THE EFFECT OF A TIME-DEPENDENT CRAZE STRESS ON CRACK GROWTH IN A LINEAR VISCOELASTIC MATERIAL

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ABSTRACT

The general equations governing plane strain crack growth in linear viscoelastic solids are derived using the crack opening displacement and energy balance fracture criteria. Account is taken of time-dependent but uniform craze stresses and it is assumed that small scale crazing conditions prevail. It is shown that the crack opening displacement and energy balance fracture criteria lead to different crack growth laws when non-steady crack propagation is taking place. The conditions which allow the steady state growth laws to be used in non-steady situations are discussed.

KEYWORDS

Crack growth laws; linear viscoelasticity; small scale crazing; crack opening displacement; energy balance; fracture criteria; steady growth; non-steady growth.

INTRODUCTION

Knauss (1974) has shown that, when a crack propagates at constant velocity through a linear viscoelastic medium having a craze stress that is uniform and time-independent, then the energy balance and crack opening displacement (COD) fracture criteria lead to the same equation governing crack growth. The author (1979) has derived the same result for a central crack in a large plate subjected to a uniform and time-independent applied stress provided that conditions of small scale crazing prevail. For such a geometry the crack tip velocity is not constant. It has been suggested that the craze stress is time-dependent, being a function either of the bulk viscoelastic properties of the material (Marshall, Coutts and Williams, 1974) or of the local rate of deformation (Wnuk and Knauss, 1970; Knauss, 1974) which must be a function of crack tip velocity. The objective of this paper is to examine the effect of a time-dependent craze stress on the crack growth equations derived from both the COD and energy balance fracture criteria.

DEFORMATION ASSOCIATED WITH THE CRAZE

Consider a crack (without a craze zone) having length $c(t)$ at time t in a fracture specimen, made of a linear viscoelastic material, which is subjected at time t=0 to a specified load history rather than a specified deflection. It is well known for plane stress or strain conditions that the resulting stress distribution corresponds precisely to the stress field that would have resulted if the fracture specimen had been made of a linear elastic material. The singularity of the stress field at the crack tip in the fracture specimen is characterized by the stress intensity factor $K(t)$ which may be time-dependent. In order that finite stresses result at the crack tip some real viscoelastic materials form a craze zone ahead of the crack. As shown in Fig. 1 the craze zone is modelled as a very thin strip of time-dependent length $R(t)$ across which the normal displacement component is discontinuous.

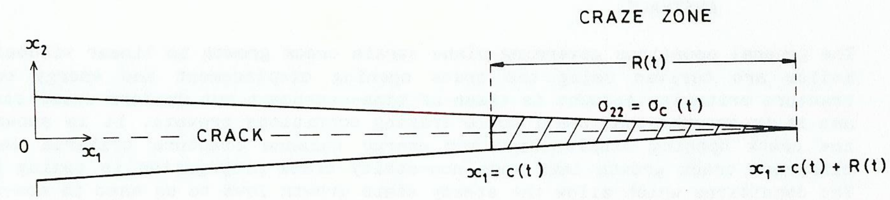

Fig. 1. Schematic diagram of a crack and its craze zone.

With respect to the Cartesian co-ordinates x_1 and x_2 shown in Fig. 1 the distribution of the displacement discontinuity is denoted by $\Delta u_2(x_1,t)$. The craze zone is capable of supporting a stress and it is assumed, for reasons of mathematical tractability, that the stress component σ_{22} has the time-dependent value $\sigma_c(t)$ at every point in the craze zone $c(t) \leq x_1 \leq c(t)+R(t)$ which is present at time t. Having specified a value of the craze stress at time t the corresponding length $R(t)$ of the craze zone is obtained by requiring that the stresses are bounded at the point $x_1 = c(t)+R(t)$, $x_2 = 0$. As for the Dugdale (1960) model of plasticity it can be shown that

$$R(t) = \pi K^2(t) \,/\, \{8\sigma_c^2(t)\} \quad , \qquad (1)$$

provided that $R(t) \ll c(t)$ so that small scale crazing conditions prevail. Such conditions will be assumed at all times for the remainder of this paper.

For plane strain conditions the distribution of the displacement discontinuity in the craze zone has the form (McCartney, 1979, eq.(2))

$$\Delta u_2(x_1,t) = 4\pi \int_0^t k(t-\tau)\, \frac{\partial \Omega(x_1,\tau)}{\partial \tau}\, d\tau \quad , \quad c(t) \leq x_1 \leq c(t)+R(t) \quad , \qquad (2)$$

$$\text{where} \qquad \Omega(x_1,t) = \frac{2\sigma_c(t)R(t)}{\pi^2}\left\{\lambda - \tfrac{1}{2}(1-\lambda^2)\ln\left(\frac{1+\lambda}{1-\lambda}\right)\right\} \quad , \qquad (3)$$

$$\text{and where} \qquad \lambda^2 = 1 - (x_1 - c(t)) \,/\, R(t) \; . \qquad (4)$$

The function $k(t)$ depends upon the deviatoric and hydrostatic creep functions which characterize the isotropic viscoelastic properties of the bulk material (see McCartney, 1978). For an elastic material and plane strain conditions $k(t) = (1-\nu^2)/E$ where E is Young's modulus and ν is Poisson's ratio. The relations (1-4) can be applied to any fracture specimen for which the stress intensity factor is known provided that small scale crazing conditions prevail.

CRACK GROWTH LAW DERIVED FROM THE COD FRACTURE CRITERION

The COD fracture criterion asserts that crack growth takes place such that the crack tip COD always has a fixed value δ . Thus the relation (2) asserts that the following equation must be satisfied whenever crack growth takes place:

$$4\pi \int_0^t k(t-\tau)\, \frac{\partial \Omega(c(t),\tau)}{\partial \tau}\, d\tau = \delta \; . \qquad (5)$$

On integrating by parts and on making use of the relation (3) it can be shown that the crack growth equation (5) may be written in the form

$$\frac{\pi\delta}{8k(0)\,\sigma_c(t)R(t)} = 1 + \int_0^t \frac{\dot{k}(t-\tau)}{k(0)}\left\{\mu - \tfrac{1}{2}(1-\mu^2)\ln\left(\frac{1+\mu}{1-\mu}\right)\right\}\frac{\sigma_c(\tau)R(\tau)}{\sigma_c(t)\,R(t)}\, d\tau \quad , \qquad (6)$$

$$\text{where} \qquad \mu^2 = 1 - (c(t) - c(\tau)) \,/\, R(\tau) \; . \qquad (7)$$

Having specified the function $\sigma_c(t)$ and the time-dependence of the applied load the equation (6) determines the dependence of the crack length on time, i.e. $c(t)$ is the unknown function of the equation. Assuming that $\delta > 8k(0)\,\sigma_c(0)R(0)/\pi$ the equation cannot be satisfied until the applied load has acted for an interval of time which is sufficient for the crack tip COD to increase to the value δ . An incubation period must therefore lapse before crack growth takes place.

The history-dependent nature of the crack growth equation (6) means that the incubation period will affect the subsequent crack growth. However the effect is expected to diminish to negligible proportions when the crack has propagated a distance which is equivalent to several craze zone lengths. It is assumed that incubation effects are negligible and that

$$\left.\begin{array}{ll} \text{(i)} & \sigma_c(\tau)R(\tau) \simeq \sigma_c(t)R(t) \; , \\ \text{(ii)} & (c(t) - c(\tau)) \,/\, R(\tau) \simeq \dot{c}(t)(t-\tau) \,/\, R(t) \; , \end{array}\right\} \quad t^* \leq \tau \leq t \; , \qquad (8)$$

where $c(t) = c(t^*)+R(t^*)$, so that $t-t^*$ is the time taken for the crack tip to move a distance $R(t^*)$. The crack growth equation (6) may then be written in the form

$$\left\{ \frac{\sigma_c(t)\delta}{k(0)} - K^2(t) \right\} \dot{c}(t) = \frac{\pi K^4(t)}{8\sigma_c^2(t)} \int_0^1 \frac{\dot{k}(\{1-\mu^2\}\alpha(t))}{k(0)} \left\{ 2\mu - (1-\mu^2)\ln\left(\frac{1+\mu}{1-\mu}\right) \right\} \mu\, d\mu , \tag{9}$$

$$\text{where} \quad \alpha(t) = \pi K^2(t) \,/\, \{8\sigma_c^2(t)\dot{c}(t)\} . \tag{10}$$

The approximations (8) will be discussed later but it should be noted that when $\sigma_c(t) = \sigma_c$, a constant, and steady state conditions prevail then the crack growth equation (9) is consistent with previous results (Kostrov and Nikitin, 1970; Knauss, 1974; Rice, 1978; McCartney, 1979). The equation (9) is capable of further reduction when the creep function $k(t)$ is specified (see McCartney (1979) for details).

CRACK GROWTH LAW DERIVED FROM ENERGY BALANCE CONSIDERATIONS

The energy balance fracture criterion (see for example Wnuk, 1971; Knauss, 1974; McCartney, 1979) may be expressed in the form

$$\sigma_c(t) \int_{c(t)}^{c(t)+R(t)} \frac{\partial \Delta u_2(x_1,t)}{\partial t}\, dx_1 = 2\Gamma \dot{c}(t) , \tag{11}$$

where Γ is the fracture energy and where it has been assumed that the craze stress is uniform in the craze zone but time-dependent. By making use of the relation (2) the crack growth equation (11) can be expressed in the form (see McCartney, 1979 eq.(7))

$$\frac{\Gamma \dot{c}(t)}{2\pi k(0)\,\sigma_c(t)} = \int_{c(t)}^{c(t)+R(t)} \frac{\partial \Omega(x_1,t)}{\partial t}\, dx_1 + \int_{t*}^{t} \frac{\dot{k}(t-\tau)}{k(0)} \left\{ \int_{c(t)}^{c(\tau)+R(\tau)} \frac{\partial \Omega(x_1,\tau)}{\partial \tau}\, dx_1 \right\} d\tau ,$$

which is valid for times t such that $c(t) \geq c(0)+R(0)$ so that $t* \geq 0$. For times t such that $c(t) < c(0)+R(0)$ the limit $t*$ of the τ integration must be replaced by zero. On integrating by parts the equation may be written

$$\frac{\Gamma \dot{c}(t)}{2\pi k(0)\,\sigma_c(t)} = \int_{c(t)}^{c(t)+R(t)} \frac{\partial \Omega(x_1,t)}{\partial t}\, dx_1 + \frac{\dot{k}(0)}{k(0)} \int_{c(t)}^{c(t)+R(t)} \Omega(x_1,t)\, dx_1 + \int_{t*}^{t} \frac{\ddot{k}(t-\tau)}{k(0)} \left\{ \int_{c(t)}^{c(\tau)+R(\tau)} \Omega(x_1,\tau)\, dx_1 \right\} d\tau , \tag{12}$$

where use has been made of the fact that $\Omega(c(\tau)+R(\tau),\tau) = 0$. Now it can be shown that

$$\int_{c(t)}^{c(t)+R(t)} \Omega(x_1,t)\, dx_1 = \frac{K^4(t)}{96\sigma_c^3(t)} , \tag{13}$$

$$\int_{c(t)}^{c(t)+R(t)} \frac{\partial \Omega(x_1,t)}{\partial t}\, dx_1 = \frac{d}{dt}\left\{ \frac{K^4(t)}{96\sigma_c^3(t)} \right\} + \frac{K^2(t)\dot{c}(t)}{4\pi\sigma_c(t)} , \tag{14}$$

$$\int_{c(t)}^{c(\tau)+R(\tau)} \Omega(x_1,\tau)\, dx_1 = \frac{K^4(\tau)}{384\sigma_c^3(\tau)} \left\{ 10\mu^3 - 6\mu + 3(1-\mu^2)^2 \ln\left(\frac{1+\mu}{1-\mu}\right) \right\} , \tag{15}$$

where μ is defined by (7). Thus the crack growth equation (12) may be written

$$\left\{ \frac{2\Gamma}{k(0)} - K^2(t) \right\} \dot{c}(t) = \frac{\pi\sigma_c(t)}{24} \frac{d}{dt}\left\{ \frac{K^4(t)}{\sigma_c^3(t)} \right\} + \frac{\pi K^4(t)}{24\sigma_c^2(t)} \frac{\dot{k}(0)}{k(0)} + \frac{\pi}{96}\sigma_c(t) \int_{t*}^{t} \frac{\ddot{k}(t-\tau)}{k(0)} \frac{K^4(\tau)}{\sigma_c^3(\tau)} \left\{ 10\mu^3 - 6\mu + 3(1-\mu^2)^2 \ln\left(\frac{1+\mu}{1-\mu}\right) \right\} d\tau . \tag{16}$$

In order to proceed analytically it is now assumed that

$$\left.\begin{aligned} &\text{(i)}\ K^4(\tau)/\sigma_c^3(\tau) \simeq K^4(t)/\sigma_c^3(t) , \\ &\text{(ii)}\ (c(t)-c(\tau)) \,/\, R(\tau) \simeq \dot{c}(t)(t-\tau) \,/\, R(t) , \end{aligned}\right\} \quad t* \leq \tau \leq t \tag{17}$$

in which case the crack growth equation (16) may be approximated as follows

$$\left\{ \frac{2\Gamma}{k(0)} - K^2(t) \right\} \dot{c}(t) = \frac{\pi\sigma_c(t)}{24} \frac{d}{dt}\left\{ \frac{K^4(t)}{\sigma_c^3(t)} \right\} + \frac{\pi K^4(t)}{24\sigma_c^2(t)} \frac{\dot{k}(0)}{k(0)} + \frac{\pi K^4(t)}{48\sigma_c^2(t)} \alpha(t) \int_0^1 \frac{\ddot{k}(\{1-\mu^2\}\alpha(t))}{k(0)} \left\{ 10\mu^3 - 6\mu + 3(1-\mu^2)^2 \ln\left(\frac{1+\mu}{1-\mu}\right) \right\} \mu\, d\mu ,$$

where $\alpha(t)$ is defined by (10). On integrating by parts the resulting crack growth equation has the approximate form

$$\left\{ \frac{2\Gamma}{k(0)} - K^2(t) \right\} \dot{c}(t) = \frac{\pi\sigma_c(t)}{24} \frac{d}{dt}\left\{ \frac{K^4(t)}{\sigma_c^3(t)} \right\} + \frac{\pi K^4(t)}{8\sigma_c^2(t)} \int_0^1 \frac{\dot{k}(\{1-\mu^2\}\alpha(t))}{k(0)} \left\{ 2\mu - (1-\mu^2) \ln\left(\frac{1+\mu}{1-\mu}\right) \right\} \mu\, d\mu , \tag{18}$$

which can be further reduced when the function $k(t)$ is specified (see McCartney, (1979) for details). A comparison of the growth laws (9) and (18) shows that they are of the same form apart from an extra term on the R.H.S. of (18). Furthermore the COD equation (9) predicts that the value of K at instability ($\dot{c} \to \infty$) is time-dependent whereas the energy equation (18) leads to a time-independent value provided that δ and Γ are constants.

DISCUSSION

The COD and energy balance fracture criteria lead respectively to the integral equations (6) and (16) which assume only that small scale crazing conditions prevail. For constant craze stresses $\sigma_c(t) = \sigma_c$ and specimens loaded so that the craze length has the constant length R during crack growth, the equations (6) and (16) assert that the crack propagates at a constant velocity $\dot{c}$ satisfying the following equation (which is exact for small scale crazing conditions when using the energy balance fracture criterion)

$$\left\{ \frac{\sigma_c \delta}{k(0)} - K^2 \right\} \dot{c} = \frac{\pi K^4}{8\sigma_c^2} \int_0^1 \frac{\dot{k}(\{1-\mu^2\}\alpha)}{k(0)} \left\{ 2\mu - (1-\mu^2)\ln\left(\frac{1+\mu}{1-\mu}\right) \right\} \mu d\mu \ , \tag{19}$$

where $\alpha = R/\dot{c} = \pi K^2/(8\sigma_c^2 \dot{c})$, and where it has been assumed that $2\Gamma = \sigma_c \delta$. As shown by Knauss (1974) and more explicitly by McCartney (1979) the COD and energy balance fracture criteria can lead to the same equation governing crack growth. It should be noted that the steady state crack growth equation (19) corresponds exactly to the relation derived by Kostrov and Nikitin (1970) using the COD fracture criterion.

For non-steady conditions the only result which is exact for small scale crazing is obtained from the energy balance equation (16) when the creep function $k(t)$ has the linear form $k(t) = k(0) + \dot{k}(0)t$. The resulting crack growth law may be written in the form

$$\left\{ \frac{2\Gamma}{k(0)} - K^2(t) \right\} \dot{c}(t) = \frac{\pi K^4(t)}{6\sigma_c^2(t)} \left\{ \frac{\dot{K}(t)}{K(t)} - \frac{3}{4}\frac{\dot{\sigma}_c(t)}{\sigma_c(t)} + \frac{1}{4}\frac{\dot{k}(0)}{k(0)} \right\} \ , \tag{20}$$

which asserts that a craze stress decreasing with time has the effect of increasing the crack tip velocity. For more general creep functions $k(t)$ approximations of the form (8) or (17) must be made in which case the COD and energy balance fracture criteria lead to the equations (9) and (18) respectively. Consider first of all the approximation (8ii) which is identical to (17ii). It is very useful to write

$$(c(t) - c(\tau)) \,/\, R(\tau) = \dot{c}(t)(t-\tau)\{1 + \epsilon(\tau,t)\} \,/\, R(t) \ , \tag{21}$$

so that the approximations (8ii) and (17ii) are valid when $\epsilon(\tau,t) << 1$ and $\dot{c}(\tau)$ is bounded for all τ such that $t^* \le \tau \le t$. For most practical applications $\ddot{c}(t) \ge 0$ and $\dot{R}(t) \ge 0$ for all times $t > 0$. It then follows that $\{c(t)-c(\tau)\}/(t-\tau) \le \dot{c}(t)$ and $R(\tau) \ge R(t^*)$ for $t^* \le \tau \le t$ in which case (21) asserts that

$$|1 + \epsilon(\tau,t)| = \left| \frac{c(t)-c(\tau)}{t-\tau} \, \frac{R(t)}{R(\tau)} \, \frac{1}{\dot{c}(t)} \right| \le \frac{R(t)}{R(t^*)} \ , \quad t^* \le \tau \le t.$$

It is therefore deduced that the approximations (8ii) and (17ii) are valid whenever the crack tip velocity is bounded and

$$\{R(t) - R(t^*)\} \,/\, R(t^*) << 1 \ . \tag{22}$$

Thus the approximations are valid when the increase in length $R(t)-R(t^*)$ of the craze zone resulting from the crack growth increment $R(t^*)$ is very much less than the original craze length $R(t^*)$. It is clear that if in addition

$$\{\sigma_c(t) - \sigma_c(t^*)\} \,/\, \sigma_c(t^*) << 1 \ , \tag{23}$$

then the conditions (8i) and (17i) are also satisfied.

It is worthwhile comparing the steady state growth law (19) with the more general equations (9) and (18) which were derived respectively from the COD and energy balance fracture criteria. Provided that the conditions (22) and (23) are satisfied the COD equation (9), valid for non-steady conditions, has precisely the same form as the steady state growth law (19). However the energy balance equation (18), valid for non-steady conditions provided that (22) and (23) are satisfied, is not of the steady state form (19) because of the presence of the first term on the R.H.S. of (18). When modelling the creep behaviour of real materials the quantity $\dot{k}(t)$ is usually a decreasing function of time and it then follows that

$$\frac{1}{3}\frac{\dot{k}(\alpha(t))}{k(0)} \le \int_0^1 \frac{\dot{k}(\{1-\mu^2\}\alpha(t))}{k(0)} \left\{ 2\mu-(1-\mu^2)\ln\left(\frac{1+\mu}{1-\mu}\right) \right\} \mu d\mu \ .$$

Thus the energy balance equation (18) has the same form as the steady state crack growth law (19) whenever

$$\left| \frac{\sigma_c^3(t)}{K^4(t)} \frac{d}{dt} \left\{ \frac{K^4(t)}{\sigma_c^3(t)} \right\} \right| << \frac{\dot{k}(R(t)/\dot{c}(t))}{k(0)} \le \frac{\dot{k}(0)}{k(0)} \ . \tag{24}$$

Consider now the special case $\sigma_c(t) = \sigma_c$, a constant, so that the condition (23) is automatically satisfied. Assuming that $\ddot{k}(t) \le 0$, a characteristic of most real materials, it follows that the condition (24) may be written

$$4\dot{K}/K << \dot{k}(R/\dot{c})/k(0) \le \dot{k}(0)/k(0) \ . \tag{25}$$

Now the condition (22) will be satisfied whenever $0 \le \dot{R}(\tau) << \dot{c}(\tau)$ for $t^* \le \tau \le t$ since

$$\int_{t^*}^{t} \dot{c}(\tau)d\tau = c(t) - c(t^*) = R(t^*) >> \int_{t^*}^{t} \dot{R}(\tau)d\tau = R(t) - R(t^*) \ .$$

When $\sigma_c(t)$ is a constant the condition $\dot{R} << \dot{c}$ may be recast in the form

$$\dot{K}/K << \dot{c}/(2R) \ , \tag{26}$$

where use has been made of the relation (1). The relevance of the conditions (25) and (26) when comparing crack growth laws for steady and non-steady conditions has been discussed recently in the literature (Knauss 1979,1980; McCartney 1980). It is deduced from the analysis presented in this paper that

(a) the condition (26) proposed by Knauss (1976,1979,1980) is sufficient to ensure that the COD equation (6), valid for non-steady conditions, can be approximated by the steady growth law (19) when applied to non-steady conditions.

(b) the condition (26) is sufficient for the energy balance equation (16) to be approximated by (18) but insufficient for (18) to be approximated by the steady state growth law (19). The additional condition (25) must also be satisfied, in keeping with earlier remarks (McCartney 1980).

Consider stress intensity factors of the form $K = Lf(c)$ where L is the applied load and f is some geometry-dependent function of the crack length. When L is held fixed it follows that the condition (26) may be expressed $R/c << f(c) \,/\, \{2cf'(c)\}$ which is normally automatically satisfied when small scale crazing conditions prevail. Thus for small scale crazing the condition (26) is placing a restriction on the rate of increase of the applied load L.

Finally it is worth noting that Schapery (1975) has proposed a localized energy fracture criterion of the form

$$2\Gamma = \int_{t*}^{t} \sigma_c(\tau) \frac{\partial \Delta u_2(c(t),\tau)}{\partial \tau} d\tau \ , \ c(t) \geq c(0)+R(0), \tag{27}$$

which reduces to the COD criterion when $\sigma_c(\tau) = \sigma_c$ a constant. For time-dependent craze stresses the relation (27) is the basis for yet another crack growth law which would reduce to the form (19) for steady conditions.

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