# THE DETERMINATION OF STRESS INTENSITY FACTORS ON ELASTIC SHELLS : A POSSIBLE METHOD

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## ABSTRACT

We describe a method to compute the stress intensity factors at the tip of a crack located on a general shell. This method permits to calculate stress intensity factors defined out of KIRCHHOFF's hypothesis by means of finite elements in the formulation of which this hypothesis has been retained. The method is used to solve the problem of the crack located in an infinite plate subjected to a uniform traction at infinity. It is used afterwards to solve a practical problem and the results are compared to those abstained in a different way.

## INTRODUCTION

Oil industry uses different types of thin shells. Vessels, pipes, offshore constructions are just examples of two-dimensional structures and it may occur that cracks are detected in some of these shells. They can generally be immediately repaired but it may occur that production or environmental conditions have the repairing delayed. In this case, it is however necessary to make sure that, until the crack can be repaired, it will not grow in fatigue to a catastrophic extend. L.E.F.M makes this prediction possible as soon as the stress intensity factors are known.

Five stress intensity factors have been defined on shells and they have been calculated for some particular geometries and loadings. On the contrary, the numerical method proposed here is not specific of a geometry and can be used as long as the region surrounding the crack tip satisfies all the thin shell theory requirements. The whole structure calculation which is necessary is performed by means of finite elements technics. Now, the most commonly used finite elements are based on KIRCHHOFF'S hypothesis which hypothesis must be abandonned to define the stress intensity factors. The present method takes this contradiction into account and makes it possible to compute the stress intensity factors by means of KIRCHHOFF type elements.

We shall, first, describe the method and we shall apply it, afterwards, to an infinite plate subjected to an uniform traction at infinity and containing a straight crack. This is just a plane problem but as the results are found to be in good agreement with the corresponding analytical solution, we shall state that the proposed method may give good results in general two dimensional problems. So we shall apply the method to calculate the stress intensity factors of a real crack. Their values will be compared to those obtained in an approximate way by means of some published results.

## DESCRIPTION OF THE METHOD

Let us consider a cracked shell. It is always possible to define a set of cartesian coordinates $x_1$ $x_2$ in the plane tangent to the middle surface of the shell at one of the crack ends. The origin of the coordinates will be located at the crack tip and $x_1$

will be tangent to the crack. In these local axis, $u_1$ and $u_2$ will denote the in-plane displacement, w the displacement along the normal, $\beta_1$ and $\beta_2$ the two components of the total rotation of the normal and r and $\theta$ the usual polar coordinates. It has been shown (1), (2) that, when the displacement and rotation of the crack front are zero, the first term of the development of the kinematic unknowns was given by :

$$u_1 = \frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left\{ K_s \left[(2x-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + K_a \left[(2x+3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]\right\}$$

$$u_2 = \frac{1+\nu}{2E}\left(\frac{r}{2\pi}\right)^{1/2}\left\{ K_s \left[(2x+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right] + K_a \left[(3-2x)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]\right\}$$

$$w = \frac{16\,(1+\nu)}{E}\left(\frac{r}{2\pi}\right)^{1/2} K_a^{**} \sin\frac{\theta}{2} \qquad (1)$$

$$\beta_1 = \frac{1+\nu}{Eh}\left(\frac{r}{2\pi}\right)^{1/2}\left\{ K_s^{*} \left[(2x-1)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right] + K_a^{*} \left[(2x+3)\sin\frac{\theta}{2} + \sin\frac{3\theta}{2}\right]\right\}$$

$$\beta_2 = \frac{1+\nu}{Eh}\left(\frac{r}{2\pi}\right)^{1/2}\left\{ K_s^{*} \left[(2x+1)\sin\frac{\theta}{2} - \sin\frac{3\theta}{2}\right] + K_a^{*} \left[(3-2x)\cos\frac{\theta}{2} - \cos\frac{3\theta}{2}\right]\right\}$$

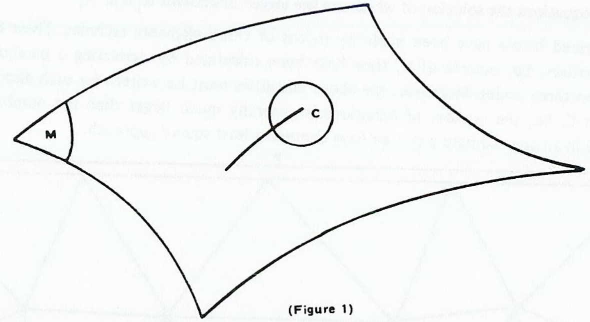
with $x = \frac{3-\nu}{1+\nu}$

$\nu$ and E are the usual elastic constants. So, the displacements and rotations originating singular stresses are perfectly known out of the five stress intensity factors $K_s$, $K_a$, $K_s^*$, $K_a^*$, $K_a^{**}$.

The above relations have been obstained by means of a shell theory rejecting KIRCHHOFF's hypothesis. In other words, to perform numerical calculations perfectly consistent with the definition of the stress intensity factors, it is necessary to use finite elements based on the same shell theory. Now, it is obvious that such elements are not available in most of the codes and, moreover, that far form the crack tip, the usual elements are satisfactory. So, the idea consists in using the usual elements out of the elastic singularity and in modeling the singularity behaviour by means of relations (1).



Let us consider (figure 1) the cracked shell and let us remove, around the crack tip a small surface bounded by a curve C, the highest dimension of C being such that, at any point of C, the singular solution prevails.




(Figure 1)

This shell is submitted to the following loadings :

- The external forces are applied and the nodes located on C are fixed (zero displacements and rotations). The calculated displacements and rotations are denoted by $V_o$.

- Let $K_i$ denote one of the five stress intensity factors. The singular displacements or rotations calculted on C with $K_i = 1$, the other K being zero, are imposed to the points of C. There are five loadings of this type and the corresponding displacements and rotations are denoted by $\overline{S_i}$.

- Let $R_i$ denote a rigid body displacement or rotation of C. This rigid body movement is applied to C with $R_i = 1$ and the corresponding calculated displacements and rotations are denoted by $\overline{T_i}$. There are six displacements of this type.

So, $K_i$ and $R_i$ being the unknown stress intensity factors and rigid body displacement and rotations of C, the displacement $\overline{V}$ of any point of $\Sigma$ is given by :

$$\overline{V} = \overline{V_o} + \sum_{i=1}^{5} K_i \overline{S_i} + \sum_{i=1}^{6} R_i \overline{T_i}$$

In the same way, with evident notations, the displacement of any point inside C is given by :

$$V = \sum_{i=1}^{5} K_i S_i + \sum_{i=1}^{6} R_i T_i$$

$S_i$ denotes the displacement calculated by (1) with $K_i = 1$, the other stress intensity factors being zero, and $T_i$ denotes a unit rigid body movement of C. So as, on C, $\overline{V_o} = 0$, $S_i = \overline{S_i}$ and $T_i = \overline{T_i}$, V and $\overline{V}$ are equal along this curve so that the displacement and its tangent derivative i.e. derivative along C are continuous across C. But these continuities do not imply the continuity of the normal derivative i.e. derivative along the external normal to C in the tangent plane to $\Sigma$. Denoting by a' the normal derivation operator, we must have, on C :

$$V' = \sum_{i=1}^{5} K_i S_i' + \sum_{i=1}^{6} R_i T_i' = \overline{V_o}' + \sum_{i=1}^{5} K_i \overline{S_i}' + \sum_{i=1}^{6} R_i \overline{T_i}' = \overline{V}'$$

These relations give :

$$\sum_{i=1}^{5} K_i (S_i' - \overline{S_i'}) + \sum_{i=1}^{6} R_i (T_i' - \overline{T_i'}) = \overline{V_o'}$$

This is a set of linear equations the solution of which are the eleven unknowns $R_i$ and $K_i$.

The calculations described below have been made by means of finite elements technics. These technics do not give the required normal derivatives. So, outside of C, they have been calculated by derivating a parabolic interpolation of the corresponding d.o.f. on three nodes. Moreover, the above equalities must be written for each displacement and rotation of each node located on C. So, the number of equations is generally much larger than the number of unknowns and the system must be solved in an approximate way ; we have choosen a least square approach.

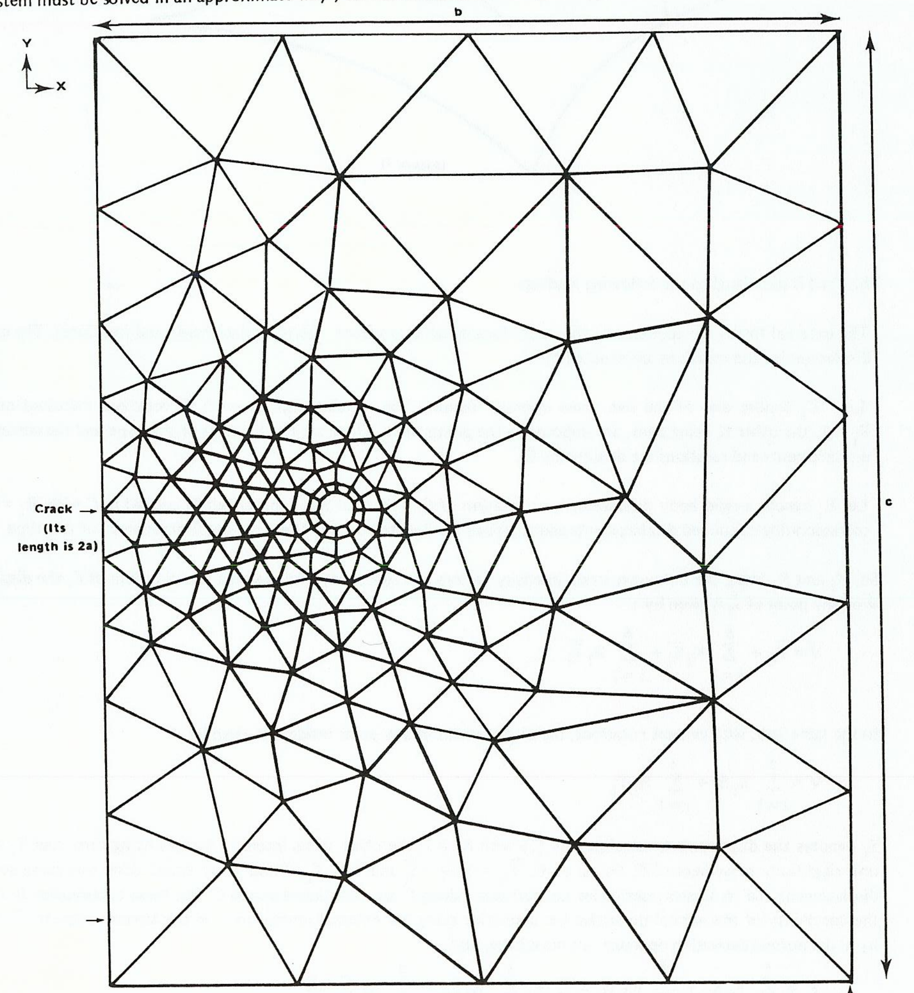



Figure 2 - Mesh used to test he method



## SAMPLE PROBLEM

To test the method, let us solve the problem of the infinite plate submitted to a uniform traction at infinity. The numerical resolution will be made by means of constant strain triangles and linear strain quadrilaterals. The plate is limited to a rectangle 2bxc the half of which, bxc, is modeled (figure 2). The crack length is equal to 2a and we have :

$$\frac{c}{a} = 4$$

$$\frac{b}{a} = 3.2$$

Comparing the calculated $K_s$ and $K_a$ to the theoretical value $\sigma\sqrt{\pi a}$ of $K_s$ we get :

$$\frac{K_s}{\sigma\sqrt{\pi a}} = 1.11$$

$$\frac{K_a}{\sigma\sqrt{\pi a}} = 0.0025$$

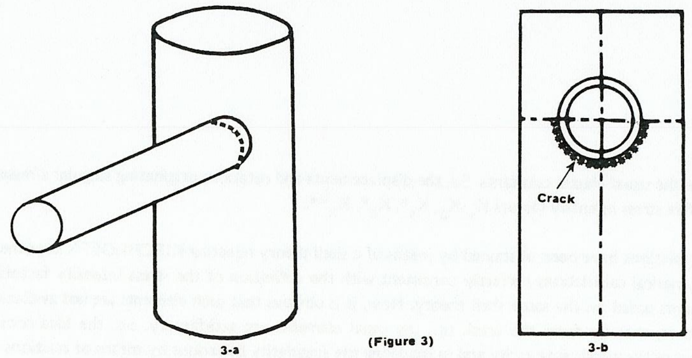
The components $u_1$ and $u_2$ of the computed rigid body displacement of the crack tip can be compared to those, $u'_1$ and $u'_2$ of the point located at the crack tip in the same uncracked plate. We get :

$$\frac{u_1}{u'_1} = 3.92 \qquad \frac{u_2}{u'_2} = 0.98$$

The limited plate we retained has a $\frac{b}{a}$ ratio of 3.2 which is a poor approximation of infinity. So, the value 1.11 calculated for $\frac{K_s}{\sigma\sqrt{\pi a}}$ is an upper bound of the real ratio. In the same way, the ratio $\frac{u_1}{u'_1}$ is meaningless due to the fact that the crack region is less stiff than an uncracked plate. So, it can be concluded from this sample problem that the proposed method is satisfactory.

## REAL PROBLEM

When inspecting the pipe intersection represented on figure 3, a crack was detected. Its length was half of the intersection and it was immediatly decided to loose the two pipes by sawing the small one as close of the intersection as possible.




(Figure 3)

But, as the main pipe was submitted to an alternated bending moment, it was necessary to make sure that the crack could not grow in fatigue unto instability until it was possible to repair the crack properly. So, it was necessary to calculate the stress intensity factors.

The crack being entirely on a geometric singularity, it is not possible to use a shell theory in the computations. However, the nature of the loading indicates that the crack will become dangerous for the safety of the structure only if it growth along a director circle i.d. perpendicularly to its ends. In that case, after growing has taken place, the crack enters the domain of the theory of shells and it becomes possible to apply the present method. So, we have imagined two crack extensions, different in length, and we have modeled the half of the corresponding geometries. The corresponding meshes are plotted on figure 4.

Figure 4 - Meshes used for finite elements calculations

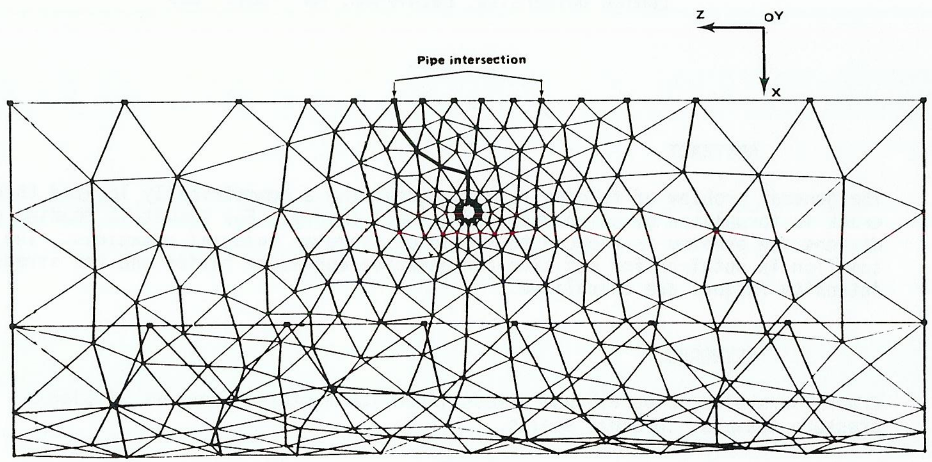
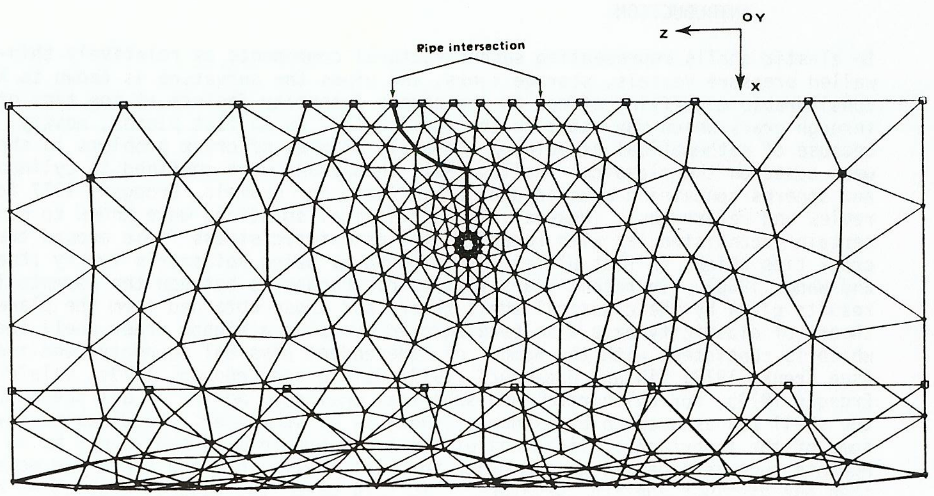



Figure 4-a




(Figure 4-b)



The external moment was applied to the structure through axial forces concentrated on the nodes located at the two ends of the cylinder. Although the two problems are identical, they have been solved in two different ways : for the smallest crack extension, the rigid body displacement parallel to the plane of symetry has not been fixed. The computation, has given a high value for this displacement and, to test its influence on the accuracy of the least square approach, we have solved the linear system a second time after having substracted this rigid body displacement from the calculated displacements. It will be noticed that the stress intensity factors have not been significantly modified. As far as the second crack is concerned, the supports have been choosen so that the rigid body displacement of the structure was zero. The results are given in table 1.

**TABLE 1 : Results of present calculations**

| Unknown | Crack n°1 | | Crack n° 2 |
|---|---|---|---|
| | Rigid body displacement Included | Rigid body displacement substracted | Rigid body displacement included |
| $K_s$ | 146.8 | 147.1 | 205.0 |
| $K_s^*$ | - 4.3 | - 4.6 | - 6.4 |
| $K_a$ | - 0.9 | - 1.1 | 1.1 |
| $K_a^*$ | 1.6 | - 1.4 | - 0.2 |
| $K_a^{**}$ | - 0.1 | 0.1 | 0.1 |
| $U_x$ | - 992.8 | 128.5 | - 0.66 |
| $U_y$ | 1 933.8 | - 250.3 | 0.96 |
| $U_z$ | 136.4 | -325.2 | 0.30 |
| $R_x$ | $1.9\,10^{-4}$ | $-1.7\,10^{-4}$ | $2.7\,10^{-5}$ |
| $R_y$ | $-0.6\,10^{-4}$ | $-0.6\,10^{-4}$ | $3.1\,10^{-5}$ |
| $R_z$ | $-2.9\,10^{-5}$ | $-0.5\,10^{-4}$ | $1.3\,10^{-3}$ |

$K_s$, $K_s^*$, $K_a$, $K_a^*$, $K_a^{**}$ are expressed in $kgf/mm^2\sqrt{mm}$

$U_x$,$U_y$,$U_z$ components of the displacement of the crack tip, are expressed in mm

$R_x$,$R_y$,$R_z$ components of the rotation of the crack tip, are expressed in radians.

The results indicate that the stress intensity factor $K_s$ corresponding to the mode I membrane forces is much larger than any other. The asymetry of the crack and the reinforcement due to the presence of the remaining part of the small pipe have not a real influence and the problem is just symetric. In a similar way, it can be noticed that the symetric bending stress intensity factor $K_s^*$ is, in both cases, close to 3 % of $K_s$ i.d. non significant.

We have compared the results with those obtained in (3). If we just consider the case of a radial crack located in a cylinder pulled in traction and if we take as the traction stress the highest value of the axial stress on the uncracked cylinder, we obtain the results given in table 2.

**TABLE 2 : Stress intensity factors computed using results given in (3)**

| | $K_s$ | $K_s^*$ | $K_s + \|K_s^*\|$ |
|---|---|---|---|
| Crack n°1 | 135.81 | - 16.56 | 152.37 |
| Crack n°2 | 167.46 | - 39.01 | 206.47 |

The maximum symetric stress intensity factor is obtained on one of the two external surfaces of the shell and is equal to $K_s + | K_s^* |$ . The values of this parameter obstained by each method differ by less than 2,4 %. However, the methods do not give the same results : the values of the membrane stress intensity factor $K_s$ differ by 18 % and the ratios $K_s/K_s^*$ are very different.

## CONCLUSION

We have described a method to calculate stress intensity factors on a general shell. This method uses Kirchhoff type plate bending finite elements and is simple in use. The results obtained are acceptable but, as any new method, it has still to be tested on other problems and compared to other methods.

## REFERENCES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