ELASTIC-PLASTIC FIELDS IN STEADY CRACK GROWTH IN A STRAIN-HARDENING MATERIAL

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ABSTRACT

For plane strain problem, the history-dependent constitutive relations and the plastic loading condition are given. The type of two-dimensional elastic-plastic equations and the existence of the inner boundary layer are discussed. For the elastic-plastic evolutionary problems with moving boundary, the four contiguity conditions are given in very simple forms. The theorem for the unloading boundary is proved. For steady-growth crack. a fifth-order partial differential equation and five contiguity conditions are derived. The asymptotic equations of predominant terms are obtained near the crack tip. The analytical solution of the asymptotic equations is obtained. The results of this paper show that for power-hardening material, if the hardening exponent is n, the singularities of stresses and strains are:

$$\sigma \sim \left(\ln \frac{A}{r}\right)^{1/(n-1)}, \qquad \varepsilon \sim \left(\ln \frac{A}{r}\right)^{n/(n-1)}$$

KEYWORDS

Steady crack growth; strain-hardening material; power-hardening material; near-tip elastic-plastic field; contiguity condition; theorem for the unloading boundary; inner boundary layer.

INTRODUCTION

The elastic-plastic field at a crack tip is one of the central problems in fracture mechanics. In different loading stages the field possesses different characters. The most important stage is that the crack grows steadily under quasi-static constant loading. In this case, the Mode III crack problem was studied by Chitaley and McClintock (1971) by restricting in elastic perfectly-plastic theory. For the same case, the Mode I crack problem was studied by Gao Yu-chen (1980), also restricting in elastic perfectly-plastic theory. As for the strain-hardening material, Amazigo and Hutchinson (1977) have studied the problems of Mode I and Mode III crack, but only for the linear strain-hardening material (n=1). When the hardening exponent n>1, the problem is essentially different from the case of n=1. For the case of n=1, in the asymptotic solution near the crack tip, the plastic strains and elastic strains possess the same order of singularity. Therefore, when n=1, the compatibility equation is elliptic. After separating variables, it becomes an ordinary differential equation wihtout any singular point. For the case of n>1, in the asymptotic solution, the plastic strains possess higher order of singularity than the elastic. Therefore, when n>1, the compatibility equation is hyperbolic asymptotically. Hence some inner boundary layers may exist inside the plastic domain. After separating variables, the asymptotic equations become ordinary differential equations with some singular properties. The contiguity conditions for different domains are formulated and the main features of the solution for n>1 are obtained.

1. THREE DIMENSTIONAL EQUATIONS

1.1 Constitutive Relations

Denoting by σ_{ij}, ε_{ij}, ε^{*e}_{ij}, ε^{*p}_{ij} the tensors of stress, strain, elastic strain and plastic strain, and by g_{ij} the metric tensor, we have the following relations for strain-hardening material:

$$\varepsilon_{ij}=\varepsilon^{*e}_{ij}+\varepsilon^{*p}_{ij} \tag{1.1}$$

$$\varepsilon^{*e}_{ij}=\frac{1+\nu}{E}-\frac{\nu}{E}\sigma^k_k g_{ij} \tag{1.2}$$

$$\dot{\varepsilon}^{*p}_{ij}=\lambda S_{ij}, \qquad S_{ij}=\sigma_{ij}-\frac{1}{3}\sigma^k_k g_{ij} \tag{1.3}$$

$$\lambda=\mu h(\sigma)\dot{\sigma} \tag{1.4}$$

$$\mu=\begin{cases}1, & \text{for plastic loading}\\ 0, & \text{for unloading}\end{cases} \tag{1.5}$$

$$\sigma=\left(\frac{3}{2}S_{ij}S^{ij}\right)^{1/2} \tag{1.6}$$

The dot denotes time-derivative, and $h(\sigma)$ is a function depending on the material. For power-hardening material, we have

$$h(\sigma)=\frac{3nc}{2\sigma}(\sigma-\sigma_0)^{n-1} \tag{1.7}$$

where σ_0 is the initial yield stress, n denotes the hardening exponent and c is a material constant. It follows from (1.7) for uni-axial tension

$$\varepsilon=\begin{cases}\dfrac{\sigma}{E} & \text{for } \sigma\le\sigma_0\\[2mm] \dfrac{\sigma}{E}+c(\sigma-\sigma_0)^n & \text{for } \sigma>\sigma_0\end{cases} \tag{1.8}$$

1.2 Type of Equations

Combining (1.1)-(1.3), we obtain

$$\dot{\varepsilon}_{ij}=\left(\frac{1+\nu}{E}g_{ik}g_{jl}-\frac{\nu}{E}g_{ij}g_{kl}+\frac{3h}{2\sigma}S_{ij}S_{kl}\right)\dot{\sigma}^{kl} \tag{1.9}$$

Besides, we have the equations of equilibrium

$$\nabla_i\dot{\sigma}^{ij}=0 \tag{1.10}$$

and the compatibility equations

$$\nabla_i\nabla_j\dot{\varepsilon}_{kl}+\nabla_k\nabla_l\dot{\varepsilon}_{ij}-\nabla_i\nabla_k\dot{\varepsilon}_{jl}-\nabla_j\nabla_l\dot{\varepsilon}_{ik}=0 \tag{1.11}$$

In order to ascertain the type of the set of equations (1.9)-(1.11), we investigate the conditions under which the second-order derivatives $\dot{\sigma}_{ij}$ with respect to coordinates can be discontinuous. Without loss of generality, we may take local Cartesian coordinates x,y,z at the point under consideration, and take the plane x=0 as plane of discontinuity. From eqs.(1.9),(1.10) and the three independent equations of (1.11) containing second derivatives with respect to x, we obtain

$$A\cdot\left(\frac{\partial^2\dot{\sigma}_{ij}}{\partial x^2}\right)+\cdots=0 \tag{1.12}$$

where... denotes terms involving derivatives of $\dot{\sigma}_{ij}$ with respect to x of order less than 2, and A is 6X6 matrix. The condition for discontinuity across the plane x=0 will be det $\|A\|$=0, or

$$\frac{1-\nu^2}{E}+\frac{3h}{2\sigma}\left\{S_y^2+S_z^2+2\nu S_yS_z+2(1-\nu)S_{yz}^2\right\}=0 \tag{1.13}$$

Eq.(1.13) holds true only when the elastic deformation can be neglected and

$$\sigma_{yz}=S_y=S_z=0 \tag{1.14}$$

But eqs.(1.14) generally cannot be satisfied on a surface or in a region. Hence for strain-hardening materials, the basic equations of the elastic-plastic incremental theory are generally elliptic. They can be hyperbolic only in extremely exceptional cases.

2. PLANE STRAIN PROBLEMS

2.1 Basic Equations

For plane strain problems we can obtain by the use of the condition $\varepsilon_z=0$

$$\sigma_z=\epsilon\sigma_p+\nu\sigma^\alpha_\alpha \qquad (\alpha=1,2) \tag{2.1}$$

where

$$\sigma_p=\frac{2E}{3}\exp\left(-\frac{2E}{3}H\right)\int_{t_0}^{t}\lambda\,\sigma^\alpha_\alpha\exp\left(\frac{2E}{3}H\right)dt \tag{2.2}$$

$$H=\int_{t_0}^{t}\lambda\,dt, \qquad \lambda=\mu h(\sigma)\dot{\sigma} \tag{2.3}$$

$$\sigma=\sqrt{\frac{3}{2}}\left\{\sigma^{\alpha\beta}\sigma_{\alpha\beta}-\frac{1}{2}\sigma^\alpha_\alpha\sigma^\beta_\beta+\frac{2}{3}\epsilon^2(\sigma^\alpha_\alpha-\sigma_p)^2\right\}^{1/2} \tag{2.4}$$

$$\epsilon=\frac{1}{2}-\nu \tag{2.5}$$

and t_0 denotes the time when the material point begins to yield. Moreover, it follows from (1.1)-(1.3)

$$\varepsilon_{\alpha\beta}=\varepsilon^e_{\alpha\beta}+\varepsilon^p_{\alpha\beta} \tag{2.6}$$

$$\varepsilon^e_{\alpha\beta}=\frac{1+\nu}{E}\sigma_{\alpha\beta}-\frac{\nu}{E}(1+\nu)\sigma^\gamma_\gamma g_{\alpha\beta} \tag{2.7}$$

$$\varepsilon^p_{\alpha\beta}=\frac{\epsilon^2}{E}\sigma_p g_{\alpha\beta}+\int_{t_0}^{t}\lambda\sigma_{\alpha\beta}dt-\frac{1}{2}g_{\alpha\beta}\int_{t_0}^{t}\lambda\sigma^\gamma_\gamma dt \tag{2.8}$$

Here $\varepsilon^e_{\alpha\beta}$ and $\varepsilon^p_{\alpha\beta}$, which are introduced formally, are different from $\varepsilon^{*e}_{\alpha\beta}$ and $\varepsilon^{*p}_{\alpha\beta}$.

Denote by φ the stress function, and then we have

$$\sigma^{\alpha\beta}=C^{\alpha\gamma}C^{\beta\delta}\nabla_\gamma\nabla_\delta\varphi \tag{2.9}$$

where $C^{\alpha\beta}=(\beta-\alpha)/\sqrt{g}$, g=det$\|g_{\alpha\beta}\|$, and ∇_α denotes the covariant derivative. The compatiblity equation will be

$$\frac{1-\nu^2}{E}\Delta\Delta\varphi+C^{\alpha\gamma}C^{\beta\delta}\nabla_\alpha\nabla_\beta\varepsilon^p_{\gamma\delta}=0 \tag{2.10}$$

or

$$\frac{1-\nu^2}{E}\Delta\Delta\dot{\varphi}+C^{\alpha\gamma}C^{\beta\delta}\nabla_\alpha\nabla_\beta\dot{\varepsilon}^p_{\gamma\delta}=0 \tag{2.11}$$

where $\Delta=\nabla_\alpha\nabla^\alpha$ is the Laplacian operator.

2.2 Type of Equations

From (2.2) follows

$$\dot{\sigma}_p = \frac{2}{3} E\lambda (\Delta\varphi - \sigma_p) \tag{2.12}$$

and from (2.4) follows

$$\dot{\sigma} = \frac{3}{2\sigma}\left\{\sigma_{\alpha\beta}\dot{\sigma}^{\alpha\beta} - \frac{1}{2}\dot{\sigma}^{\alpha}_{\alpha}\left[\sigma^{\beta}_{\beta} - \frac{4}{3}\epsilon^2(\sigma^{\beta}_{\beta} - \sigma_p)\right] - \frac{2}{3}\epsilon^2(\sigma^{\alpha}_{\alpha} - \sigma_p)\dot{\sigma}_p\right\} \tag{2.13}$$

From (2.12) and (2.13) one can obtain for plastic loading

$$\lambda = h\dot{\sigma} = (\nabla_\nu \nabla^\mu \dot{\varphi})\left[\frac{2\sigma}{3h} + \frac{4}{9}E\epsilon^2(\sigma^{\alpha}_{\alpha} - \sigma_p)^2\right]^{-1} B^{\nu}_{\mu} \tag{2.14}$$

where

$$B^{\alpha}_{\beta} = \nabla^{\alpha}\nabla_{\beta}\varphi - \frac{1}{2}\delta^{\alpha}_{\beta}\left[\Delta\varphi(1 - \frac{4}{3}\epsilon^2) + \frac{4}{3}\epsilon^2\sigma_p\right] \tag{2.15}$$

The compatibility equation (2.11) can be written in the form

$$\frac{1-\nu^2}{E}\Delta\Delta\dot{\varphi} + B^{\alpha}_{\beta}\nabla_{\alpha}\nabla^{\beta}\lambda + (\nabla_{\alpha}\lambda)\left[\nabla^{\alpha}(\Delta\varphi)(1 + \frac{4}{3}\epsilon^2) - \frac{4}{3}\epsilon^2\nabla^{\alpha}\sigma_p\right]$$
$$+ \frac{1}{2}\lambda\left[\Delta\Delta\varphi(1 + \frac{4}{3}\epsilon^2) - \frac{4}{3}\epsilon^2\Delta\sigma_p\right] = 0 \tag{2.16}$$

Substitution of (2.14) into (2.16) gives

$$A^{\alpha\mu}_{\beta\lambda}\nabla_{\alpha}\nabla^{\beta}\nabla_{\mu}\nabla^{\lambda}\dot{\varphi} + \ldots = 0 \tag{2.17}$$

where ... denotes lower derivatives of $\dot{\varphi}$, and

$$A^{\alpha\mu}_{\beta\lambda} = (1-\nu^2)\delta^{\alpha}_{\beta}\delta^{\mu}_{\lambda} + B^{\alpha}_{\beta}B^{\mu}_{\lambda}\left[\frac{2\sigma}{3Eh} + \frac{4}{9}\epsilon^2(\Delta\varphi - \sigma_p)^2\right]^{-1} \tag{2.18}$$

Now consider the coefficient of the highest derivative of $\dot{\varphi}$ with respect to x^{α} in (2.17), which for α =1 will be

$$A^{11}_{11} = (1-\nu^2) + \left[\frac{1}{2}(\nabla_1\nabla^1\varphi - \nabla_2\nabla^2\varphi) + \frac{2}{3}\epsilon^2(\Delta\varphi - \sigma_p)\right]^2 \cdot \left[\frac{2\sigma}{3Eh} + \frac{4}{9}\epsilon^2(\Delta\varphi - \sigma_p)^2\right]^{-1} \tag{2.19}$$

It is easy to see that generally A^{11}_{11} is not vanishing, and so eq.(2.17) is elliptic.

The situation is, however, somewhat different near the crack tip which is a point of stress singnlarity. Suppose the hardening exponent n>1, then A^{11}_{11} can be asymptotically much smaller than the other coefficients (for example, $A^{11}_{11}/A^{12}_{21} \sim 1/Ec\sigma^{n-1} \ll 1$, if $\nabla_1\nabla^1\varphi - \nabla_2\nabla^2\varphi = 0$), and eq. (2.17) can be asymptotically hyperbolic at the point of stress singularity. Owing to this fact, there can occur certain kinds of inner boundary layer. The line of discontinuity Γ is expected to be the limit of an inner boundary layer.

2.3 The Contiguity Conditions of φ

We assume that Γ is an arbitrary curve on the elastic-plastic plane. In the vicinity of Γ we take the parallel-curves family of Γ and their straight normals to be the coordinate-lines, and denote the coordinates by s and n. The contiguity conditions of stresses for Γ can be written as

$$[\varphi]_\Gamma = \left[\frac{\partial\varphi}{\partial n}\right]_\Gamma = 0 \tag{2.20}$$

where $[\psi]_\Gamma$ denotes the gap of ψ across Γ.

We can prove[1] that the contiguity conditions of displacements can be written as

$$[\varepsilon_s]_\Gamma = 0 \tag{2.21}$$

$$2\frac{d}{ds}[\varepsilon_{ns}]_\Gamma - \left[\frac{\partial\varepsilon_s}{\partial n}\right]_\Gamma + \frac{d\vartheta}{ds}[\varepsilon_n]_\Gamma = 0 \tag{2.22}$$

in which ϑ is the angle from x axis to the normals of Γ. We can also prove that for hardening material, in general, the strong discontinuity (of the strains or stresses) cannot exist, while the weak discontinuity (of the derivatives of strains or stresses) is usually the boundary between elastic and plastic domains. For the weak discontinuity, (2.21) and (2.22) can be written as

$$\left[\frac{\partial^2\varphi}{\partial n^2}\right]_\Gamma = 0 \tag{2.23}$$

$$\frac{1-\nu^2}{E}\left[\frac{\partial^3\varphi}{\partial n^3}\right]_\Gamma - \frac{1}{V}[\lambda]_\Gamma\left\{\frac{1}{2}(\sigma_s - \sigma_n) + \frac{2}{3}\epsilon^2(\Delta\varphi - \sigma_p)\right\} = 0 \tag{2.24}$$

where V is the normal component of the velocity of moving Γ.

$$\frac{1}{V} = \frac{\partial t^*}{\partial n} \tag{2.25}$$

Here t* denotes the time when the line of discontinuity Γ "sweeps" the point under consideration.

2.4 Contiguity Conditions of $\dot{\varphi}$ for Weak Discontinuity

In some cases, it is necessary to take $\dot{\varphi}$ as the basic unknown guantity. By use of the same method as in another paper[2], we can obtain three contiguity conditions of $\dot{\varphi}$ frome (2.20),(2.23) and (2.24).

$$[\dot{\varphi}]_\Gamma = \left[\frac{\partial\dot{\varphi}}{\partial n}\right]_\Gamma = 0 \tag{2.26}$$

$$\frac{1-\nu^2}{E}\left[\frac{\partial^2\dot{\varphi}}{\partial n^2}\right]_\Gamma + [\lambda]_\Gamma\left\{\frac{1}{2}(\sigma_s - \sigma_n) + \frac{2}{3}\epsilon^2(\Delta\varphi - \sigma_p)\right\} = 0 \tag{2.27}$$

When (2.26),(2.27) and (2.11) are satisfied, in order that φ and $\varepsilon^p_{\alpha\beta}$ may satisfy (2.10), the following supplementary condition must be satisfied by $\dot{\varphi}$ and $\dot{\varepsilon}^p_{\alpha\beta}$

$$\frac{1-\nu^2}{E}\left[\frac{\partial^3\dot{\varphi}}{\partial n^3}\right]_\Gamma + \left[\frac{\partial\dot{\varepsilon}^p_s}{\partial n}\right]_\Gamma - \frac{d\vartheta}{ds}[\lambda]_\Gamma\left\{\frac{1}{2}(\sigma_n - \sigma_s) + \frac{2}{3}\epsilon^2(\Delta\varphi - \sigma_p)\right\} - 2\frac{d}{ds}\left\{\sigma_{ns}[\lambda]_\Gamma\right\} = 0 \tag{2.28}$$

The proof is omitted here.

2.5 Theorem for the Unloading Boundary

The location of the loading plastic boundary can be determined by $\sigma = \sigma_0$, but the unloading boundary cannot be determined in such a way. Hence we give the following theorem.

THEOREM: If Γ is an unloading plastic boundary, then

$$\dot{\sigma}|_{\Gamma(e)} = \dot{\sigma}|_{\Gamma(p)} = \dot{\sigma}|_\Gamma = 0, \qquad \lambda|_{\Gamma(p)} = 0 \tag{2.29}$$

where the subscripts Γ(e) and Γ(p) are used to denote the values on the boundary Γ at the sides of plastic unloading and loading domains.

PROOF: From (2.2),(2.20),(2.23) and (2.26) we have

$$[\sigma_p]_\Gamma = 0, \qquad [\sigma_{\alpha\beta}]_\Gamma = [\dot{\sigma}_{ns}]_\Gamma = [\dot{\sigma}_n]_\Gamma = 0 \tag{2.30}$$

Further, using (2.12),(2.13) we can obtain

1,2 See Gao Yu-chen and Hwang Keh-chih, On the formulation of plane strain problems for elastic perfectly-plastic medium (unpublished work).

$$2\,[\sigma\dot\sigma]_\Gamma = 2\sigma\,[\dot\sigma]_\Gamma = 3\left\{\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right\}[\dot\sigma_s]_\Gamma - \tfrac{4}{3}\epsilon^2 E(\Delta\varphi-\sigma_p)^2[\lambda]_\Gamma \tag{2.31}$$

Using (2.27) and noting that $[\dot\sigma_s]_\Gamma=[\partial^2\dot\varphi/\partial n^2]_\Gamma$, we have

$$2\sigma[\dot\sigma]_\Gamma = -\frac{3E}{1-\nu^2}\left\{\left[\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right]^2 + \tfrac{4}{9}\epsilon^2(1-\nu^2)(\Delta\varphi-\sigma_p)^2\right\}[\lambda]_\Gamma \tag{2.32}$$

On the other hand, from (1.4) we have

$$2\sigma\dot\sigma\big|_{\Gamma(p)} = \frac{2\sigma}{h}\lambda\big|_{\Gamma(p)}, \qquad \lambda\big|_{\Gamma(e)}=0 \tag{2.33}$$

hence,

$$2\sigma\dot\sigma\big|_{\Gamma(e)} = \left\{\frac{3E}{1-\nu^2}\left[\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right]^2 + \tfrac{4}{3}\epsilon^2E(\Delta\varphi-\sigma_p)^2+\frac{2\sigma}{h}\right\}\lambda\big|_{\Gamma(p)} \geqslant 0 \tag{2.34}$$

The unloading condition demands

$$\dot\sigma\big|_{\Gamma(e)}\leqslant 0 \tag{2.35}$$

Comparing (2.34) and (2.35), we obtain

$$\lambda\big|_{\Gamma(p)}=0,\qquad \dot\sigma\big|_{\Gamma(e)}=0 \quad \text{and} \quad \dot\sigma\big|_{\Gamma(p)}=0$$

Q.E.D.

3. THE STEADY FIELD

3.1 Equations

In the case of a crack growing steadily, if we observe the tip-field from a reference system moving together with the crack-tip, the pattern of the field will not vary with time. Hence in the fixed reference system, for any quantity given in Cartisian coordinates, we have

$$\dot{(\ )} = \frac{\partial}{\partial t}(\ ) = -\frac{\partial}{\partial x}(\ ) \tag{3.1}$$

Here we assume that the crack grows along the x-direction. By using (3.1), the eqs. (2.2),(2.3) and (2.8) can be written as

$$\begin{cases} \sigma_p = -\frac{2}{3}E\exp(-\frac{2}{3}EH)\int_x^{x_A} h\,\Delta\varphi\,\exp(\frac{2}{3}EH)\frac{\partial\sigma}{\partial x}\,dx \\ H=-\int_x^{x_A} h\frac{\partial\sigma}{\partial x}dx, \qquad \lambda=-\mu h\frac{\partial\sigma}{\partial x} \end{cases} \tag{3.2}$$

$$\varepsilon^p_{\alpha\beta} = \frac{\epsilon^2}{E}\sigma_p g_{\alpha\beta}+\int_x^{x_A}\lambda\sigma_{\alpha\beta}dx-\frac{1}{2}g_{\alpha\beta}\int_x^{x_A}\lambda\Delta\varphi\,dx, \qquad \alpha,\beta=x,y \tag{3.3}$$

in which x_A denotes the value of x at the front plastic boundary Γ_A (Fig. 1). The compatibility equation (2.16) becomes

$$\frac{1-\nu^2}{E}\Delta\Delta\frac{\partial\varphi}{\partial x}+(\nabla^\alpha\nabla^\beta\varphi)\nabla_\alpha\nabla_\beta\left(h\frac{\partial\sigma}{\partial x}\right)-\frac{1}{2}\Delta\left(h\frac{\partial\sigma}{\partial x}\right)\left\{\Delta\varphi\left(1-\tfrac{4}{3}\epsilon^2\right)+\tfrac{4}{3}\epsilon^2\sigma_p\right\}$$
$$+\nabla_\alpha\left(h\frac{\partial\sigma}{\partial x}\right)\left\{\nabla^\alpha(\Delta\varphi)\left(1+\tfrac{4}{3}\epsilon^2\right)-\tfrac{4}{3}\epsilon^2\nabla^\alpha\sigma_p\right\}+\frac{1}{2}h\frac{\partial\sigma}{\partial x}\left\{\Delta\Delta\varphi\left(1+\tfrac{4}{3}\epsilon^2\right)-\tfrac{4}{3}\epsilon^2\Delta\sigma_p\right\}=0 \tag{3.4}$$

3.2 Contiguity Conditions for Weak Discontinuity

The first three contiguity conditions of φ are given by (2.20) and (2.23). The fourth (2.24) can be written as

$$\frac{1-\nu^2}{E}\left[\frac{\partial^3\varphi}{\partial n^3}\right]_\Gamma+\frac{1}{V}h(\sigma)\left[\mu\frac{\partial\sigma}{\partial x}\right]_\Gamma\left\{\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right\}=0 \tag{3.5}$$

Noting

$$\begin{cases} V=\cos\vartheta \\ \dfrac{\partial\sigma}{\partial x}=\dfrac{\partial\sigma}{\partial n}\cos\vartheta-\dfrac{\partial\sigma}{\partial s}\sin\vartheta \end{cases} \tag{3.6}$$

and $\sigma|_\Gamma=\sigma_o$ =const at loading plastic boundary, we have

$$\frac{1-\nu^2}{E}\left[\frac{\partial^3\varphi}{\partial n^3}\right]_\Gamma+h(\sigma)\left[\mu\frac{\partial\sigma}{\partial n}\right]_\Gamma\left\{\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right\}=0 \tag{3.7}$$

for loading boundary. As for the unloading plastic boundary, (2.24) becomes by virtue of (2.29)

$$\left[\frac{\partial^3\varphi}{\partial n^3}\right]_\Gamma=0 \tag{3.8}$$

The fifth contiguity condition (2.28) of φ (the fourth of $\dot\varphi$) can be written as

$$\frac{1-\nu^2}{E}\left[\frac{\partial^4\varphi}{\partial n^4}\right]_\Gamma-\frac{1-\nu^2}{E}\tan\vartheta\frac{d}{ds}\left[\frac{\partial^3\varphi}{\partial n^3}\right]_\Gamma+\frac{1}{\cos\vartheta}\left[\frac{\partial}{\partial n}\left\{\mu h(\sigma)\frac{\partial\sigma}{\partial x}\left(\tfrac{1}{2}(\sigma_s-\sigma_n)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right)\right\}\right]_\Gamma-\frac{1}{\cos\vartheta}h(\sigma)\frac{d\vartheta}{ds}\left\{\tfrac{1}{2}(\sigma_n-\sigma_s)+\tfrac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)\right\}\left[\mu\frac{\partial\sigma}{\partial x}\right]_\Gamma$$
$$-\frac{2}{\cos\vartheta}\frac{d}{ds}\left\{\sigma_{ns}h(\sigma)\left[\mu\frac{\partial\sigma}{\partial x}\right]_\Gamma\right\}=0 \tag{3.9}$$

3.3 Simplification of the Equations

As is well known, both the stresses and the strains have singularities at the crack tip. If we are intereated only in the principal singular term of each quantity, the equations can be greatly simplified. Hereafter we shall be confined only to the case of power-hardening, and assume n>1. Owing to the singularity of stress, the term σ_o can be neglected in (1.7), and then we have

$$h(\sigma)=\tfrac{3}{2}nc\sigma^{n-2} \tag{3.10}$$

Now we would expect

$$\varepsilon^p_{\alpha\beta}\gg\frac{\sigma}{E} \tag{3.11}$$

On the other hand[3],

$$\varepsilon^e_{\alpha\beta}\sim\frac{\sigma}{E}, \qquad \frac{\sigma_p}{E}\sim\frac{\sigma}{E} \tag{3.12}$$

Hence we may neglect $\varepsilon^e_{\alpha\beta}$ and σ/E in the total strains, and have from (2.6)-(2.8)

$$\varepsilon_y=-\varepsilon_x=\frac{3}{4}nc\int_x^{x_A}\sigma^{n-2}(\sigma_x-\sigma_y)\frac{\partial\sigma}{\partial x}dx$$
$$\varepsilon_{xy}=-\frac{2}{3}nc\int_x^{x_A}\sigma^{n-2}\sigma_{xy}\frac{\partial\sigma}{\partial x}dx \tag{3.13}$$

where σ is given by (2.4). From (2.12) one obtains

$$cE\sigma^{n-1}(\Delta\varphi-\sigma_p)\sim\sigma_p\sim\sigma \tag{3.14}$$

and then

$$\frac{\Delta\varphi-\sigma_p}{\sigma}\sim\frac{1}{cE\sigma^{n-1}}\ll 1 \tag{3.15}$$

Hence we can neglect the term involving $\sigma^\alpha_\alpha-\sigma_p$ in (2.4) and obtain

$$\sigma=\sqrt{3}\left[\tfrac{1}{4}(\sigma_x-\sigma_y)^2+\sigma_{xy}^2\right]^{1/2} \tag{3.16}$$

Finally, after neglecting elastic strains, the compatibility equations (3.4) can be reduced to

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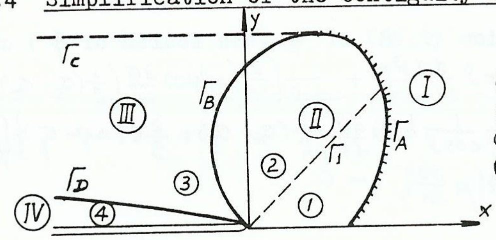
[3] We use the symbol "$\sim$" for quantities with the same asymptotic order of magnitude.

$$\frac{1}{2}(\nabla_1\nabla^1-\nabla_2\nabla^2)\left(\sigma^{n-2}\frac{\partial\sigma}{\partial x}\right)\cdot(\nabla_1\nabla^1-\nabla_2\nabla^2)\varphi+2\nabla_1\nabla^2\left(\sigma^{n-2}\frac{\partial\sigma}{\partial x}\right)\cdot\nabla_2\nabla^1\varphi$$
$$+\nabla_\alpha\left(\sigma^{n-2}\frac{\partial\sigma}{\partial x}\right)\cdot\nabla^\alpha(\Delta\varphi)+\frac{1}{2}\sigma^{n-2}\frac{\partial\sigma}{\partial x}\Delta\Delta\varphi=0 \qquad (3.17)$$

for orthogonal coordinates x^α.

3.4 Simplification of the Contiguity Conditions



I – Elastic domain
II – Primary plastic domain, loading
III – Plastic domain, unloading
IV – Secondary plastic domain, reverse loading

Fig. 1

The equations for the case of hardening materials are elliptic. Hence, the solution inside plastic region will be sufficiently smooth and there will generally not exist any lines of discontinuity. There can exist weak discontinuities at the boundary between various domains (Fig. 1). From(2.20) and (2.23) the first three contiguity condition at Γ must be

$$[\varphi]_\Gamma=\left[\frac{\partial\varphi}{\partial n}\right]_\Gamma=\left[\frac{\partial^2\varphi}{\partial n^2}\right]_\Gamma=0 \qquad (3.18)$$

The fourth and the fifth contiguity conditions are (3.5) and (3.9).

For the unloading boundary Γ_B , the fourth contiguity condition is (3.8), i.e.

$$\left[\frac{\partial^3\varphi}{\partial n^3}\right]_{\Gamma_B}=0 \qquad (3.19)$$

With the predominant singular terms retained, the fifth condition (3.9) is reduced to

$$\left.\frac{\partial\lambda}{\partial n}\right|_{\Gamma_B(p)}=0, \quad \text{or} \quad \left.\frac{\partial}{\partial n}\left(\frac{\partial\sigma}{\partial x}\right)\right|_{\Gamma_B(p)}=0, \quad \text{if} \quad \sigma_s-\sigma_n\neq 0 \qquad (3.20)$$

Form (2.29), the unloading condition at Γ_B is

$$\lambda|_{\Gamma_B}=0 \quad \text{or} \quad \left.\frac{\partial\sigma}{\partial x}\right|_{\Gamma_B}=0 \qquad (3.21)$$

Finally, at the reloading boundary Γ_D , the fourth and the fifth conditions (3.5) and (3.9) reduce to

$$\lambda|_{\Gamma_D(p)}=\left.\frac{\partial\lambda}{\partial n}\right|_{\Gamma_D(p)}=0, \quad \text{or} \quad \left.\frac{\partial\sigma}{\partial x}\right|_{\Gamma_D(p)}=\left.\frac{\partial}{\partial n}\left(\frac{\partial\sigma}{\partial x}\right)\right|_{\Gamma_D(p)}=0, \quad \text{if} \quad \sigma_s-\sigma_n\neq 0 \qquad (3.22)$$

Besides, the following reloading condition at Γ_D must be satisfied

$$\sigma|_{\Gamma_D}(y)=\sigma|_{\Gamma_B}(y) \qquad (3.23)$$

4. THE FIRST VERSION OF ASYMPTOTIC ANALYSIS

We will first try the power expansion, and be confined only to the first-order approximation. Let r, θ denote polar coordinates and put

$$\varphi=r^{2-\delta}f(\theta) \qquad 0<\delta<0.5 \qquad (4.1)$$

then

$$\begin{cases}\sigma_r=r^{-\delta}[f''+(2-\delta)f], & \sigma_\theta=r^{-\delta}(1-\delta)(2-\delta)f \\ \sigma_{r\theta}=-r^{-\delta}(1-\delta)f' & \sigma=r^{-\delta}\sqrt{3}[g(\theta)]^{1/2}\end{cases} \qquad (4.2)$$

in which

$$g(\theta)=\frac{1}{4}[f''+\delta(2-\delta)f]^2+(1-\delta)^2(f')^2 \qquad (4.3)$$

From the σ expression in (4.2), one obtains

$$\lambda=\frac{1}{2}nc\cdot 3^{(n+1)/2}r^{-(1+\mu)}G_1(\theta) \qquad (4.4)$$

where

$$\begin{cases}G_1=\delta\cos\theta\cdot g^{(n-1)/2}+\frac{1}{2}\sin\theta\cdot g^{(n-3)/2}g' \\ \mu=(n-1)\delta\end{cases} \qquad (4.5)$$

Substituting (4.1) and (4.4) into (3.17), we are led to a lengthy expression for f^V:

$$f^V=F(n,\delta,\theta,f,f',f'',f''',f^{iv}) \qquad (4.6)$$

The contiguity conditions are

$$\begin{cases}[f]_{\Gamma_B}=[f']_{\Gamma_B}=[f'']_{\Gamma_B}=[f''']_{\Gamma_B}=0 \\ G_1|_{\pi-\beta-0}=\partial G_1/\partial\theta|_{\pi-\beta-0}=0\end{cases} \qquad (4.7)$$

$$\begin{gathered}[f]_{\Gamma_D}=[f']_{\Gamma_D}=[f'']_{\Gamma_D}=0 \\ G_1|_{\pi-\gamma+0}=\partial G_1/\partial\theta|_{\pi-\gamma+0}=0, \qquad g(\pi-\beta)(\sin\beta)^{2\delta}=g(\pi-\gamma)(\sin\gamma)^{2\delta}\end{gathered} \qquad (4.8)$$

For the Mode I crack, f(θ) is even function, so the boundary conditions are

$$\begin{cases}f(\pi)=f'(\pi)=0 \\ f'(0)=f'''(0)=0\end{cases} \qquad (4.9)$$

Moreover, since coefficient of the highest derivative in (4.6) vanishes at θ =0, in order that the solution may be regular at θ =0, we must demand

$$f^{IV}(0)=\frac{1}{2+n\delta}\Big\{n^2\delta^4(2-\delta)f(0)-\delta\big[\!\big[(2-\delta)(2+\mu)-\delta[2(2n-1)+$$
$$+(n^2-4n+1)\delta]\big]\!\big]f''(0)-4(2+\mu)(1-\delta)^2[f''(0)+\delta(2-\delta)f(0)]^{-1}(f''(0))^2\Big\} \qquad (4.10)$$

Using the previous conditions, for fixed value of n, we can calculate the corresponding value of δ . This is an eigenvalue problem of nonlinear fifth-order equation. We take the value of n, δ , f"(0),f(0) as follows

$$n=1.5\ ,\ 2\ ,\ 3,\ 4,\ 5.5\ , \qquad 0.01<\delta<0.48$$
$$-(2-\delta)<f''(0)<-0.9\,\delta(2-\delta), \qquad f(0)=1$$

The calculation was performed, but the results show that there is no proper set of n, δ and f"(0) that make the first two of (4.9) to be satisfied. Therfore we think to guess that the assumed singularity $\sigma\sim r^{-\delta}$ is unreasonable.

It should be noted that here we have not taken into consideration the possibility of discontinuity across characteristic line for (4.6) $(\sigma_r-\sigma_\theta=0)$ within the plastic domain. In our numerical calculation performed, nowhere did $\sigma_r-\sigma_\theta$ happen to be zero in the plastic domain. This confirms the continuity of f(θ) and its first four derivatives in the plastic domains, since discontinuity can occur only across characteristic line.

5. THE SECOND VERSION OF ASYMPTOTIC ANALYSIS

The unsuccessful attempt made for the power expansion in the first version of asymptotic analysis suggests the logarithmic expansion. We shall also consider the discontinuity across characteristic line Γ_1 within the plastic domain (Fig.1,2).

5.1 Contiguity Conditions for Line of Discontinuity within Plastic Domain

As mentioned before, with elastic strains neglected in the neighborhood of the crack tip, the equation (2.17) is asymptotically hyperbolic with respect to λ (or respect to φ . Similarly (3.17) is hyperbolic with respect to $\sigma^{n-2}\partial\sigma/\partial x$). Hence there can exist discontinuity across characteristic line Γ_1 within the plastic domain in the asymptotic sense. By the asymptotic sense we mean that the discontinuity results only as the crack tip is approached. The line of discontinuity is the limit to which an inner boundary layer shrinks as the crack tip is approached. In the crude approximation, the inner boundary layer can be replaced by this line of discontinuity. It can be proved[4] that the equation for characteristic line Γ_1 is

$$\frac{1}{2}(\sigma_s-\sigma_n)+\frac{2}{3}\epsilon^2(\Delta\varphi-\sigma_p)=0 \tag{5.1}$$

The first three contiguity conditions at Γ_1 are (3.18). The fourth condition (2.22) can be written for steady field as[4]

$$\frac{1-\nu^2}{E}\left[\frac{\partial^3\varphi}{\partial n^3}\right]_{\Gamma_1}+2\frac{d\vartheta}{ds}\tan\vartheta\,[\varepsilon^p_{ns}]_{\Gamma_1}-2\frac{d}{ds}[\varepsilon^p_{ns}]_{\Gamma_1}=0 \tag{5.2}$$

The fifth condition can be obtained from (2.28) by adding a term $2V(d^2\vartheta/ds^2)\cdot[\varepsilon^p_{ns}]_{\Gamma_1}$, or from (3.9) by adding a term $-2(d^2\vartheta/ds^2)[\varepsilon^p_{ns}]_{\Gamma_1}$ to the left side[4].

5.2 Asymptotic Expansion

For simplicity we shall be confined to the case $\nu=\frac{1}{2}$, hence $\epsilon=0$. Assume the logarithmic expansion

$$\varphi=r^2\left(\ln\frac{A}{r}\right)^{\alpha}\sum_{n=0}^{\infty}\left(\ln\frac{A}{r}\right)^{-n}f_n(\theta) \tag{5.3}$$

Then the stress components in polar coordinates will be

$$\begin{aligned}\sigma_r&=\left(\ln\frac{A}{r}\right)^{\alpha}\left\{f_0''+2f_0+\sum_{n=1}^{\infty}\left(\ln\frac{A}{r}\right)^{-n}\left[f_n''+2f_n-(\alpha-n+1)f_{n-1}\right]\right\}\\ \sigma_\theta&=\left(\ln\frac{A}{r}\right)^{\alpha}\left\{2f_0+\left(\ln\frac{A}{r}\right)^{-1}(2f_1-3\alpha f_0)\right.\\&\qquad\left.+\sum_{n=2}^{\infty}\left(\ln\frac{A}{r}\right)^{-n}\left[2f_n-3(\alpha-n+1)f_{n-1}+(\alpha-n+1)(\alpha-n+2)f_{n-2}\right]\right\}\\ \sigma_{r\theta}&=\left(\ln\frac{A}{r}\right)^{\alpha}\left\{-f_0'-\sum_{n=1}^{\infty}\left(\ln\frac{A}{r}\right)^{-n}\left[f_n'-(\alpha-n+1)f_{n-1}'\right]\right\}\end{aligned} \tag{5.4}$$

Substitution of (5.4) into (2.4) gives

$$\sigma=\sqrt{3}K^{1/2}\ ,\qquad K=\frac{1}{4}(\sigma_r-\sigma_\theta)^2+\sigma_{r\theta}^2 \tag{5.5}$$

whose expansions are

$$\begin{aligned}K&=\left(\ln\frac{A}{r}\right)^{2\alpha}\left\{K_0(\theta)+\left(\ln\frac{A}{r}\right)^{-1}K_1(\theta)+\cdots\right\}\\ K_0(\theta)&=\frac{1}{4}f_0''^2+f_0'^2\\ K_1(\theta)&=\frac{1}{2}f_0''(f_1''+2\alpha f_0)+2f_0'(f_1'-\alpha f_0')\end{aligned} \tag{5.6}$$

and

$$\sigma=\sqrt{3}\left(\ln\frac{A}{r}\right)^{\alpha}\left\{[K_0(\theta)]^{1/2}+\frac{1}{2}\left(\ln\frac{A}{r}\right)^{-1}K_1(\theta)/[K_0(\theta)]^{1/2}+\cdots\right\} \tag{5.7}$$

Substituting (5.7) into expression for λ in (3.2) (for $\mu=1$) and using (3.10), we obtain

$$\lambda=\frac{1}{r}\left(\ln\frac{A}{r}\right)^{\alpha(n-1)}\left\{\lambda_0(\theta)+\left(\ln\frac{A}{r}\right)^{-1}\lambda_1(\theta)+\cdots\right\} \tag{5.8}$$

where

$$\lambda_0(\theta)=\frac{1}{4}nc\cdot3^{(n+1)/2}\sin\theta\,K_0^{(n-3)/2}K_0' \tag{5.9}$$

$$\lambda_1(\theta)=\frac{1}{4}nc\cdot3^{(n+1)/2}K_0^{(n-5)/2}\left[\sin\theta\left((n-3)K_0'K_1+K_0K_1'\right)+2\alpha\cos\theta K_0^2\right] \tag{5.10}$$

The compatibility equation (2.16) can be written in the form

$$\frac{1-\nu^2}{E}\Delta\Delta\frac{\partial\varphi}{\partial x}+(\Delta\lambda)\Delta\varphi-(\nabla^\alpha\nabla^\beta\lambda)\nabla_\alpha\nabla_\beta\varphi-\frac{1}{2}\Delta(\lambda\Delta\varphi)=0 \tag{5.11}$$

We anticipate that the elastic-strain term in (5.11) has a less order of singularity than the plastic-strain terms involving λ. This anticipation gives

$$\alpha=\frac{1}{n-1} \tag{5.12}$$

The compatibility equation (2.10) for the plastic unloading domain III (Fig. 1) can be written as

$$\frac{1-\nu^2}{E}\Delta\Delta\varphi+\frac{d^2}{dy^2}\varepsilon^p_x(y)=0 \tag{5.13}$$

where $\varepsilon^p_x(y)$ is the plastic strain in domain III . Eq.(5.13) suggests the expansion for the plastic strains

$$\varepsilon^p_{\alpha\beta}=\left(\ln\frac{A}{r}\right)^{\alpha+1}\left\{\varepsilon_{\alpha\beta0}(\theta)+\left(\ln\frac{A}{r}\right)^{-1}\varepsilon_{\alpha\beta1}(\theta)+\cdots\right\} \tag{5.14}$$

$$\varepsilon^p_x(y)=\left(\ln\frac{A}{y}\right)^{\alpha+1}\left\{a_0+\left(\ln\frac{A}{y}\right)^{-1}a_1+\cdots\right\} \tag{5.15}$$

5.3 The First Approximation

Substitution of (5.3) and (5.8) into (5.11) gives for the first approximation

$$\left(\frac{d^2}{d\theta^2}+1\right)\left(f_0''(\theta)\lambda_0(\theta)\right)=0 \tag{5.16}$$

Its solution is

$$f_0''(\theta)\lambda_0(\theta)=A_i\sin\theta+B_i\cos\theta \tag{5.17}$$

where A_i, B_i are constants of integration with i=1,2,4 for plastic domains ①, ②, ④ respectively (Fig. 2). Substituting (5.9) into (5.17), one obtains

$$\frac{1}{8}nc\cdot3^{(n+1)/2}\sin\theta\,K_0^{(n-3)/2}f_0''^2(f_0'''+4f_0')=A_i\sin\theta+B_i\cos\theta \tag{5.18}$$

For domain ①, letting θ=0, we obtain B_1=0. Consideration of symmetry leads further to A_1=0. With f'(0)=f'''(0)=0 as reguired in (4.9), the solution of (5.18) is

$$f_0(\theta)=a+\frac{1}{2}F\cos2\theta\ ,\qquad K_0(\theta)=F^2\quad\text{for domain ①} \tag{5.19}$$

The characteristic line Γ_1 will then be located at $=\pi/4$ (Fig. 2). (5.4) gives the first approximation for stresses in domain ①.

$$\sigma^{①}_{r0}=2a-F\cos2\theta\ ,\qquad\sigma^{①}_{\theta0}=2a+F\cos2\theta\ ,\qquad\sigma^{①}_{r\theta0}=F\sin2\theta \tag{5.20}$$

or, in Cartesian coordinates,

$$\sigma^{①}_{xo}=2a-F\,,\qquad \sigma^{①}_{yo}=2a+F\,,\qquad \sigma^{①}_{xyo}=0 \qquad (5.21)$$

Couditions (3.20), (3.21) at the unloading boundary Γ_B, and conditions (3.22) at the reloading boundary Γ_D can be expressed respectively as

$$\lambda_o\big|_{\Gamma_B}=0\,,\qquad \frac{d\lambda_o}{d\theta}\Big|_{\Gamma_B}=0 \qquad (5.22)$$

and

$$\lambda_o\big|_{\Gamma_D}=0\,,\qquad \frac{d\lambda_o}{d\theta}\Big|_{\Gamma_D}=0 \qquad (5.23)$$

Form (5.17) and (5.22) follows $A_2=B_2=0$, and from (5.17) and (5.23) follows $A_4=B_4=0$. Hence the solution of (5.17) which satisfies the contiguity conditions (3.18) at Γ_1, i.e. $[f_o]_{\Gamma_1}=[f_o']_{\Gamma_1}=[f_o'']_{\Gamma_1}=0$, will be

$$f_o(\theta)=a-F(\theta-\pi/4)\,,\qquad K_o(\theta)=F^2 \qquad \text{for domain ②} \qquad (5.24)$$

And the solution of (5.17), satisfying the traction-free crack surface condition $f(\pi)=f'(\pi)=0$ in (4.9) and the reloading condition (3.23), will be

$$f_o(\theta)=\frac{1}{2}F(1-\cos 2\theta)\,,\qquad K_o(\theta)=F^2 \qquad \text{for domain ④} \qquad (5.25)$$

Owing to $K_o(\theta)$ being constant, $\lambda_o(\theta)$ is uniformly zero in all plastic domains ①, ② and ④, as can be seen from (5.9). Then the expansion (5.8) for λ will begin from the term $\lambda_1(\theta)$.

$$\lambda=\frac{1}{r}\Big\{\lambda_1(\theta)+\Big(\ln\frac{A}{r}\Big)^{-1}\lambda_2(\theta)+\dots\Big\} \qquad (5.26)$$

where

$$\lambda_1(\theta)=\frac{1}{4}nc\cdot 3^{(n+1)/2}F^{n-3}\big\{\sin\theta\cdot K_1'+2\alpha\cos\theta\cdot F^2\big\} \qquad (5.27)$$

We mention in passing that the vanishing of $\lambda_o(\theta)$ does not invalidate our anticipation that the elastic-strain term in (5.11) has a less order of singularity than the plastic-strain terms, since the elastic term $\frac{1-\nu^2}{E}\Delta\Delta\frac{\partial\varphi}{\partial x}$ also vanishes to the first approximation for the expressions of $f_o(\theta)$ (5.19), (5.24),(5.25) for domains ①, ②, ④. In Fig. 2 are shown the stresses in multiples of $(\ln A/r)^{\alpha}$ and the fields of characteristic lines ($\sigma_s-\sigma_n=0$, from(5.1)) in domains ①, ②, and ④.

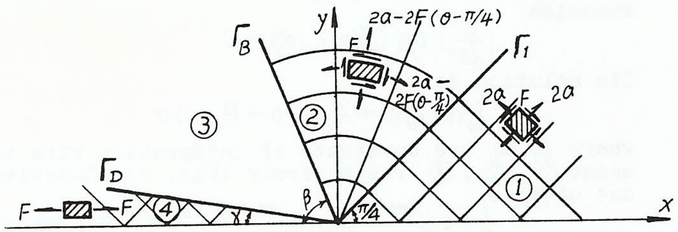


Fig. 2

For the unloading domain ③ we can obtain the first approximation for the compatibility equation (5.13) by substitution of (5.3) and (5.15)

$$\frac{1-\nu^2}{E}\big(f_o^{iv}+4f_o''\big)+\frac{(\alpha+1)a_o}{\sin^2\theta}=0 \qquad (5.28)$$

Its solution is

$$f_o(\theta)=h_1+h_2\theta+h_3\cos 2\theta+h_4\sin 2\theta+f_o^*(\theta) \qquad (5.29)$$

where h_1, h_2, h_3, h_4 are constants of integration, and

$$f_o^*(\theta)=-\frac{(\alpha+1)a_oE}{4(1-\nu^2)}\Big\{(\cos 2\theta-1)\ln\sin\theta+\Big(\theta+\frac{1}{2}\cot\theta\Big)\sin 2\theta\Big\} \qquad (5.30)$$

We now have eight constants to be determined, namely: $a, h_1, h_2, h_3, h_4, a_o$ and two angles β, γ. On the other hand, we have only seven conditions for their determination: (3.18),(3.19) at Γ_B and (3.18) at Γ_D:

$$[f_o]_{\Gamma_B}=[f_o']_{\Gamma_B}=[f_o'']_{\Gamma_B}=[f_o''']_{\Gamma_B}=0 \qquad (5.31)$$

$$[f_o]_{\Gamma_D}=[f_o']_{\Gamma_D}=[f_o'']_{\Gamma_D}=0 \qquad (5.32)$$

The one lacking condition will be supplemented by strain analysis (see eq. (5.43) in the next paragraph 5.4).

5.4 Strain Analysis

The expansion for stresses $\sigma_{\alpha\beta}$ in Cartesian coordinates is from (5.4)

$$\sigma_{\alpha\beta}=\Big(\ln\frac{A}{r}\Big)^{\alpha}\Big\{\sigma_{\alpha\beta o}(\theta)+\Big(\ln\frac{A}{r}\Big)^{-1}\sigma_{\alpha\beta 1}(\theta)+\cdots\Big\} \qquad (5.33)$$

Using (2.8),(3.1) and (3.3),(5.33), we obtain

$$\sin\theta\frac{d}{d\theta}\varepsilon_{\alpha\beta o}(\theta)=\lambda_o(\theta)\Big\{\sigma_{\alpha\beta o}(\theta)-\frac{1}{2}g_{\alpha\beta}\sigma^{\gamma}_{\gamma o}(\theta)\Big\} \qquad (5.34)$$

Since $\lambda_o(\theta)=0$, it follows that $\varepsilon_{\alpha\beta o}(\theta)$ is constant for each plastic domain.

$$\varepsilon_{\alpha\beta o}(\theta)=const=\varepsilon^{(i)}_{\alpha\beta o} \qquad (5.35)$$

where i=1,2,4 for domains ①,②, ④. For the domain ①, $\varepsilon^{①}_{\alpha\beta o}$ can be determined from the strains along the x-axis, where the stresses are asymptotically proportional, with fixed principal directions. Along the x-axis ($\theta=0$), we have from (3.3) to the first approximation

$$-\frac{d}{dx}\Big\{\varepsilon^{①}_{\alpha\beta o}\Big(\ln\frac{A}{x}\Big)^{\alpha+1}\Big\}=\Big\{\lambda_1(0)\frac{1}{x}\Big\}\cdot\Big\{\Big(\sigma^{①}_{\alpha\beta o}-\frac{1}{2}g_{\alpha\beta}\sigma^{\gamma①}_{\gamma o}\Big)\Big(\ln\frac{A}{x}\Big)^{\alpha}\Big\} \qquad (5.36)$$

where $\lambda_1(0)$ can easily be found from (5.27).

$$\lambda_1(0)=\frac{1}{2}\alpha nc\cdot 3^{(n+1)/2}F^{n-1} \qquad (5.37)$$

Substitution of stresses (5.21) for domain ① into (5.36) gives

$$\varepsilon^{①}_{yo}=-\varepsilon^{①}_{xo}=\frac{1}{2}3^{(n+1)/2}\cdot cF^n\,,\qquad \varepsilon^{①}_{xyo}=0 \qquad (5.38)$$

The r-derivative of strain gap across Γ_1 appears in the fourth condition (5.2) across Γ_1. Assume that the angle ϑ has an expansion of the form

$$\vartheta=\vartheta_o+\Big(\ln\frac{A}{r}\Big)^{-1}\vartheta_1+\cdots \qquad (5.39)$$

Then the second term $2(d\vartheta/ds)\tan\vartheta\,[\varepsilon^{p}_{ns}]_{\Gamma_1}$ in (5.2) may be neglected for the first approximation, and (5.2) reduces to

$$\frac{1-\nu^2}{E}[f_o''']_{\Gamma_1}+2(\alpha+1)[\varepsilon_{r\theta o}]_{\Gamma_1}=0 \qquad (5.40)$$

which, together with (5.19) and (5.24), gives

$$[\varepsilon_{r\theta o}]_{\Gamma_1}=\varepsilon_{r\theta o}\Big(\frac{\pi}{4}+0\Big)-\varepsilon_{r\theta o}\Big(\frac{\pi}{4}-0\Big)=\frac{2(1-\nu^2)}{E(\alpha+1)}F \qquad (5.41)$$

A suggestion may be made that the strain gap (5.41) is negligibly small as compared with strains themselves (5.38), $[\varepsilon_{r\theta o}]_{\Gamma_1}\approx 0$. This can be explained by the fact that the contiguity condition (5.2) gives only the r-derivative of the strain gap. However, since the strain gap exists only in a very small near-tip neighborhood, with its scale r_o negligibly small as compared with the scale of plastic region, e.g. the constant A in ln(A/r), the strain gap, which is obtained by integration of its derivative over very short interval, will be small. The strains $\varepsilon^{②}_{\alpha\beta o}$ for the domain ② will be

$$\varepsilon_{yo}^{②} = -\varepsilon_{xo}^{②} = \frac{1}{2} 3^{(n+1)/2} \cdot c F^n + [\varepsilon_{r\theta o}]_{\Gamma_1} \quad , \quad \varepsilon_{xyo}^{②} = 0 \tag{5.42}$$

where $[\varepsilon_{r\theta o}]_{\Gamma_1}$ is given by (5.41) or taken to be zero. Since the strains are continuous across the unloading boundary Γ_B, which is a line of weak discontinuity, we obtain by comparison between (5.15) and (5.42)

$$a_o + \frac{1}{2} 3^{(n+1)/2} \cdot c F^n + [\varepsilon_{r\theta o}]_{\Gamma_1} = 0 \tag{5.43}$$

(5.43) is the supplementary condition mentioned at the end of the foregoing paragraph 5.3. The strains in the domain ④ can be similarly determined[5].

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5 The numerical calculation for the determination of constants is being undertaken and we hope that some results will be presented at the ICF 5.