,1978) for cracks in elastodynamics with the use of an arbitrary contour Γ . The path-independent J-integral for cracks can be extended to the damage model ($h \neq 0$) as follows :

$$(22)\quad J = \int_{\Gamma} \{Wn_1 - \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i n_1 - \sigma_{ij}n_j u_{i,1} - \rho\dot{u}_i u_{i,1} V n_1\} ds + \frac{d}{dt}\int_{A(\Gamma)} \rho\dot{u}_i u_{i,1} dv$$

where Γ is a contour the end points of which are on the straight lines BC, B'C', and $A(\Gamma)$ is the area bounded by Γ and ∂Z. The proof of the path independency property of Bui (1978) can be extended, word for word, to the present model. Let us consider now the contour Γ on the boundary ∂Z. The area-integral of Eq. (22) vanishes. Taking into account $\dot{u}_i = -Vu_{i,1}$, $\sigma_{ij}n_j = 0$, we obtain :

$$(23)\quad J = \int_{BB'} [Wn_1 + \frac{1}{2}\rho\dot{u}_i\dot{u}_i n_1] ds$$

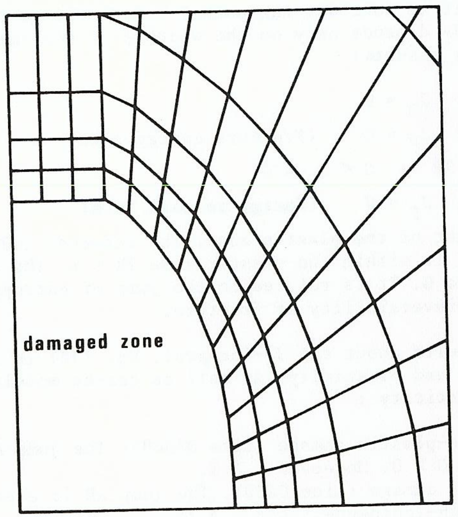

Fig. 8. Damage front in mode I by the use of a finite element method.

In mode III, Eq. (23) can be written in a more convenient form :

$$(24)\quad J = \frac{\beta}{2\mu}\, \mathrm{Im} \int_{bb'} \tau^2 z'(\tau)\, d\tau$$

when adding the lines bc and b'c' to the path bb', the value of the J-integral does not change, since $\tau^2 dz$ is real on bc and b'c'. This allows the calculation of J by residue technique.In view of Eqs. (10), (16), we obtain $J = K^2/2\mu\beta$ for the small-scale model, and $J = \mu w^2/H$ for the strip model. These results explain why the damage fronts presented in Fig. 4 and 5 have nearly the same form, since they are associated with the constant flux of energy into BB'.

The analysis of the "generalized force" acting on the damaged zone, by means of Eqs. (22) to (24), is very similar to the result already known for flat notches, Rice (1968), Eshelby (1969). As pointed out by Eshelby (1969), the weak point of the "generalized force" theory, when applied to notches, is that the motion of a notch is meaningless, since the removal of material is quite artificial. Such a difficulty disappears in a damage theory in which there is only a change of state of the material behind the front BB'.
Let us notice that in Eq. (23) the density W is constant along BB'. The kinetic term $\frac{1}{2}\rho\dot{u}_i\dot{u}_i$ is expected to increase with the velocity V. Therefore, a constant flux of energy dissipated through BB' implies a decrease of the thickness 2h with increasing velocity. This point has been confirmed by exact solutions of mode III. The limiting value $h/h_o = 1/2$ means that the equality

$$\int_{BB'} \frac{1}{2}\rho\dot{u}_i\dot{u}_i dx_2 = \int_{BB'} W dx_2 \text{ holds for the limit velocity } V = c. \quad (25)$$

7. DUCTILE FRACTURE AND DAMAGE.

We have just developed very simple models of damage in order to get some close form solutions. It would be interesting to have similar analyses for more complex behaviour, for example elastic-perfectly plastic material. A reasonable assumption on the damage state would be that it occurs when a finite value of strain ε_R has been reached, Fig. 1.c. Of course, it is out of question to derive analytical solutions for such a model. Nevertheless, we expect that some general features of the elastic-damage models, concerning characteristic dimension, finite stress and strain, etc..., are still valid, while the situation is more complex due to plasticity. These points were confirmed by an explicit elasto-plastic solution recently obtained by Bui (1980 b) for a stationary damaged zone in the quasi-static mode III, see Appendix.

In what follows, we derive some general results on energy rates and their associated integrals for moving damaged zone.
For isothermal loading condition, the energy-release rate VG per unit time is defined by :

$$(26)\quad VG = P_{ext} - \dot{U} - \dot{K}$$

where P_{ext} is the power of external loads, $U = \int_\Omega W(\varepsilon^e)dv$ is the internal energy, $W(\varepsilon^e) = W(\varepsilon - \varepsilon^p)$ is the free energy density and K is the kinetic energy $K = \frac{1}{2}\int_\Omega \rho\dot{u}_i\dot{u}_i dv$.

The isothermal loading assumption has been relaxed in recent works of Bui, Ehrlacher, Nguyen (1979) who gave the analysis of crack propagation in general materials by coupled equations between the mechanical fields u, ε, σ...and the temperature field T. The energy release rate in elastic material was interpreted as a moving heat source. In thermo-elasto dynamics, the temperature singularity was found to be logarithmic $T \simeq - \mathcal{G}V\log(r/d)/2k\pi$. Such a weak singularity does not change the dominant singularities of the mechanical field.

As already known in crack theory, see also NGUYEN (1980), the dissipative rate $V\mathcal{G}$ consists of two parts, plastic work rate $D_p = \int_\Omega \sigma.\dot{\varepsilon}^p dv$ and energy rate VG consumed by fracture at the tip, if Ω is a cracked body. For damage models, we can see that the same interpretation holds, $V\mathcal{G} = D_p + VG$, where G is the fracture energy rate with respect to the damaged zone length.

Let us recall some results already known for "perfect" materials, i.e., the materials which have some infinite characteristic. For "perfect" elastic body, i.e. elastic material ($\varepsilon^p = 0$) having infinite strength ($\sigma_o = \infty$), one has $D_p = 0$ and $\mathcal{G} = G$, so making a distinction between the symbols $\mathcal{G}$ and G is superfluous indeed. The quantities $\mathcal{G}$, G are equivalent to the path-independent J-integral Eq. (22). A quite different situation appears in "perfect" plasticity, in a double sense that σ_o = constant, and that $\varepsilon_R = \infty$. The material has an infinite ductility. After Rice (1966), the fracture energy rate consumed at the crack tip is $G = 0$, see also Nguyen (1980), and thus the energy release rate equals the plastic work rate $V\mathcal{G} = D_p$.

Let us consider the damage models for establishing the balance equation of energy. The dynamic equation, in integral form over $\Omega(t)$, is written as follows :

$$(27) \qquad \int_\Omega \rho\dot{u}.\ddot{u}\, dv + \int_\Omega \sigma.\dot{\varepsilon}^e\, dv + \int_\Omega \sigma.\dot{\varepsilon}^p dv = \int_{\partial\Omega} T.\dot{u}\, dv \equiv P_{ext}$$

where $\dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p = \frac{1}{2}(\nabla\dot{u} + \nabla'\dot{u})$. Since $T_i = 0$ on ∂Z, the integral over $\partial\Omega$, i.e. over the outer contour S of the body is nothing but the power of external loads P_{ext}. We introduce the convected differentiation of integrals $\mathcal{U}$, $\mathcal{K}$ over time-dependent domain $\Omega(t)$. Then, assuming linear elasticity $\sigma.\dot{\varepsilon}^e = \dot{W}(\varepsilon^e)$, the Eq.(27) is re-written as :

$$(28) \qquad \frac{d}{dt}\int_\Omega \frac{1}{2}\rho\dot{u}_i\dot{u}_i dv + \frac{d}{dt}\int_\Omega W(\varepsilon^e)dv + D_p = P_{ext} + \int_{\partial\Omega} \{W(\varepsilon^e) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\}\vec{V}.\vec{\nu}\, ds$$

where $\vec{\nu} = -\vec{n}$ is the outward normal to $\partial\Omega$, $V_i = 0$ on S, $\vec{V}.\vec{n} = 0$ on B'C' and BC, and $\vec{V}.\vec{n} = Vn_1$ on BB'. Thus comparing Eqs. (28) and (26), we obtain :

$$(29) \qquad V\mathcal{G} = D_p + \int_{BB'} \{W(\varepsilon^e) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\}Vn_1\, ds$$

Hence it is proved that $V\mathcal{G}$ also consists of two parts, plastic rate D_p and fracture energy rate VG through the front BB'. The rate G with respect to unit length is :

$$(30) \qquad G = \int_{BB'} \{W(\varepsilon^e) + \frac{1}{2}\rho\dot{u}_i\dot{u}_i\}n_1 ds$$

We notice the similarity between Eq.(23) and Eq.(30). In fact they are the same because $W(\varepsilon^e)$ is defined as the elastic strain energy density. In elastic-damage model the J-integral, Eq.(22) (equivalent to Eq. (30)) is path-independent. In Eq. (30), the constant critical state $\sigma_{tt} = \sigma_o$ along BB' implies that $W(\varepsilon^e)$ is constant along the front too, $W(\varepsilon^e) = (1 - \nu^2)\sigma_o^2/2E$ in plane strain mode I.

Let us prove that the thickness is non zero. Suppose $h = 0$, then, because of $T_i = 0$ on the lines B'C', BC, we have a crack in the body for which the strain should be infinite at the tip $B \equiv B'$, see Rice (1968). Thus, the assumption of a finite value of the strain ε_R along BB' implies necessarily $h > 0$, and consequently, from Eq. (30), we get the result $G > 0$, which is proved to be valid for

quasi-static as well as for dynamic loading.

In the quasi-static case, the formula for G is :

$$(31) \qquad G = (1 - \nu^2)\sigma_o^2 h / E \qquad \text{(Plane strain, mode I)}$$

where h is one half of the damaged zone thickness. We cannot determine the characteristic dimension h by energy considerations because of plasticity. Numerical analysis is the only way to know how far h may depend on σ_o, ε_R and the loading condition. The following result is expected : $h \to 0$ when $\varepsilon_R \to \infty$ (crack model).

Our result $G > 0$ obtained in damage theory differs fundamentally from that of Rice in crack theory, $G = 0$ (Rice, 1966). In fact, there is no contradiction at all between damage and crack models, since we have interpreted the crack in a body as the limiting case when $h \to 0$ ($\varepsilon_R \to \infty$) so that $G \to 0$. A comparison between the model of damage and the crack model will be made later by considering the J-integral. For the sake of simplicity, let us confine ourself to the quasi-static loading for examining the J-integral in plasticity.

There are many definitions of the J-integral in plasticity, according to the choice of the density W. A path-independent integral was obtained by Rice with the definition $W(\varepsilon) = \int^\varepsilon \sigma.d\varepsilon$ and the assumption of monotonic loading. Such an integral is valid for a stationary crack, not for a propagating crack. Another choice is the elastic strain energy $W(\varepsilon - \varepsilon^p)$, also see Nguyen Q.S. (1980). Because of the term ε^p, the integral

$$(32) \qquad J_\Gamma = \int_\Gamma \{W(\varepsilon - \varepsilon^p)n_1 - T_i u_{i,1}\}ds$$

is path-dependent. If we consider the path Γ homothetic to the contour ∂Z of the damaged zone, then J_Γ depends only on the width ϕ of the contour Γ. Evidently, we obtain the following results:

$\phi < 2h$: $J_\Gamma = 0$

$\phi = 2h$: $J_\Gamma = G$ (Fracture energy rate)

$2h < \phi < 2R$: $G < J_\Gamma < \mathcal{G}$

$2R < \phi$: $J_\Gamma = \mathcal{G}$ (Energy release rate)

where $2R$ is the height of the plastic zone. The integral (32) is independent of the path, whenever Γ is within the elastic zone $2R < \phi$. The discontinuity of J_Γ at $\phi = 2h$ is equal to G. It is related to the jump of entropy through ∂Z which characterizes the irreversibility of fracture.

We summarize the results about the J_Γ-integral, Eq. (32) in Fig. 9, for damage models in plasticity and elasticity, as well as cracks models in perfect plasticity and perfect elasticity :

1. Elastic-plastic damage (Line OABCD). The jump AB characterizes the fracture energy rate $G > 0$. Moreover $\mathcal{G} \geq G$.
2. Elastic damage (Line OAED). The jump AE is equal to $G = \mathcal{G} = J$. The J-integral is path-independent for $\phi > 2h$.
3. Crack in perfect plasticity (Curve OBCD). One has $V\mathcal{G} = Dp$, $G = 0$ and $J(\Gamma \to 0) = 0$.
4. Crack in elasticity (Line OFD). $G = \mathcal{G} = J$ (for all paths).

Finally, our interpretation of the integral (32) differs from that of Eshelby (1968), Roche (1975). The important feature of the general model of elastic-plastic damage is the jump AB. We notice that the balance equation (29) in the static case can be written as $VJ_\Gamma = D_p + VJ_{BB'}$ for large contour Γ ($2R<\phi$).

DISCUSSION ON THE RESULT $G > 0$.

The relationship between the damage theory and the crack theory has been established by a limiting procedure, when $h \to 0$. The result $G \to 0$ for a crack in perfect plasticity coincides with the paradox of Rice $G^\Delta \to 0$, when $\Delta a \to 0$. Kfouri and Rice (1977) proposed a finite growth step $\Delta a \neq 0$ ahead the crack tip, which has been considered as a characteristic of the material, and they obtained by this way a non zero crack-separation energy $G^\Delta = \Delta W / \Delta a$ ($-\Delta W$ being the work of unloading the stress in the segment Δa prior to rupture). Although the models are different, their curve G^Δ versus Δa is similar to our curve OBCD for the J_Γ-integral versus ϕ (crack in perfect plasticity). In damage theory, we obtain a finite fracture energy rate $G > 0$, Eq. (30) by classical balance equation without a priori assumption on a characteristic dimension. The fracture energy rate G turns out to be proportional to the thickness of the damaged zone, h (σ_o, ε_R loading conditions)... which is entirely determined by the material characteristics, and the loading conditions. For an elastic-plastic body in small-scale yielding, the elastic field is governed by the K_I-factor (symmetric loading), i.e. $\mathcal{G} = (1 - \nu^2)K_I^2/E$. The plastic zone size R is proportional to $R \simeq (K_I^2/\sigma^2) \simeq \mathcal{G}$. Then, the ratio $\mathcal{G}/G$ is roughly proportional to R/h, and appears to be an important parameter.

8. CONCLUDING REMARKS

A simple model of fracture by the propagation of damage has been presented to elastic material or plastic material. Perhaps the elastic models of mode III have little importance for engineering problems. However, the close form of solutions are very helpful in understanding some general features of fracture. The theory predicts a characteristic dimension h which depends on the fracture stress, the velocity and the loading condition. It is exactly a characteristic of the structure, not of the material. The damaged zone has a non-zero thickness, and it is getting thinner and sharper with increasing velocity. An interesting point is that we obtain the linear fracture mechanics when the fracture stress is (mathematically) infinitely large. However, it is sufficient to consider a fracture stress higher than 3 or 4 times the yield stress to get classical results. The model does not involve any singularities of the mechanical fields. Stress and strain are always finite, in elastic-damage model and in elastic-plastic damage models.

The model of damage in perfect plasticity is discussed with the basic assumption that the material cannot undergo certain limit strain ε_R (octahedral strain). The theory predicts again a characteristic dimension $h > 0$ and consequently a positive fracture energy rate $G > 0$. The last result is new, compared to the crack theory in perfect plasticity.

A qualitative analysis of the dissipative energy rates has been presented. It clearly shows the necessity to make a distinction between the symbols $\mathcal{G}$ (energy-release rate available from the whole body) and G (fracture energy rate for separation energy rate or energy release rate near the process zone). According to the path Γ, the J_Γ-integral, Eq. (32), can be related to either $\mathcal{G}$ or G. In the future, it would be interesting to have quantitative numerical analyses of the dissipation rates, in plasticity and in fracture by damage. The important quantity to be determined is the thickness h which is directly related to the fracture energy rate G, Eq. (31).

Progress in this topic would be helpful for a better understanding of ductile fracture.

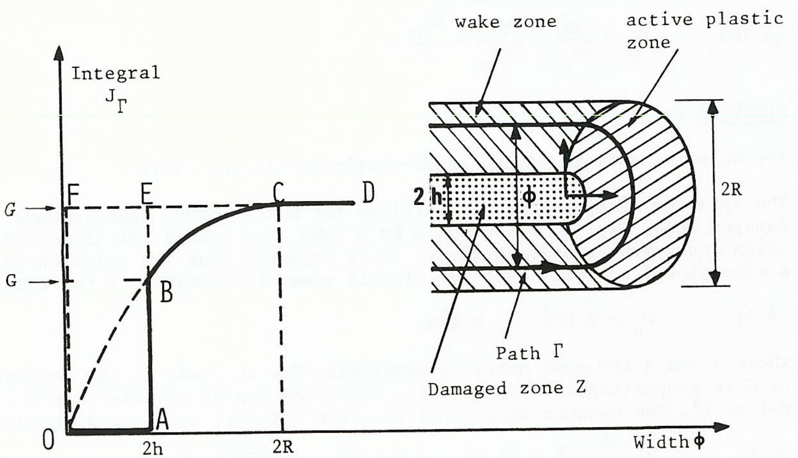

Fig. 9 Variation of the path-integral J with the width of the contour, for different models, elastic-plastic-damage OABCD, elastic-damage OAED, crack in perfect plasticity OBCD, crack in elasticity OFD. The characteristic values are G (fracture energy rate) and $\mathcal{G}$ (energy-release rate).

APPENDIX A

An explicit solution of the quasi-static problem of damage in elastic-plastic solid under antiplane shear.

Statement of the problem.

Consider a strip shaped damaged zone Z, of thickness 2h, the front of which is the cusped cycloïd BB', Eq. (14). This model can idealize a fatigue crack which has been propagating at a very low stress level, say $K_o = \tau_o\sqrt{2h}$, so that the undamaged material is still elastic. Now, assume that the damaged zone is not propagating and that a large monotonic load is applied to the body, the behaviour of which is the elastic-plastic-damage law, with the yield stress τ_o, and the limit strain ε_R. Determine the unknown boundary of the elastic and plastic zones and the limit load K_R.

Along the previous front BB', one requires that the stress-vector is null, the yield stress is reached ($K_o \leqslant K$), the strain ε is constant, $|\varepsilon| < \varepsilon_R$ for $K < K_R$ and $|\varepsilon| = \varepsilon_R$ for the limit load K_R corresponding to plastic damage.

At first sight, it seems that too much boundary conditions have to be met along BB'. Fortunately, the problem was completely solved, i.e. a complete solution was found by Bui (1980 b),satisfying both static and kinematic conditions. In what follows, we write down the solution without proof.

Solution.

The stress and displacement fields in the elastic zone are respectively :

(A.1) $$z = \frac{K^2}{2\pi\tau_o^2}\left(\frac{\tau_o^2}{\tau^2} + 1\right) - \frac{2h}{\pi}\log\frac{\tau}{\tau_o}$$

(A.2) $$w = \frac{1}{\mu}\,\mathrm{Im}\left\{\frac{K^2}{\pi}\left(\frac{1}{\tau} - \frac{1}{\tau_o}\right) - \frac{2h}{\pi}(\tau - \tau_o)\right\}$$

where τ lies within the semi-circle $|\tau| \leqslant 1$, $-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$. Thus, the elastic-plastic boundary is searched in the following form $(\phi = -\theta)$:

$$\text{(A.3)} \qquad \begin{cases} M_1 = \dfrac{K^2}{2\pi\tau_o^2}(\cos 2\phi + 1) \\[2ex] M_2 = \dfrac{K^2}{2\pi\tau_o^2}\sin 2\phi + 2\dfrac{h}{\pi}\phi \end{cases}$$

The curve (A.3) is a <u>curled cycloid</u>, fig. (10).

In the plastic zone (plastic flaw rule $\dot{\varepsilon}^p_{3i} = \lambda\,\sigma_{3i}$) the family of characteristics $\vec{\alpha}$ consists of the straight lines $\vec{\alpha}(\cos\phi\ ,\sin\phi)$, the center of rotation of which is the variable point ω :

$$\text{(A.4)} \qquad \begin{cases} x_\omega = -\dfrac{2h}{\pi}\cos^2\phi \\[2ex] y_\omega = \dfrac{2h}{\pi}\phi - \dfrac{h}{\pi}\sin 2\phi \end{cases}$$

The β-lines contain a particular line which is precisely the <u>cusped cycloid</u> BB' :

$$\text{(A.5)} \qquad \begin{cases} N_1 = \dfrac{h}{\pi}(\cos 2\phi + 1) \\[2ex] N_2 = \dfrac{h}{\pi}(\sin 2\phi + 2\phi) \end{cases}$$

The stresses are $\sigma_{3\beta} = \tau_o$, $\sigma_{3\alpha} = 0$. The strain $\varepsilon_{3\beta}$ is found to be constant along the β-line BB' :

$$\text{(A.6)} \qquad \varepsilon_{3\beta}(N) = \frac{1}{8h\tau_o\mu}\left[K^2 + 2h\tau_o^2\right]$$

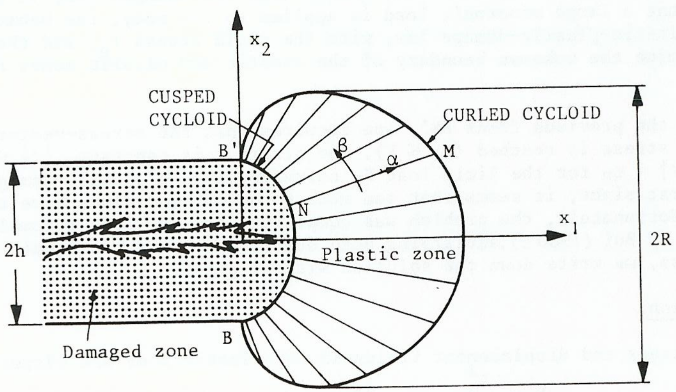

Fig. 10 Elastic, plastic and damaged zone in mode III from Bui's solution. The front is a cusped cycloid and the elastic-plastic boundary is a curled cycloid.

The displacement field is constant along the α-lines :

$$\text{(A.7)} \qquad w = w(M) = \frac{1}{\mu\pi}\left\{\frac{K^2}{\tau_o} + 2h\tau_o\right\}\sin\phi$$

The solution given by Eqs. (A.1) to (A.7) is valid when $\varepsilon_o = \frac{\tau_o}{2\mu} \leqslant \varepsilon_{3\beta} \leqslant \varepsilon_R$, i.e.:

$$K_o^2 \leqslant K^2 \leqslant 2h\tau_o\,(4\mu\varepsilon_R - \tau_o)\ .$$

Hence the limit load K_R is related to the material characteristics and the thickness h by the simple formula :

$$\text{(A.8)} \qquad K_R = \sqrt{2h\tau_o\,(4\mu\varepsilon_R - \tau_o)}$$

or equivalently :

$$K_R^2 = 2\,h\,\tau_o^2\left(2\frac{\varepsilon_R}{\varepsilon_o} - 1\right).$$

(It is very tempting to anticipate that a similar formula holds for mode I :

$$K_R^2 = h\,\sigma_o^2\left(2\frac{\varepsilon_R}{\varepsilon_o} - 1\right)\ ,\ \varepsilon_o = \frac{\sigma_o}{E}$$

and $K_o^2 = h\,\sigma_o^2$).

Physically, the finiteness of the limit load K_R implies that $h \to 0$ when $\varepsilon_R \to \infty$. The limiting result is nothing but the crack problem studied by Hult and Mc Clintock (1956), Rice (1968), for which the elastic-plastic boundary is a circle, see Eq. (A.3), h = 0. Crack model in plasticity is not compatible with an assumption $\varepsilon_R \neq \infty$. For the general case, $\varepsilon_R \neq \infty$, $h \neq 0$. The plastic zone height R is found to be :

$$\text{(A.9)} \qquad R = \frac{K^2}{2\pi\tau_o^2}\sin\left\{\operatorname{arc\,cos}\left(-\frac{2h\tau_o^2}{K^2}\right)\right\} + \frac{h}{\pi}\operatorname{arc\,cos}\left(-\frac{2h\tau_o^2}{K^2}\right)$$

The plastic dissipation rate is :

$$\text{(A.10)} \qquad \mathcal{D}_p = \frac{2K\dot{K}}{\mu\tau_o}\left(\frac{K^2}{2\pi\tau_o} - \frac{h\tau_o}{\pi}\right)$$

<u>APPENDIX B</u>

Approximate analysis of the quasi-static resistance curve.

The solution of appendix A is obtained for a stationary damaged zone. For a moving damaged zone, Fig. 9, there would be a wake zone behind the active plastic zone which changes the result. However, it is expected from the solution (A.10) and from a dimensional analysis that the plastic power is probably of the following form

$$\text{(B.1)} \qquad \mathcal{D}_p = A\,K^3\dot{K} - B\,K\dot{K}h$$

where A and B are some material constants. For the mode I, the fracture energy rate G is proportional to h, Eq.(31), while the energy release rate $\mathcal{G}$ is proportional to K^2. The balance equation (29), where V=da/dt, can be rewritten as

$$\text{(B.2)} \qquad \frac{(1-\nu^2)}{E}K^2 = \{A\,K^3 - B\,Kh\}\frac{dK}{da} + \frac{(1-\nu^2)}{E}\sigma_0^2 h$$

The bracket of the right hand side of Eq.(B.2) must be a positive quantity ($\mathcal{D}_p > 0$):

$$\text{(B.3)} \qquad K^2 \geqslant Bh/A$$

a) Tearing modulus model.

Let us assume that $B/A=\sigma_0^2$. Then the differential equation (B.2) gives

$$(B.4)\qquad (K^2 - h\sigma_0^2)\{\frac{2(1-\nu^2)}{EA} - \frac{dK^2}{da}\} = 0$$

From (B.3) the first bracket is positive. Thus we have :

$$(B.5)\qquad \frac{dK^2}{da} = \frac{2(1-\nu^2)}{EA} \qquad (K^2 > h\sigma_0^2)$$

This is a linear dependence between K^2 and $\Delta a=a-a_0$ beyond the threshold value K_0^2 = (Bh/A), the tearing modulus depending only on the plastic power analysis.

b) General model.

A more complex model can be derived from Eq.(B.2) by introducing a weaker assumption that the damaged zone thickness increases during stable growth. For example, let us assume a linear dependence on K^2

$$(B.6)\qquad h = \alpha + \beta K^2$$

The condition $\mathcal{D}_p>0$ implies a threshold value of K^2, namely

$$(B.7)\qquad K_1^2 = \frac{B\alpha}{A-B\beta}$$

Thus, Eq. (B.2) may be rewritten as :

$$(B.8)\qquad \frac{dK^2}{da} = 2(\frac{1-\nu^2}{E})(\frac{1-\beta\sigma_0^2}{A-B\beta})(\frac{K^2-K_2^2}{K^2-K_1^2})$$

where

$$K_2^2 = \frac{\sigma_0^2\alpha}{1-\beta\sigma_0^2}$$

From Eq.(B.8) we derive the R-curve of the a-K^2 plane :

$$(B.9)\qquad a-a_0 = \frac{E(A-B\beta)}{2(1-\nu^2)(1-\beta\sigma_0^2)}[K^2 - K_1^2 + (K_2^2-K_1^2)\log\frac{K_2^2-K^2}{K_2^2-K_1^2}]$$

which increases monotonically from K_1^2 to infinity. We remark that $K_2\leq K_1$, or equivalently $\sigma_0^2\leq B/A$. Otherwise, there would be a load K^2 such that $Bh/A<K^2<\sigma_0^2 h$, for which the plastic power is positive, dK/da>0, $\dot{a}$>0, the balance equation (B.2) is not satisfied.

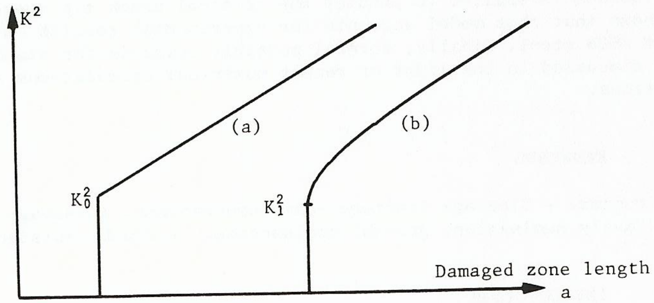

Fig. 11 Resistance curves : (a) Tearing modulus model
(b) General model with increasing h.

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