ANALYTIC ASYMPTOTIC SOLUTION OF THE KINKED CRACK PROBLEM

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ABSTRACT

By the use of complex potentials and conformal mapping, we derive the asymptotic equations of the kinked crack problem when the length of the kink goes to zero. We give the analytic solution of the problem by means of a series which is proved to be strongly convergent. This solution agrees very well with those obtained by Bilby and others (1977) or Wu (1978) and it enables us to discuss some criteria of rupture in mixed mode.

KEYWORDS

Kinked crack; branched crack; crack growing; mixed mode; energy release rate; stress-intensity factors.

INTRODUCTION

There are many works devoted to the branched (or kinked) crack problem in a linear isotropic elastic body in the condition of plane strain (Dudukalenko and Romalis, 1973; Hussain and others, 1974; Palaniswamy and Knauss, 1974; Chatterjee, 1975; Bilby and others, 1975 and 1977; Wu, 1978; Lo, 1978; ...). The knowledge of the stress-intensity factors k_1, k_2 at the tip of the secondary branch and their relation with the remote stress-intensity factors K_I, K_{II} of the main crack of length 2ℓ (fig. 1.a) are of great importance for the analysis of the fatigue crack propagation in mixed mode.

The asymptotic case when the length s of the secondary branch goes to zero has already been analysed by Bilby and others (1975 and 1977) and Wu (1978). In this paper we study the same problem by another approach. We will start from the equation derived by Hussain and others (1974) for a finite length s. By means of the double scale technique (as usual in fluid mechanics) we derive the asymptotic equation in a simple manner and we solve it by the series technique.

INTEGRAL EQUATION OF THE PROBLEM

Let the infinite body containing a stress-free kinked crack, be loaded at infinity by the stress $S = \sigma_{22} - i\,\sigma_{12}$, for which the stress intensity factors of a straight crack of length 2ℓ would be $K_I - i\,K_{II} = S\sqrt{\pi\ell}$. Using the notations of Dudukalenko and Romalis (1973) we consider the function $z = \omega(\xi)$ which maps the exterior (Ω^-) of the unit circle $|\xi| = 1$ onto the physical plane with a cut along the kinked crack with the angle $m\pi$ between the two branches

$$(1)\quad \begin{cases} z = \omega(\xi) = R\,e^{im\alpha}\,(\xi - e^{i\alpha})\,(\xi - e^{-i\alpha})\,\xi^{-1}[\,(\xi - e^{-i\alpha})/(\xi - e^{i\alpha})]^{m} \\ \ell = 2R\,[\cos(\frac{\alpha+\beta}{2})]^{1-m}\;[\cos(\frac{\alpha-\beta}{2})]^{1+m} \\ s = 4R\,[\sin(\frac{\alpha+\beta}{2})]^{1+m}\;[\sin(\frac{\alpha-\beta}{2})]^{1-m} \\ \sin\beta = m\sin\alpha \end{cases}$$

The crack and its tips or corners are mapped into the points a, b, c, d of the circle (fig. 1a,b). From the complex potentials of Muskhelishvili $\Phi(z)$, $\Psi(z)$, we define new complex potentials $\varphi(\xi) = \Phi[\omega(\xi)]$, $\psi(\xi) = \Psi[\omega(\xi)]$. The boundary conditions at the free surfaces and at infinity are all satisfied by the function $\varphi(\xi)$ solution of the integral equation (Hussain and others, 1974)

$$(2)\quad \varphi'(\xi) = \Gamma\xi R e^{im\alpha} - (\Gamma + \overline{\Gamma}')Re^{-im\alpha}/\xi + \frac{(1 - e^{2i\pi m})}{2i\pi}\int_{C_1}\frac{(\sigma - e^{i\alpha})(\sigma - e^{-i\alpha})\overline{\varphi'(\sigma)}\,d\sigma}{\sigma(\sigma + e^{-i\beta})(\sigma - e^{i\beta})(\sigma - \xi)}$$

where $\Gamma = S/4$, $2\Gamma + \overline{\Gamma}' = \sigma_{22} - i\,\sigma_{12}$, ξ belonging to Ω^- and $e^{i\beta}$ belonging to Ω^- and the arc C_1. In other words, the contour of integration in (2) consists of the arc C_1 and the small clockwise semi-circle around $e^{i\beta}$ (fig. 1.b). The stress intensity factors at the tip of the secondary branch are defined by

$$(3)\quad k_1(s,\pi m) - i\,k_2(s,\pi m) = 2\sqrt{\pi}\,\varphi'(e^{i\beta})\,[\,e^{i\pi m}\,\omega''(e^{i\beta})]^{-1/2}$$

As a first approach one can define the limiting stress-intensity factors by $k_1^*(\pi m) - i\,k_2^*(\pi m) = \lim\,(s\to 0)\;k_1(s,\pi m) - i\,k_2(s,\pi m)$. A more attractive method is to derive directly from eq. (2) the asymptotic equation $(s\to 0)$ for the function $\varphi(\xi)$, then to compute the stress intensity factors from the new function. Wu (1978) has obtained the same equation by a different method from ours.

ASYMPTOTIC EQUATION

Since $\alpha\to 0$ when $s\to 0$ we make use of the double scale technique which magnifies both variable and function

$$(4)\quad \begin{cases} \xi \to t = \frac{1}{i\alpha}\log\xi \quad \text{or} \quad \xi = e^{i\alpha t}\,, \quad t\in[-1,\,1] \\ \varphi(\xi) \to g(t) = [\varphi(\xi) + \overline{\Gamma}'\ell/2]\;/\;i\alpha\ell \end{cases}$$

Inserting (4) in eq. (2) and dropping the terms of higher order than $O(\alpha)$, we obtain the following equation

$$(5)\quad g(t) = S\,(t+m)/2 + \frac{1 - e^{2i\pi m}}{4i\pi}\int_{-1}^{1}\frac{(\lambda^2-1)\overline{g'(\lambda)}\,d\lambda}{(\lambda-m)(\lambda-t)}$$

where the real points $\{t,m\}$ are taken in the lower half plane Π^- (fig. 1.c).

Let us introduce the conjugate function $\overline{g}'(\lambda) = \overline{g'(\overline{\lambda})}$, holomorphic in the upper half plane Π^+ , which allows us to change the contour of integration $[-1, +1]$ into the clockwise semi-circle Γ^+. With $h(t) = (t^2 - 1)g'(t)$ and after differentiation, the eq. (5) is rewritten as follows

$$(6)\quad h(t) = S\,(t^2-1)/2 + \frac{(1-e^{2i\pi m})(t^2-1)}{4i\pi}\int_{\Gamma^+}\frac{\overline{h}(\lambda)\,d\lambda}{(\lambda-m)(\lambda-t)^2}\,,\quad t\in\Pi^-$$

Equation (6) is the same as Wu's equation, with slightly different notations. The stress-intensity factors for $s\to 0$ become

$$(7)\quad k_1^*(\pi m) - i\,k_2^*(\pi m) = 2\sqrt{\pi\ell}\;e^{-i\pi m}\,[\,(1-m)/(1+m)]^{m/2}\;g'(m)$$

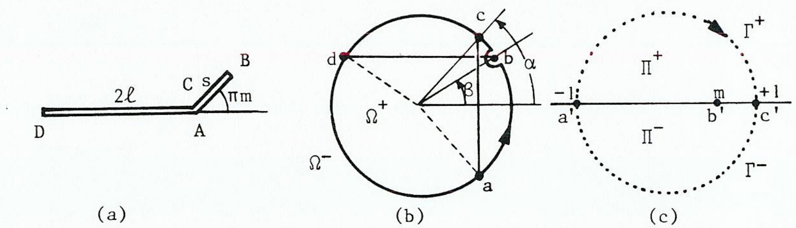


Fig. 1 : (a) Physical z-plane of the kinked crack.
(b) Mapped ξ-plane and path-integration C_1=arc(abc).
(c) Auxiliary t-plane magnifying the neighbourhood of $\xi=1$ and path-integration Γ^+ .

SOLUTION

Let us write the equation (6) in the form $h(t) = h_o(t) + \mathcal{L}h(t)$ for $t\in\Gamma^-$ with $h_o(t) = S(t^2-1)/2$ and $\mathcal{L}$ the integral operator. Following the work of Amestoy and others (1979), introducing the norm of uniform convergence over Γ^- , we can prove that the operator $\mathcal{L}$ satisfies the inequality

$$(8)\quad \|\mathcal{L}h\| \leqslant \frac{\sin\pi m}{\pi(1-m)}\;\|h\|$$

where the constant $A = \sin\pi m/\pi(1-m)$ is less than 1, since $0 \leqslant m < 1$. Inequality (8) shows that the operator $\mathcal{L}$ is a contraction, hence the solution of (6) can be expressed in term of the strongly convergent series

$$(9)\quad h = h_o + \mathcal{L}h_o + \mathcal{L}^2\,h_o + \ldots\ldots + \mathcal{L}^p\,h_o + \ldots$$

where $\mathcal{L}^p = \mathcal{L}(\mathcal{L}^{p-1})$ is the p-iterated integral operator. Once the density $h(t)$ along the lower semi-circle Γ^- is determined, we calculate the value $h(m)$ from (6) and we express the stress-intensity factors in the linear form

$$\begin{pmatrix} k_1^* \\ k_2^* \end{pmatrix} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} K_I \\ K_{II} \end{pmatrix}$$

where the K_{ij} only depend on the value of the angle πm.

RESULTS

We compute the series (9) by dividing the arc Γ^+ in N = 100 nodes in such a way that their angle seen from the point m varies linearly. We obtain a very good convergence of the series to within three digits with a seriesof ten terms if $m\pi \geqslant 150°$ and two terms if $m\pi \leqslant 50°$. The same accuracy was practically obtained with N = 150 or N = 200 nodes. The results shown in Table 1 and in Fig. 2, 3 are in very good agreement with those obtained by Bilby and others (1977) and Wu (1978). We also have plotted in dotted lines of the Fig. 2 and 3 the values of K_{ij} corresponding to one term (h_o) of series(9)

$$(10) \quad \begin{cases} K_{11}^{(o)} = \left(\frac{1-m}{1+m}\right)^{m/2} \left(\cos \pi m - \frac{\sin \pi m . L}{2\pi}\right) , & K_{12}^{(o)} = \left(\frac{1-m}{1+m}\right)^{m/2} \left(-\frac{3}{2} \sin \pi m\right) \\ K_{21}^{(o)} = \left(\frac{1-m}{1+m}\right)^{m/2} \left(\frac{1}{2} \sin \pi m\right) , & K_{22}^{(o)} = \left(\frac{1-m}{1+m}\right)^{m/2} \left(\cos \pi m + \frac{\sin \pi m . L}{2\pi}\right) \end{cases}$$

with $L = \text{Log } [(1-m)/(1+m)] - 2m/(1-m^2)$.

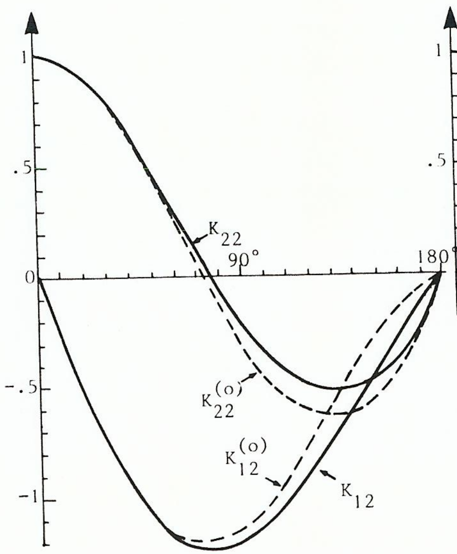
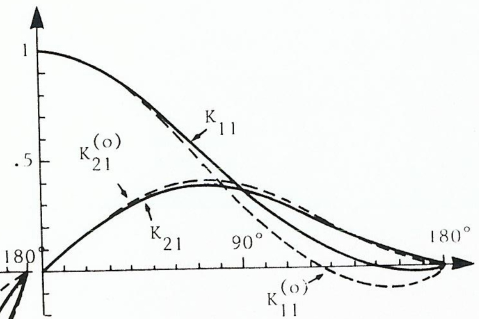


Fig. 2. Stress intensity factors K_{12}, K_{22} in mode II.



$K_{11}(-m)=K_{11}(m)$, $K_{21}(-m)=-K_{21}(m)$

$-K_{12}(-m)=K_{12}(m)$, $K_{22}(-m)=K_{22}(m)$

Fig. 3. Stress intensity factors K_{11}, K_{21} in mode I.

kink angle	K_{11}	K_{21}	K_{12}	K_{22}	kink angle	K_{11}	K_{21}	K_{12}	K_{22}
0°	1	0	0	1	50°	0.7479	0.3431	-1.0665	0.4872
10°	0.9886	0.0864	-0.2597	0.9764	60°	0.6559	0.3696	-1.1681	0.3077
20°	0.9552	0.1680	-0.5068	0.9071	70°	0.5598	0.3788	-1.2220	0.1266
30°	0.9018	0.2403	-0.7298	0.7972	80°	0.4640	0.3718	-1.2293	-0.0453
40°	0.8314	0.2995	-0.9189	0.6540	90°	0.3722	0.3507	-1.1936	-0.1988

Table 1

It is worthwhile to notice that eqs.(10) agree very well with the exact numerical solution for $0 \leqslant m \leqslant 1/2$. The accuracy is within 2% up to kink angles as large as 40°. This agreement can be compared with that obtained by using Nuismer's formulae (1978) solving an approximate problem (see also Howard, 1978).

For high value $m \simeq 1$ we observe that $K_{11} < 0$, hence the crack closure must be considered at the tip of the secondary branch in this case.

ENERGY RELEASE RATE

We refer to the excellent analysis of Wu (1978) and we confirm the numerical equality $G = (1-\nu^2)(k_1^{*2} + k_2^{*2})/E$ with a higher accuracy (0.2% instead of 0.7% at α = 90°, 4% instead of 10% at 150°). The discrepancy at high value seems to be the result of a singular integration of the crack closure energy. The curves $k_1^{*2} + k_2^{*2}$ (normalized quantities) versus the angle α are presented in the figure 4, for mode I loading ($K_I = 1$, $K_{II} = 0$) and mode II loading ($K_I = 0$, $K_{II} = 1$).

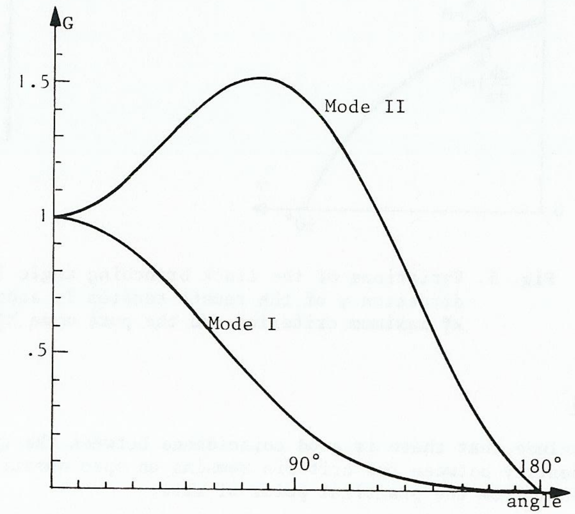


Fig. 4. Variation of the energy release rate G with the kink angle in modes I and II.

CRITERION OF CRACK BRANCHING

The present analysis gives a linear relationship between the initial stress-intensity factors K_I, K_{II} and the stress-intensity factors k_1^*, k_2^* just after crack branching. Hence any criterion of crack branching angle may be expressed in terms of either K_I, K_{II} or k_1^*, k_2^*. However it seems more natural to formulate a criterion in terms of the local parameters k_1^*, k_2^* rather than of the loading parameters K_I, K_{II}.

Two interesting criteria may be expressed as follows :

- the crack branching angle corresponds to the maximum of energy release rate ;
- the crack locally propagates in pure mode k_1^* ($k_2^* = 0$).

Since G is proportional to $k_1^{*2} + k_2^{*2}$, a possible equivalence between the mentioned criteria requires that

$$k_1^{*\prime}(\pi m) = 0 \quad \Leftrightarrow \quad k_2^*(\pi m) = 0 \tag{11}$$

In mode II loading, we found that $k_1'^*(76.6°) = 0$ while $k_2^*(77.3°) = 0$. In the case of uniaxial tension at infinity, in the direction γ with respect to the main crack, the crack branching angles $\alpha = \pi m$ predicted by the energy criterion and the pure mode I criterion are presented in figure 5, with a deviation less than 1%. Similar results were obtained by Bilby and Cardew (1975) ; their figure 2 (for $\lambda = 0$) agrees closely with figure 5.

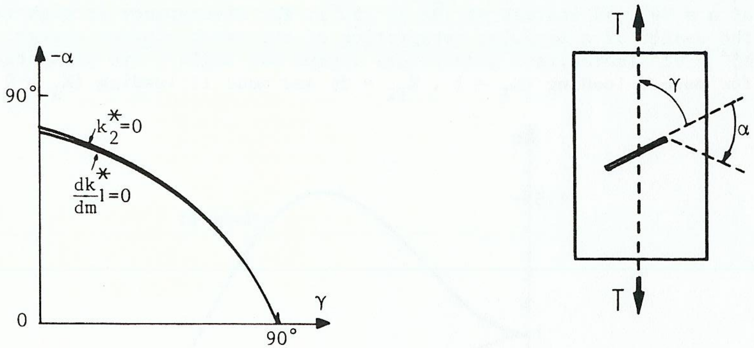


Fig. 5. Variations of the crack branching angle ($-\alpha$) with the direction γ of the remote tension T, according to the k_1^* maximum criterion and the pure mode k_1^* criterion.

CONCLUSION

We conclude that there is good coincidence between the two criteria. The mathematical identity between the criteria remains an open question, but not a fundamental question, from the practical point of view.

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