DEFECT FORCES, DEFECT COUPLES AND PATH INTEGRALS

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ABSTRACT

Definition and meaning of concepts like "J integral" are given without any assumption about material behaviour. The key of the work is the field of "defect forces" and "defect couples" in a continuous media. These forces and couples, which can also be called "material forces" and "material couples" are related to the work done by a particle moving through a solid. It is shown that the resultant of all the defect forces included in a volume is the J_k integral computed on the surface surrounding this volume. A similar result is obtained about the moment resultant. Conventional form of the principle of virtual work is not applicable to fractures mechanics because equations of compatibility are not satisfied. A generalized form is given, which is valid when (virtual) crack propagation is considered. The virtual work of "material" forces is included in the generalized form, and can be used as a new definition of J concept.

As an illustration application, a simple procedure is described which allows to obtain the curve J-Δa (the so called J-R curve) from only one experimental.

KEY WORDS

Fracture mechanics, Elastic plastic material, J integral, Path integral, Material forces, Virtual work, Energy momentum tensor, J-R curve, J measurement.

INTRODUCTION

The J integral method suggested by RICE (1968) has received considerable attention. This concept is not only used as a criterion of onset of crack propagation, but also used to study propagation stability (PARIS 1977). The definition and meaning of J concept can be considered in different ways : the path integral, the variation of energy with the crack length (Sumpter and Turner, 1976), the factor characterizing the crack type singularity (Hutchinson, 1968). This concept can also considered to be derivated from notions coming from the electromagnetic field theory and extended to the mechanics of elastic solids by ESHELBY (1970). Except for special circumstances (BILBY, 1977), the J integral is path independent only for elastic material (linear or non linear) although Rice's definition of J is applicable to more general materials and J is calculated for elastic plastic fields. The aim of this paper is to give another discussion of J without making assumptions about the behaviour of the material (ROCHE, 1976, 1977).

The method can be considered as a generalization of the Energy momentum tensor approach to solid mechanics (Eshelby, 1975) but with very significant changes made to avoid any assumption about the behaviour of the material. It is based on the definition in a continuous medium of a field of vectors having the dimension of a force. These vectors are in direct connexion with the work done by a material particle moving through the solid. Such a displacement through the solid may be called "material displacement" and the related vectors "material forces" or "defect forces". Such a concept of force was also considered for the elastic behaviour only, by ROGULA (1977) and by CASAL (1978) who names it "Suction force". In the elastic case ESHELBY (1970) has also given a related discussion applied particularly to forces on interfaces.

DEFINITION OF DEFECT FORCES

The aim of this section is to give the definition and the mathematical expression of defect forces. Like conventional volume body forces and surface body forces (corresponding to conventional or spatial displacement) it will be given expressions of volume defect forces j_k and surface defect forces $\bar{j}_k$ which are volume or surface density of defect (or material) forces. But forces alone are not sufficient and for a more complete discussion a defect couple must be also considered, hence expressions of volume defect couples l_k and surface defect couples $\bar{l}_k$ will be also given. Defect couples are like conventional body couples in iron submitted to magnetic field. Movement of a material particle will be considered through the neighboring particles (Eshelby cutting and welding argument, 1975). Due to that movement there is a variation of the stress working density of the particle. Part of this variation is only a consequence of the geometrical change of the particle position. The balance can be attributed to the modifications in the particles order (material displacement). This part is related to defect forces or defect couples. Its expression is the scalar product of the defect force by the translation of the particle through the solid and/or the scalar product of the couple defect by the rotation of the particle in relation to the solid.

A point of the material will be identified by its cartesian coordinates x_i in the initial state (Lagrangian formulation). By the action of body forces X_i (per unit of volume) and $\bar{X}_i$ (surface forces), the point is displaced to reach $x_i + u_i$ (u_i is the spatial displacement), exhibiting a state of strain ε_{ij} and of stress σ_{ij}. Obviously the internal forces (stresses) have done work, and the stress working density is W.

From a rigourous point of view, attention must be given to the definition of strain and stress. If u_i and ε_{ij} are small, conventional definitions can be used. If they are not small, it is necessary to choose the displacement gradient as the strain $\varepsilon_{ij} = u_{i,j}$ and[1] the Boussinesq nominal stress tensor as the stress, so that the variation of stress working density can be written $\delta W = \sigma_{ij}\ \delta u_{i,j}$.

In the strained state, if a translation δx_k of a material particle is considered, (δx_k is a material displacement, it is to say a variation of initial coordinates), there is a variation of the particle stress working density equal to $\sigma_{ij}\ \varepsilon_{ij,k}\ \delta x_k$, which is greater than the increase $W_{,k}\ \delta x_k$ corresponding to geometrical change only. Hence the increase δW in stress working density due intrinsically to material translation δx_k can be written :

$$\delta W = -\ j_k\ \delta x_k \tag{1}$$

$$j_k = W_{,k} - \sigma_{ij}\ \varepsilon_{ij,k} \tag{2}$$

where j_k is the defect force per unit of volume. By the same procedure, the expression of surface defect force is found to be written :

$$\bar{j}_k = -\ (W n_k - T_i\ u_{i,k}) \tag{3}$$

where $T_i = \sigma_{ij}\ n_j$ and n_j the normal in the jump direction (in case of discontinuity)[2] or the outside normal (body boundary).

Now, if a rotation $\delta\omega_k$ of the particle is considered the variation δW of stress working density due intrinsically to this material rotation can by written :

$$\delta W = -\ l_k\ \delta\omega_k \tag{4}$$

$$l_k = -\ e_{kmi}\ (\sigma_{ip}\ \varepsilon_{mp} + \sigma_{pi}\ \varepsilon_{pm}) \tag{5}$$

where l_k is the defect couple per volume unit, the notation e_{kmi} expressing the alternating tensor which is completely antisymetric (cartesian coordinates).

There are also surface defect couples which can be written :

$$\bar{l}_k = -\ e_{kmi}\ u_m\ T_i \tag{6}$$

PATH INTEGRALS AS THE RESULTANTS OF MATERIAL FORCES

In two dimensional problems, the value of J is given by a path integral. In three dimensional problems J is a vector (having the

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[1] A comma followed by suffixes will denote differentiation with respect to x, so that, for example $u_{i,j} = \dfrac{\partial u_i}{\partial x_j}$

[2] In case of surface discontinuity, W is the "jump" of stress working, and $u_{i,k}$ the "jump" of strain gradient.

dimensions of a force), and the values of its components are given by a surface integral. A very simple computation give the following result :

$$J_k = - \int_S \bar{j}_k \, ds = \int_V j_k \, dv + \int_\Sigma \bar{j}_k \, ds \qquad (7)$$

The resultant of all the defect forces contained in a volume V (including defect forces connected with surface discontinuities Σ) is the J_k integral computed on the surface S surrounding the volume V.

In the same way, it is possible to compute the moment resulting from defect forces and defect couples (relating to the origine for instance)

$$L_k = - \int_S (e_{kmn} x_m \bar{j}_n + \bar{l}_k) \, ds = \int_V (l_k + e_{kmn} x_m j_n) \, dv + \int_\Sigma (\bar{l}_k + e_{kmn} x_m \bar{j}_n) \, ds \qquad (8)$$

L_k is one of the integrals considered by Günther (1962), Knowles and Sternberg (1972) and Eshelby (1975).

At this time, it must be pointed out that J and L are defined without any hypothesis about material behaviour. It appears that these integrals are only the resultants of defect forces and couples which are expressions of the trend of material movements in the body. As cracking is a typical material movement, it transpires that defect forces and couples can be more representative fracture criteria than J. The field of defect forces in the vicinity of crack tip is more significant of local fracture conditions than path integrals which may include other phenomena like the effect of thermal stresses (ROCHE 1979a, 1979b, 1979c).

As far as fracture mechanics is concerned, it is valuable to examine the distribution of defect force $\bar{j}$ along a notch (with free boundaries). This surface defect force is normal to the surface, and its value is equal to the stress working density W (Rice and Drucker, 1967, Eshelby, 1970). Due to the strain concentration at the bottom of the notch, this area show high values of $\bar{j}$ and the resultant J is mainly characterising the intensity of defect forces in that area. The smaller is the tip radius, the higher is $\bar{j}_k$, and J gives an indication of the intensity of defect forces at the notch tip or at the crack tip when the radius is very small.

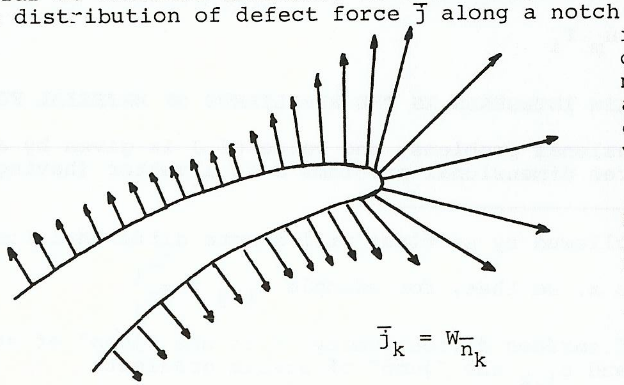

FIGURE 1.

ENERGY MOMENTUM TENSOR AND ITS COUPLE STRESS COMPANION

Equations (7) and (8) can be interpreted as follow : the resultants of all the defect forces and couples applied to a solid (or part of it), are equal to zero. They are like equilibrium equations of the body, but defect (or material) forces play the role of conventional forces. Hence an attempt can be made to introduce what correspond to a stress tensor. Usually, the energy momentum tensor Θ_{kj} (well known in the electromagnetic field theory) is introduced (Eshelby, 1970)

$$\Theta_{kj} = W\delta_{kj} - \sigma_{ij} u_{i,k} \qquad (9)$$

where δ_{kj} is KRONECKER's tensor. Such a method allow to write equilibrium equations, but a more comprehensive analysis shows that is is necessary to add a couple stress :

$$\Lambda_{pi} = e_{ijk} \sigma_{ip} u_k \qquad (10)$$

in order to obtain the complete set of equilibrium equations for COSSERAT media (1909).

$$\begin{cases} \Theta_{kj,j} = j_k \\ \Lambda_{pi,p} = l_i - e_{ijk} \Theta_{kj} \end{cases} \qquad \begin{cases} \Theta_{kj} n_j + \bar{j}_k = 0 \\ \Lambda_{pi} n_p + \bar{l}_i = 0 \end{cases} \qquad (11)$$

Obviously, these equilibrium equations can be translated to a material form of the principle of virtual work (CASAL, 1979). The material virtual displacement (translation δx_k and rotation $\delta\Omega_k$) is the displacement of material particles through the body. In other words, material displacement is the flow of material properties through the body including flowing of holes, discontinuities, heterogeneities, cracks... .

GENERALIZED PRINCIPLE OF VIRTUAL WORK

The conventional form of the principle of virtual work is only related to conventional forces X_i, and only conventional (spatial) displacements u_i are considered. It must be pointed out that virtual displacements δu_i must satisfy equations of compatibility. Unfortunately, in Fracture Mechanics, it is necessary to consider virtual crack propagation and such a displacement do not satisfy compatibility equations. Conventional form of the principle of virtual work is not applicable to Fracture Mechanics. A more general form, taking into consideration material displacement must be used. This form is easily deduced from the preceeding results :

$$\int_V \delta W \, dv = \int_S X_i \, \delta u_i \, ds - \int_V (j_i \, \delta x_i + l_i \, \delta \Omega_i) \, dv - \int_\Sigma (\bar{j}_i \, \delta x_i + \bar{l}_i \, \delta \Omega_i) \, d\sigma \quad (12)$$

where Σ means the discontinuity surfaces inside the volume V (like holes, cracks).

A well known particular form of this equation can be written when the only material displacement is a uniform translation δa along x_1 axis. Such a simplification is too restrictive for crack propagation is not identical to crack translation and a more general approach is preferable. In practical cases it can be assumed that material displacement distribution can be defined by a finite number of parameters a_α and the material work variation can be written $J_\alpha \delta a_\alpha$, where J_α are parameters defining defect forces distribution (J_α is dual of a_α). The general form of the principle of virtual work can be written :

$$\int_V \delta W \, dv = X_\alpha \, \delta u_\alpha - J_\alpha \, \delta a_\alpha \quad (13)$$

where u_α are conventional displacement parameters (or generalized displacements) X_α conventional force parameters (generalized forces), a_α are material displacement parameters (or generalized material displacements) and J_α defect forces parameters (or generalized defect forces). This equation give another definition of J as a set of scalars J_α connected to a set of geometrical parameters a_α describing crack propagation.

The current practice is to describe crack propagation with only one parameter a (crack extention) and consequently to consider only one parameter J. Such a simplification seems working fairly in many practical problems, but cannot be applied in every cases. As an example, in order to get an estimation of the effect of sample thickness it can be useful to define crack propagation by two parameters: a_m in the middle plan and a_b at the free surface, consequently J concept is a set of two quantities J_m (plane strain) and J_b (plane stresse).

PRACTICAL DETERMINATION OF J VALUE

Examination of the current practice

If crack propagation can described by one parameter only, there is only one J quantity to know, related to the simplified equation :

$$\int_V \delta W \, dv = \delta V = X \delta u - J \delta a \quad (14)$$

In such an equation the geometrical state of the cracked structure is defined by two variables u and a. There are two other variables (dual variables) X and J and the scalar function V. A first question arise about these variables and functions : "Are they well defined functions of u and a ?" in other words "the values of V, X, J are they dependent only on the values of a and u ? is there an effect of the path used in the u-a field ?". If such an assumption is true for elastic materials, it is not proven for elastic plastic materials, but it is approximately verified in most of the practical cases. Experimental verification can be achieved concerning the path dependence of X value by comparing values obtained at the some a and u values with different methods of crack advance (natural propagation or machining).

To obtain the J value as a function of the increase in crack length (J - R curves) by experimental testing of specimens, such an assumption is made (V is only a function of u and a). Therefore V can be measured for constant crack length and J is obtained by derivation :

$$V = \int_0^x X \, du \Big|_{a = \text{constant}} \qquad J = \frac{\partial V}{\partial a}\Big|_{u = \text{constant}} \quad (15)$$

Historically these equations are the principle of the first method used to obtain experimental value of J (Landes and Begley, 1972). Such a method is costly and difficult to perform, therefore other methods have been proposed (Rice and co workers1973, Merkle and Corten 1979). The analysis of these methods show that they are based on the same type of assumption : If load displacement curve (X-u) is known for one given value of crack length a, all other load displacement curves (for other values of crack length) are known (ROCHE, 1979).

As an example, it is often assumed that the load is given as the product of a known function A of the crack length by a function φ of the displacement (depending on the material), this assumption give the expression of J

$$X = A(a) \, \varphi(u) \qquad J = -\frac{1}{B}\frac{A'}{A} \int_0^u X du \quad (16)$$

(B being the thickness). This result is only valid if the crack length does not vary during the test. If crack propagation occurs, some correction must be taken into account (see Roche 1979 for more general laws).

Procedure to got J. R. curve with only one sample

It must be pointed out that the parameter a describing the crack propagation is often considered as the "crack length", but such an assimilation is not obvious. From a theoritical point of view, this assimilation is arbitrary and from an experimental point of view, crack length is not easy to define. During crack propagation, new created surface is not always flat and crack tip is not often straight. As a consequence, great difficulties arise about crack length measurement. This is especially true for determination of J-R curve giving J as a function of Δa. Different methods have been proposed (compliance, electrical properties) but they are not easy to use and their reliability is questionnable.

Recently, it has been noted that the hypothesis used for the determination of J value can be also used for determination of the value of the crack length (Ernst 1979, Milne and Chell 1979, Roche 1979). What has to be done is to obtain experimentally the function φ corresponding to the sample tested. For a given shape this function is only depending on the material itself. For a given type of material, the general features of the function φ are known (from the results of preceeding tests) but there is a need for adjustements taking into account the material properties variations. This can be done with the help of the results of one experimental test.

Experiment on one sample give two indications about crack length a : first the initial value a_o (before propagation) and then the final value a_f. As the propagation does not occur immediately, the beginning of the curve φ (u) is known. The final point corresponding to a_f is also known. These indications are sufficient ot draw the curve (if the type of material is known). Practically it is more conveniant to use the "corrected load" X^* which is proportional to φ :

$$X^* = X \frac{A\ (a_o)}{A\ (a)} = A\ (a_o)\ \ \varphi\ (u) \qquad (17)$$

The procedure is indicated on figure 2. It has been successfully applied and validated as a very efficient, quick and inexpensive method. The J-R curve can be obtained from only one test on one sample, if the type of material is already known.

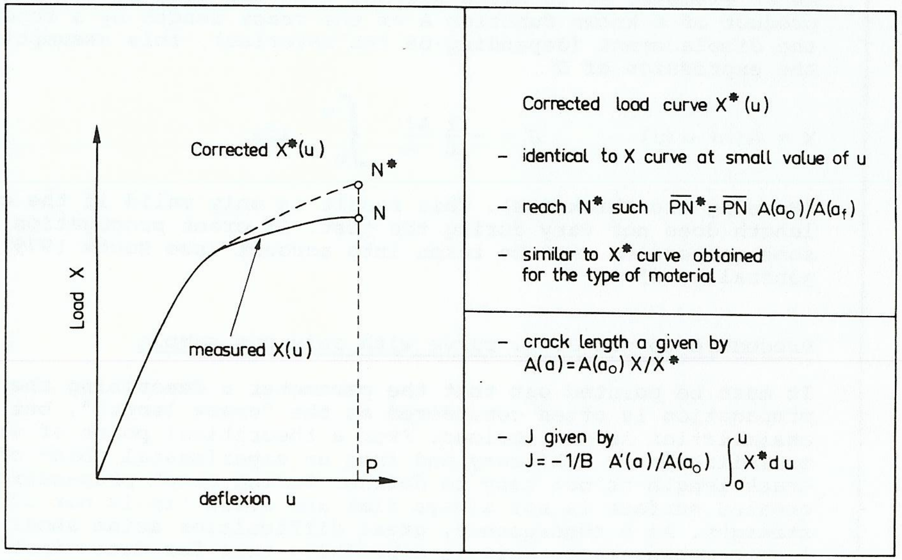

Fig. 2 Procedure J-R curve

CONCLUSIONS

J can be introduced on the basis of the concept of defect forces j_i (and defect couples) which are connected to the displacement of the material particles through the body (material displacement). The meaning of the defect forces is more general than that of the J integral. Fracture Mechanics requires a more general form of the principe of virtual work than the conventional one. A complementary term must be introduced in order to take into account the material displacement effects. From this point of view, the increase of crack lenght in only a parameter representing the material displacement field and J is the "defect load" parameter associated with it, obviously such a lecture is only a simplification. Representation of crack growth by a set of several parameters can be also considered, as a consequence J is extended to a set of several scalars. The practical use of the J concept in Fracture Mechanics implies several assumptions about material and structural behavior. This is especially true for the measure of J values. Nevertheless, it is possible to extend their application to measure crack growth, in order to get the J-R curve from only one experimental test.

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