Fracture 1977, Volume 4, ICF4, Waterloo, Canada, June 19 - 24, 1977

VARIATIONAL BOUNDS AND QUALITATIVE METHODS IN FRACTURE MECHANICS

R. V. Goldstein* and V. M. Entov*

INTRODUCTION

Fracture mechanics has given rise to the problem of calculating values of stress intensity factor, K, in elasticity, i.e. the value of K at square root type singularities of stress near a crack contour. This problem, not simple by itself for complicated shapes of crack contour, becomes almost irresolvable if variations in K have to be investigated for the sequence of contours assumed by the crack as it develops (as is necessary in an analysis of the kinetics of crack growth). Hence, a problem naturally arises regarding the estimation of the values of K from which it would be possible to derive sufficient conditions for fracture (or non-failure) of a solid. The problem here is to find upper and lower bounds for the maximum and minimum values of K for a given contour or set of contours.

The present paper reports recent results obtained in this area, and also some other bounds of integral characteristics of the solutions of elasticity which are closely associated with these upper and lower bounds.

1. BOUNDS OF STRESS INTENSITY FACTORS FOR A PLANE OPENING-MODE CRACK IN AN ELASTIC SPACE

Consider an elastic space with a crack occupying the domain G in the plane $x_3 = 0$ bounded by a piecewise smooth curve Γ. It may be assumed that the external load is represented by wedging tractions symmetrical with respect to the plane $x_3 = 0$:

$$\sigma_{33} = -q(x_1,\ x_2),\ x_3 = 0 \qquad (x_1, x_2)\in G \tag{1.1}$$

Only normal stresses act in the plane $x_3 = 0$, while the vertical component of displacement on this plane outside the crack is zero:

$$\tau_{13} = \tau_{23} = 0\ (x_3 = 0),\ w(x_1,\ x_2) = 0 \qquad (x_1, x_2)\notin G$$

In the plane $x_3 = 0$ along the smooth parts of crack contour the stress σ_{33} and displacement w have singularities of the type:

$$\sigma_{33} = \frac{N}{\sqrt{5}}\ ,\quad w = \frac{4(1-\nu^2)}{E} N\sqrt{S} \tag{1.2}$$

where S is the distance along the normal from the crack contour; $N(t)$, $(x_1(t),\ x_2(t))\in\Gamma$ is the stress intensity factor at a given point

*Institute of Mechanical Problems, U.S.S.R. Academy of Science, 1250 Moscow, U.S.S.R.

on the contour corresponding to the value of the parameter t.

The following *theorem of comparison* is essential for further discussion.

Suppose that there are two crack domains G and ($G' < G$) with contours Γ and Γ' having a common region $\Gamma'' = \Gamma\ \Gamma'$. Assume that Γ'' consists of a certain number of smooth arcs and (or) isolated points of contact of the contours Γ and Γ'. Further assume that the corresponding loads $q(x_1,x_2)$ and $q'(x_1,x_2)$ satisfy the conditions:

$$q(x_1,x_2) \geq 0 \quad (x_1,x_2)\in G/G', \quad q(x_1,x_2) \geq q'(x_1,x_2), \quad (x_1,x_2)\in G \tag{1.3}$$

Then, at those points of Γ'', where the normals to the contours coincide, the stress intensity factor N for the crack occupying the domain G is not less than the stress intensity factor N' for the crack occupying the domain G':

$$N'(M) \leq N(M) \quad , \quad M\in\Gamma'' \tag{1.4}$$

The proof of this theorem and examples of its application are given in [1].

Of the corollaries to this theorem, we may mention the following assertion, which is valid within the framework of quasi-statical growth of cracks. Let there be two contours Γ_0 and Γ_0', of which the first envelops the second at the initial moment, and a system of wedging loads q and q' such that $q \geq q'$ at any instant and at every point. In this case a crack developing from the contour Γ_0' will always remain inside the bounds of a crack developing from the contour Γ_0. Hence, we have two simple, but important, conclusions:

(a) If the crack bounded at the initial instant by the enveloping contour is not critical for a given history of loading (i.e. it does not lead to failure), then the crack bounded at the initial instant by the enveloped contour is also not critical.

(b) If the enveloped crack is critical, then the enveloping crack is also.

Because of these assertions, the need to analyze complicated "irregular" curves is eliminated, and thus it is possible to restrict consideration to relatively simple crack contours. Indeed, these assertions are quite natural, and it may be assumed that they hold true for opening-mode cracks, not only in the whole elastic space, but also in bodies of other shapes. Unfortunately, this statement has so far not been proved, and we have therefore to limit ourselves to far less general statements.

It is not difficult to perceive that in all these arguments we have made use only of the property of *positiveness* rather than any particular solution of the elastic problem: positive (wedging) loads give rise to positive displacements of points on the crack surface and positive (stretching) stresses outside the crack in its plane. Thus, in order to widen the field of applicability of these assertions, it suffices to establish the positiveness of the corresponding class of elastic problems. It is natural to expect that the opening-mode crack problem will be a positive one not only for an infinite body, but also for bodies with boundaries sufficiently remote from the crack. This assertion is proved below for the particular case of a layer with a crack in the midplane.

2. OPENING-MODE CRACK IN A THICK LAYER

Suppose that the crack under consideration occupies a plane domain G of diameter d in the mid-plane $x_3 = 0$ of an elastic layer of thickness 2h. Assume that there are no tractions on the layer faces and the crack surface is acted upon by normal forces:

$$\sigma_{33} = -q(x_1,x_2) \quad , \quad (x_1,x_2)\in G \tag{2.1}$$

The problem can be reduced to the following pseudo-differential equation (see Appendix A):

$$Ar = -\frac{2(1-\nu)}{\mu}q \quad , \quad (x_1,x_2)\in G \tag{2.2}$$

where A is a pseudo-differential operator with symbol $A(\xi)$

$$Ar = \frac{1}{4\pi^2}\iint_{-\infty}^{\infty} A(\xi)\tilde{r}(\xi)e^{-i(x,\xi)}d\xi, \quad A(\xi) = -2\pi\ |\xi| + 2\pi K(h|\xi|)$$
$$K(h|\xi| \equiv \frac{1}{h}K(h|\xi| = \frac{2|\xi|\left(1+2h|\xi|+2h^2|\xi|^2-e^{-2h|\xi|}\right)}{4h|\xi|+e^{2h|\xi|}-e^{-2h|\xi|}} \tag{2.3}$$

Here $\xi = (\xi_1,\xi_2)$ stands for the parameter of the Fourier transformation in (x_1,x_2) $\tilde{r}(\xi)$ is the Fourier transform of the displacement of the points on the surface in the direction of x_3, and ν, μ are the Poisson's ratio and shear modulus of the material, respectively.

Let $R_h(x_1,x_2,u_1,u_2) = R_h(x,u)$ denote the resolving operator of Eq. (2.2), so that

$$r(x_1,x_2) = \int_G R_h(x,u)q(u)du \tag{2.4}$$

For an infinite medium ($h = \infty$) we have $K(h|\xi|) = 0$, and the left-hand side of Eq. (2.2) contains only one term. Hence, by the assertion proved in Section 1, we find that the posi-function q(x) in the right-hand side corresponds to the positive solution of Eq. (2.2) regarded as an equation in r(x). Thus, $R_\infty(x,u)>0$, i.e. the operator R_∞ is positive. In order to show that the operator R_h is positive for sufficiently large h, we shall construct the solution of Eq. (2.2) by iteration, assuming that

$$r_0 = R_\infty q \quad , \quad r_K = R_\infty(q+q_K) \quad , \quad K \geq 1$$

$$q_K = \frac{\mu}{2(1-\nu)}\frac{1}{(2\pi)^2}\iint_{-\infty}^{\infty} (2\pi)K(h|\xi|\tilde{r}_{K-1}(\xi)e^{-i(x,\xi)}d\xi \tag{2.5}$$

The quantitites q_K may be looked upon as fictitious additional loads which have to be applied to the surface of a crack in an infinite space so that crack opening may coincide with crack opening in this layer to the same approximation.

Clearly, positiveness of R_h will be proved if it can be shown that $(q+q_K)$ is positive for any K.

Let $q' = \lim_{K\to\infty} q_K$. From the convergence of successive approximations (see Appendix B) it follows that for sufficiently large values of the ratio h/d, the correction term in Eq. (2.5) will be close to

$$q_1 = \frac{\mu}{2(1-\nu)} \frac{1}{(2\pi)^2} \iint_{-\infty}^{\infty} (2\pi)K(h|\xi|)\tilde{r}_0(\xi)\, e^{-i(x,\xi)}\, d\xi \tag{2.6}$$

In turn, using the smallness of h/d, this expression can be asymptotically represented by

$$q_1 \simeq q_1^0 = \frac{\mu}{2(1-\nu)} \frac{1}{(2\pi)^2 h^3} \iint_{-\infty}^{\infty} (2\pi)K(\eta)\tilde{r}_0(0)\, d\eta \tag{2.7}$$

This relationship defines in the domain G a constant additional load acting in the same direction as that of the initial loads applied to the crack surface. The fact that this additional load, irrespective of the initial load q distribution, does not vanish anywhere in the domain G, shows by itself that for sufficiently large h/d, the quantity q' and the sum $(q_i + q')$ are strictly positive in the domain G. By estimating the contribution of successive terms in the expansion in powers of d/h (see Appendix C), we find that this assertion remains valid at least for d/h<0.7 which proves that the operator R_h is positive for d/h<0.7 (i.e. for a sufficiently thick layer) and that all those assertions enunciated in Section 1 and [1] can also be applied for a thick layer with a crack. At the same time, it has been demonstrated that the opening and stress intensity factor for a crack in a thick layer may be asymptotically represented (with an error of the order of $O(d^3/h^3)$) in the form:

$$r(x) = r_0 + r_{10} \quad , \quad N = N_0 + N_{10} \tag{2.8}$$

where r_0 is the opening of a crack under the action of given loads q in a body occupying the whole space; r_{10} is the opening of the same crack under the action of a constant load q_1^2 defined by (2.7). In the range of validity of the bounds (d/h<0.7), crack opening and stress intensity factor at each point diminish on passing from the layer to the space when the load and crack configuration remain unaltered.

3. EXTREMAL CRACK CONTOURS

3.1 The following approach may be of use in constructing the bounds of stress intensity factors.

Consider an elastic body whose plane of symmetry $x_3 = 0$ contains an opening-mode crack wedged by symmetrical normal loads $q(x_1,x_2)$. Take an arbitrary closed domain G_0 (of area S_0) in the plane $x_3 = 0$. Consider the set of all domains G containing G_0 as a subdomain and having an area S, $S>S_0$. To each such domain we may attribute an energy W defined as the potential energy of the body having a crack occupying this domain in the given field of loads. The quantity W is a functional of the contour Γ of the domain G. Further assume that among all these contours, there exists one

(or several) contour on which W attains its maximum. Such a contour is hereafter called the extremal contour. The extremal contour consists of two parts: Γ'' coinciding with a part of the boundary of the domain G_0, and Γ' lying outside G_0 (the part Γ'' may, in general, be absent). The main property of the extremal contour is that the stress intensity factor N is constant on the part Γ'. Indeed, along with Γ', consider an arc Γ^* close to Γ' such that its end points coincide with the end points of Γ'. Then Irwin's formula (see [2]) shows that the variation in energy on passing from the contour $\Gamma''+\Gamma'$ to the contour $\Gamma''+\Gamma^*$ is

$$\sigma W = \frac{\pi(1-\nu)}{\mu} \int_{\Gamma'} N^2 (\vec{n}\sigma\vec{r})\, d\ell \tag{3.1}$$

where $(\vec{n}\sigma\vec{r})$ is the distance between the arcs Γ^* and Γ' along the normal to Γ'. On the other hand, the corresponding change in the area is

$$\sigma S = \int_{\Gamma'} (\vec{n}\sigma\vec{r})\, d\ell \tag{3.2}$$

For the attainment of conditional extremum of W we should have

$$N^2 - \lambda = 0 \quad , \text{ where } \lambda = \text{const on } \Gamma' \tag{3.2}$$

Thus, the invariance of N over Γ' is the necessary condition that the contour $\Gamma''+\Gamma'$ may be extremal.

If the arc Γ'' does not vanish, then the stress intensity factor on it is not greater than N* over the remaining part of the extremal contour. In the contrary case, indeed, by deforming the contour Γ' such that the area decreases so that the area inside $\Gamma'+\Gamma''$ remains constant, we find that the energy increases, the area remaining constant. Using asymptotic expressions for the stresses near corner points of the contour, we find that there cannot exist angular points on the free part Γ' of the contour, at any case for N≠0 on the free part of the contour.

Suppose that we can find a sequence of embedded extremal contours $\Gamma(S)$ corresponding to increasing the parameter S from S_0, and thus determine the corresponding stress intensity factors N(S) on the "free" part Γ' of the contour. Introduce a loading parmeter P, such that

$$q = Pq_0(x_1,x_2) \tag{3.4}$$

Then for each $S>S_0$, there exists such a value P for which N = N* = const.

$$P = \frac{N^*}{N_0(S)} \quad , \quad N_0 = N(S,q_0) \tag{3.5}$$

If N* is identified with the critical value of stress intensity factor $(N^*=K/\pi$, K is the cohesion modulus of the material) at which limit equilibrium is attained at the crack edge, then the extremal contour of area is W(S). We have obviously

$$N^2 = \frac{\mu}{\pi(1-\nu)} \frac{dW}{dS} \tag{3.6}$$

Since the true opening of a crack with a given contour can be found from the condition of minimum potential energy of an elastic body, the problem of finding the extremal crack contour can be reduced, in a general case, to the problem of finding the minimum, i.e. to finding, for a given area S, a contour enveloping the domain such that $S_0<S$, for which W(S) attains the value

$$W_0(S) = \max_{\text{mes}G=S} \min_{G} W$$

3.2 We shall now give an example of an extremal crack contour. Consider an infinite medium with a crack occupying the domain G_0 in the plane $x_3=0$. Assume that the extremal loads are normal stresses acting on the crack surface:

$$\sigma_{33} = P(1+\varepsilon x_1^2) \tag{3.7}$$

where ε is small. If $\varepsilon = 0$, in (3.7), then a circular contour Γ will be the extremal contour without common points with the initial contour Γ_0, since the stress intensity factor does not vary over the contour of a penny-shaped crack under uniform wedging loads. Therefore, by virtue of the smallness of ε and symmetry of load (3.7) with respect to the axes x_1, x_2, it is natural to search for an extremal contour among elliptical contours close to circular ones. Let the ellipse be defined by the parametric equations:

$$x_1 = a_1\cos\theta \quad , \quad x_2 = B_1\sin\theta$$
$$a_1 = a(1+\sigma_1) \quad , \quad B_1 = a(1+\sigma_2) \tag{3.8}$$

where σ_1 and σ_2 are small quantities to be determined.

The stress intensity factor N on an elliptical crack contour under the load (3.7) is given by the formula [4]:

$$N = -\mu\left(\frac{2}{a_1B_1}\right)^{\frac{1}{2}}\left(a_1^2\sin^2\theta+B_1^2\cos^2\theta\right)^{\frac{1}{4}}\left[\frac{4C_{00}}{a_1B_1} - \frac{16C_{02}\sin^2\theta}{a_1B_1^3} - \frac{16C_{20}\cos^2\theta}{a_1B_1^3}\right] \tag{3.9}$$

where the constants C_{00} , C_{02} , and C_{20} are determined from the solution of the system of linear algebraic equations:

$$||M||\begin{pmatrix} C_{00} \\ C_{20} \\ C_{02} \end{pmatrix} = \frac{1}{2\mu}\begin{pmatrix} 1 \\ \varepsilon \\ 0 \end{pmatrix}$$

$$||M|| = \left\| \begin{matrix} \dfrac{4E(k)}{a_1B_1^2} \ , & \dfrac{8[k'^2K(k)-(1-2k^2)E(k)]}{a_1^3B_1^2k^2} \ , & \dfrac{8[(1+k^2)E(k)-k'^2K(k)]}{a_1B_1^4k^2} \\ 0 \ , & \dfrac{8L(k,k')}{a_1^5B_1^2k^2} & \dfrac{8U(k,k')}{a_1^3B_1^4k^4} \\ 0 \ , & \dfrac{8U(k,k')}{a_1^3B_1^4k^4} & \dfrac{8Q(k,k')}{a_1^3B_1^4k^4} \end{matrix} \right\| \tag{3.10}$$

$$L = (3-8k^2+2/k^2)E(k)-2k'^2(2+11k^2)K(k)$$
$$U = k'^2(2-k^2)K(k)-2(k'^2+k^4)E(k)$$
$$Q = 2(3k^2-1)K(k)+(3k^2+\frac{2-10k^2}{k'^2})E(k)$$

Here K(k) and E(k) are total elliptical integrals of the first and second kind respectively; $k^2 = 1-B_1^2/a_1^2$, $k'^2 = B_1^2/a_1^2$. From (3.10), we find

$$C_{00} = \frac{a^3P}{4\pi\mu} \ , \ C_{02} = \frac{3a^7P}{460\pi\mu}\varepsilon \ , \ C_{20} = -\frac{29a^7P}{7380\pi\mu}\varepsilon \tag{3.11}$$

accurate to the order of ε. By virtue of (3.11) and the smallness of σ_1, σ_2, from (3.9) we obtain

$$N = \frac{P\sqrt{2a}}{\pi}\left\{\left(1-\sigma_1-\sigma_2\right)-\sin^2\theta\left(\frac{\sigma_2}{2}+\frac{3a^2\varepsilon}{230}\right)+\cos^2\theta-\left(\frac{\sigma_1}{2}+\frac{29a^2\varepsilon}{690}\right)\right\} \tag{3.12}$$

Hence, it is seen that for

$$\sigma_1 = \varepsilon a^2\frac{29}{345} \quad , \quad \sigma_2 = -\varepsilon a^2\frac{3}{115} \tag{3.13}$$

the stress intensity factor does not change over the contour. Thus, ellipses with semi-axes given by (3.8) and (3.13) will be the extremal contours for a load of the type (3.7).

This family of extremal contours may be used in estimation of the limiting equilibrium of a crack of arbitrary configuration in a field of loads of the type (3.7). We shall construct a family of ellipses with semi-axes a_1 and B_1 given by Eqs. (3.8) and (3.13) respectively for different a. From these ellipses, take that ellipse Γ^* to which the given critical stress intensity factor $N = N^*$ corresponds. Then all the cracks enveloped by the ellipse Γ^* will be non-critical; all the cracks enveloping Γ^* will be critical. In a similar manner, as noted in 3.1, we may use the family of extremal contours constructed for any (symmetric with respect to the plane of crack) load field.

4. ENERGETIC BOUNDS OF AN INHOMOGENEOUS BODY

4.1 The following theorems on the variation of deformation energy with the elastic constants of a body will be of use in further discussion.

Consider an isotropic, and in general, inhomogeneous elastic body. A body is regarded as inhomogeneous in the sense that its elastic constants are different at different points: $a = \lambda(x_i)$; $\mu = \mu(x_i)$; or $E = E(x_i)$; $\nu = \nu(x_i)$. Let $S = S_0+S_1+\ldots S_i$ be its boundary, where S_0 is the outer boundary, while $S_1,\ldots,S_i$ are the inner boundaries (cavities, cracks, etc.) Assume that the external loads act only on the part S' of the surface S, while the displacements on the remaining portion S'' are fixed:

$$\vec{\sigma}_n(\vec{x}) = f(\vec{x}) \quad , \quad \vec{x} = (x_1,x_2,x_3)\varepsilon S' \tag{4.1}$$

$$\vec{U}(\vec{x}) = g(\vec{x}) \quad , \quad \vec{x}\varepsilon S'' \tag{4.2}$$

and that the body is in equilibrium under the action of these loads. The following theorem holds true:

Theorem 4.1: On increasing (decreasing) the elastic constants λ and (or) μ or the Young's modulus E of the material at any region of the body, the quantity

$$Q = \iint_{S''} \vec{f}(\vec{x})\vec{U}d\sigma - \iint_{S''} \vec{\sigma}_n(\vec{x})\vec{g}(\vec{x})d\sigma \tag{4.3}$$

does not diminish (increase).

Proof:

Consider the potential energy of a system in equilibrium [5]:

$$W(\vec{U}) = \int_V A d\tau - \int_{S''} \vec{\sigma}_n(\vec{x})\vec{U}_s(\vec{x})d\sigma \tag{4.4}$$

where $\vec{U}$ is the displacement field corresponding to the equilibrium state of the body, and integration is taken over the volume of the body V, $\vec{U}_s$ is the displacement of points on the surface S, and A is the elastic energy density.

$$A = \frac{1}{2}[\lambda(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33})^2+2\mu(\varepsilon_{11}^2+\varepsilon_{22}^2+\varepsilon_{33}^2+2\varepsilon_{12}^2+2\varepsilon_{23}^2+2\varepsilon_{31}^2)] \tag{4.5}$$

Diminish the elastic constants λ and (or) μ in some domain V^1 (with the boundary S^1) to a value λ_1 and (or) μ_1 without altering the displacement field. Evidently here the first positive term in (4.4) will diminish, therefore,

$$W(\vec{U}) \geq W_1(\vec{U}) \tag{4.6}$$

where $W_1(\vec{U})$ is the potential energy of the deformed body that corresponds to the displacement field $\vec{U}$. Note that the states with displacement field $\vec{U}$ are admissible [5] for a deformed elastic body, i.e. the displacements $\vec{U}$ take the same values on S'' as the displacements $\vec{U}_1$ corresponding to the equilibrium state under the previous loads (4.1). By virtue of the minimum potential energy principle, we have

$$W_1(\vec{U}) \geq W_1(\vec{U}_1) \tag{4.7}$$

For the equilibrium states of the initial and deformed bodies, by the Clapeyron formula [5], we have

$$W(\vec{U}) = \frac{1}{2}\left[-\iint_{S''} \vec{f}(\vec{x})\vec{U}_s(\vec{x})d\sigma + \iint_{S''} \vec{\sigma}_n(\vec{x})\vec{g}(\vec{x})d\sigma\right]$$

$$W_1(\vec{U}_1) = \frac{1}{2}\left[-\iint_{S''} \vec{f}(\vec{x})\vec{U}_{1s}(\vec{x})d\sigma + \iint_{S''} \vec{\sigma}_{n_1}(\vec{x})\vec{g}(\vec{x})d\sigma\right] \tag{4.8}$$

where $\vec{\sigma}_n(\vec{x})$ and $\vec{\sigma}_{n_1}(\vec{x})$ are the tractions developed on that part of the surfaces of the initial and deformed bodies on which displacements are given.

Due to (4.8), from (4.4) and (4.6), it follows that

$$Q = \iint_{S''} \vec{f}_n(\vec{x})\vec{U}_s(\vec{x})d\sigma - \iint_{S''} \vec{\sigma}n(\vec{x})\vec{g}(\vec{x})d\sigma \leq$$
$$\iint_{S''} \vec{f}_n(\vec{x})\vec{U}_{1s}(\vec{x})d\sigma - \iint_{S''} \vec{\sigma}_{n_1}(\vec{x})\vec{g}(\vec{x})d\sigma = Q_1 \tag{4.9}$$

Evidently, on interchanging the initial and deformed bodies, we find that Q does not increase with the increasing constants λ and (or) μ. Thus, the theorem has been proved.

Corollary 1

On the whole surface, if the external loads (S = S'') are given or zero displacements ($\vec{g}(\vec{x}) = 0$) are defined on a part of the surface S'', then from (4.9) it follows that

$$\iint_{S'} (\vec{f})(\vec{x})\vec{U}_s(\vec{x})d\sigma \leq \iint_{S'} \vec{f}\vec{U}_{1s}d\sigma \tag{4.10}$$

i.e. the work done by the external forces does not diminish with the decreasing of elastic constants λ and (or) μ in an arbitrary domain of the body.

Corollary 2

If there are no loads on a part S of the boundary, i.e. $\vec{f} = 0$, or displacements are given on the whole surface (S = S''), then from (4.9), we have

$$\iint_{S''} \vec{\sigma}_n(\vec{x})\vec{g}(\vec{x})d\sigma \geq \iint_{S''} \vec{\sigma}_{n_1}(\vec{x})\vec{g}(\vec{x})d\sigma \tag{4.11}$$

i.e. the integral of stresses with weight $\vec{g}(\vec{x})$ over that part of the surface where the displacements are defined does not increase with decreasing of the elastic constants λ and (or) μ in an arbitrary part of the body.

From this corollary it immediately follows that an increase (decrease) in Young's modulus gives rise to an increase (or decrease) in Q, as the constants λ and μ are proportional to Young's modulus.

Notes: 1: A few cases are mentioned for which Theorem 4.1 holds true and its proof is practically unchanged.

1. Moments but not forces are applied to the surface of the body.

2. The body consists of several parts in contact and there is no friction at the contact surface.

3. The body is composite and total adhesion exists at the contact surfaces. If partial adhesion or a slipping condition is substituted for total adhesion, then Q increases.

2: As the problem (4.1), (4.2) is linear, its fields of stresses and displacements can be represented in the form of the sum $\sigma_{ij}=\sigma^f_{ij}+\sigma^g_{ij}$, $U_i=U^f_i+U^g_i$ where the superscripts f and g denote the solutions of the elasticity problem under the following boundary conditions:

$$a)\quad \vec{\sigma}_n(\vec{x})=\vec{f}(\vec{x}) \quad , \vec{x}\in S'' \quad , \quad \vec{\sigma}_n(\vec{x})=0 \quad , \quad \vec{x}\in S''$$

$$b)\quad \vec{\sigma}_n(\vec{x})=0 \quad , \vec{x}\in S'' \quad , \quad \vec{U}(\vec{x})=\vec{g}(\vec{x}) \quad , \quad \vec{x}\in S''$$

respectively. With the help of this partition, the expression (4.3) for Q may be rewritten as follows:

$$Q = \iint\limits_{S'} \vec{f}(\vec{x})\vec{U}^f(\vec{x})d\sigma - \iint\limits_{S''} \vec{\sigma}^g_n(\vec{x})\vec{g}(\vec{x})d\sigma$$

where account has been taken of the fact that, by virtue of the reciprocity theorem, we have

$$\iint\limits_{S'} \vec{f}(\vec{x})\vec{U}^g(\vec{x})d\sigma = \iint\limits_{S''} \vec{\sigma}^f_n(\vec{x})\vec{g}(\vec{x})d\sigma$$

Thus, the quantity Q is the difference between the work done by external forces in the problem (a) and the quantity which may be called the work of pre-straining which has to be applied so that the displacement defined in the problem (b) may be attained on S''.

4.2 In the case of a crack subjected to uniform internal pressure P, the work done by external forces is, evidently, equal to the product of pressure P and the volume of opened crack (or increment in volume due to deformation if the crack was opened initially). Therefore, from the theorems proved above, it follows that a decrease in the rigidity of a body in some of its parts leads to an increase in the increment of the volume of internal crack in the body under a given internal pressure, whereas an increase in rigidity leads to a decrease in the volume increment*.

*Hereafter, the term "volume of crack" is used everywhere, and it is implicit that the crack is closed in the absence of stresses.

For an infinite body, in the case of uniformly distributed loads acting normal to the crack plane, circles are obviously the extremal free contours and the energy of a body with a crack equals half the product of stress and crack volume. Hence, it follows that the volume of a crack occupying an arbitrary two-dimensional domain G of area S, opened by internal pressure P does not exceed the volume of a circular crack of the same area. This inequality

$$V \leq \frac{P}{E} \frac{16(1-\nu^2)}{3\pi^{3/2}} S^{3/2}$$

is equivalent to the inequality known for the case of capacitance of a plane domain [6].

5. APPLICATION OF ENERGY BOUNDS TO PROBLEMS IN FRACTURE MECHANICS

In certain cases, the energy bounds given in Section 4 are directly applicable to problems of fracture mechanics, i.e. to the estimation of stress intensity factors and assessment of the possibility of failure of a cracked body. We shall give some examples.

5.1 Consider an axisymmetric piecewise homogeneous body having the symmetry plane $x_3=0$ normal to the symmetry axis ($x_1=x_2=0$). In the plane $x_3=0$ let there be a penny-shaped crack of radius R with centre at the origin. Let uniform wedging loads act on the crack surface. Let $d_1,\ldots,d_i$ denote the characteristic dimensions* of the body, and $\nu_1,\ldots,\nu_K$ and $E_1,\ldots,E_K$ be the elastic constants of its homogeneous parts. As the problem is linear, from dimensional considerations it follows that the total potential energy of a body with a crack is

$$W = \frac{P^2R^3}{E_1}\Phi\left(\frac{d_1}{R},\ldots,\frac{d_K}{E_1},\frac{E_2}{E_1},\ldots,\frac{E_K}{E_1},\nu_1,\ldots,\nu_K\right)$$

$$= \frac{P^2R^3}{E_1}\Phi\left(\xi_m,\frac{E_n}{E_1},\nu_t\right) \qquad (5.1)$$

m = 1,...,i
n = 1,...,K-i
t = 1,...,K

where the Young's modulus of that part containing the crack edge is denoted by E_1 .

Differentiating (5.1) with respect to R, we obtain

$$\frac{\partial W}{\partial R} = \frac{3P^2R^2}{E_1}\Phi\left(\xi_m,\frac{E_n}{E_1},\nu_t\right) - \frac{P^2R}{E_1}\sum_{m=1}^{i}\frac{\partial\Phi}{\partial\xi_m}dm \qquad (5.2)$$

*Here in considering the boundaries of an axisymmetric body or cavity it is assumed that the characteristic dimensions $d_1,\ldots d_i$ are so chosen that they do not diminish under all possible axisymmetric expansions of the body or cavity.

We shall distinguish two particular cases:

1. $\partial\Phi/\partial\xi_m > 0$, m = 1,...,i (5.3)

2. $\partial\Phi/\partial\xi_m < 0$, m = 1,...,i (5.4)

Note that the signs of $(\partial\Phi/\partial\xi_m)$ for many practically interesting cases can be directly determined using Theorem 4.1.

In the first case, obviously we have

$$\frac{\partial W}{\partial R} \le \frac{3P^2R^2}{E_1}\Phi \frac{3W}{R} \quad (5.5)$$

and in the second case

$$\frac{\partial W}{\partial R} \ge \frac{3P^2R^2}{E_1}\Phi \frac{3W}{R} \quad (5.6)$$

According to Irwin's formula

$$N^2 = \frac{E_1}{2(1-\nu_1^2)\pi}\frac{1}{2\pi R}\frac{\partial W}{\partial R} \quad (5.7)$$

Hence, by virtue of (5.5) and (5.6) we obtain

$$N^2 \le \frac{3E_1}{4(1-\nu_1^2)\pi^2}\frac{W}{R^2}, \quad \frac{\partial\Phi}{\partial\xi_m} > 0$$
$$N^2 \ge \frac{3E_1}{4(1-\nu_1^2)\pi^2}\frac{W}{R^2}, \quad \frac{\partial\Phi}{\partial\xi_m} < 0 \quad (5.8)$$

Thus, the upper bounds of stress intensity factor for the first case and the lower bounds for the second case can be derived either from the energy W, or what is more important, from the upper and lower bounds of W respectively, which may be derived from Theorem 1* We shall illustrate these with the help of examples:

1. Consider a space with an axisymmetric cavity surrounded by a crack of radius R in the plane $x_3=0$, assuming that a pressure P acts in the cavity and on the crack. Let d_1 denote the diameter of the cavity in the plane of the crack. From Theorem 4.1 it follows that W increases with the increasing cavity size for a fixed R.

*The closeness of the estimated values of $\partial W/\partial R$ in (5.5) and (5.6) to the true values depends on the ratio between the retained and discarded terms in (5.2). Inequalities (5.5) and (5.6) become equalities for a penny-shaped crack in a homogeneous space. It is not difficult to show that the terms discarded from (5.2) will be small compared to the retained terms if $R/d_m \ll 1$ or $R/d_m \gg 1$. In these cases, the bounds of stress intensity factor may be expected to be very accurate. In other cases, the bounds are only a rough approximation.

Therefore, in this case we have $\partial\Phi/\partial\xi_m > 0$ and (5.8) can be used to estimate the stress intensity factor. Note that by virtue of Theorem 1, we have $\Phi(\xi_1,\ldots,\xi_m) \le \Phi(1,\xi_2,\ldots,\xi_m)$ where $W(R,1,\xi_2,\ldots,\xi_m,P,E) = \frac{P^2R^3}{E}\Phi(1,\xi_2,\ldots,\xi_m)$ is the energy of an elastic space with a cavity obtained by any axisymmetric expansion of the initial cavity in such a way that the cavity intersection contour with the $x_3=0$ coincides with the crack contour. Thus, from (5.8) we have

$$N^2 = \frac{3E}{4(1-\nu^2)\pi^2R^2} W(R,1,\xi_2,\ldots,\xi_m,P,E) \quad (5.10)$$

In particular, for a spherical cavity, from (5.10) we have

$$N^2 \le \frac{3E}{4(1-\nu^2)\pi^2R^2} W(R,1,P,E) = \frac{3P^2R}{4\pi(1-\nu)} \quad (5.11)$$

where W(R,1,P,E) is the energy of a space with a spherical cavity of radius R. For $\nu = 0.25$, we find that $N \le 0{,}56P\sqrt{R}$, and for $\nu = 0.5$, we have $N < 0{,}7P\sqrt{R}$. Note that for a penny-shaped crack of radius r_0, $N = \frac{\sqrt{2r_0}}{\pi}P \approx 0.45\sqrt{r_0}$. Thus, a crack of radius R which protrudes out of a spherical cavity is less hazardous than a penny-shaped crack of radius $r_{ef} = \frac{3\pi}{8(1-\pi)}R$ (for example, $r_{ef} = \frac{1}{2}\pi R$ when $\nu = 0.25$).

2. Under the assumptions made at the beginning of 1, consider a homogeneous body of finite dimensions with a penny-shaped crack. For this case, Theorem 1 gives that $\partial\Phi/\partial\xi_m < 0$, and thus the lower bound of N can be deduced from (5.9). Consider a sphere with centre at the origin and diameter equal to the diameter of the initial body and a right circular cylinder, its generator being parallel to the symmetry axis. The cylinder envelops the sphere. From Theorem 1 it is evident that

$$W \ge W_s(R,\frac{d_1}{R},P,E,\nu) \ge W_c(R,\frac{d_1}{R},P,E,\nu) \quad (5.12)$$

where W_s and W_c are the energies of the sphere and the cylinder with the same crack, respectively. From (5.9), therefore it follows that

$$N^2 \ge \frac{3E}{4\pi^2(1-\nu^2)}\frac{W_s}{R^2} \ge \frac{3E}{4\pi^2(1-\nu^2)}\frac{W_c}{R^2} \quad (5.13)$$

The quantitites W_s and W_c can be calculated from the solutions of the axisymmetric problem for a crack in a sphere [7] and a cylinder [8], respectively. Using (5.13) we shall estimate the stress intensity factor N_s on the contour of a penny-shaped crack in a sphere under no load.

Table 1 lists the exact values of N_s/N_∞ taken from [7], and estimated values corresponding to the middle and right-hand side terms in (5.13) denoted by X_{Bs} and X_{Bc}, and the values calculated from the data reported

Table 1

R/d_1	0.5	0.7	0.8
N_s/N_∞ [7]	1.156	1.32	1.47
X_{Bs} [7]	1.08	1.17	1.25
X_{Bc} [8]	1.03	1.12	1.21

in [7,8]*.

From the table it is seen that the estimated values are quite close to the true values. For $R/d_1 < 0.5$, the agreement, as would be expected, is even better.

3. In an infinite body with elastic constants E_1 and ν_1, let there be a spherical inclusion of radius ρ with centre at the origin in perfect adhesion with the medium; let the elastic constants of the inclusion be E_2, ν_1. Assume that a circular crack, with constant pressure P acting on its surface, envelopes the inclusion in the plane $x_3=0$ and occupies the domain $\rho \leq r \leq R$.

If the inclusion is more rigid than the matrix $E_2>E_1$, then we have $\partial\Phi/\partial\xi<0$ $(\xi=\rho/R)$, and the stress intensity factor on the contour of the crack, N_{R_1}, is bounded from below by (5.9). Clearly, the quantity

$W\left(\frac{\rho}{R}, \frac{E_2}{E_1}, P, \nu_1\right) \geq W\left(\frac{\rho}{R}, \frac{E_1}{E_2}, P, \nu_1\right)$ in (5.9) is the potential energy of a homogeneous body with an annular crack and elastic constants equal to the constants of the inclusion. Therefore, from (5.9) we have

$$N_{R_1}^2 \geq \frac{3E_1}{4\pi^2(1-\nu_1^2)R^2} W\left(\frac{\rho}{R}, \frac{E_2}{E_2}, P, \nu_1\right) \tag{5.14}$$

If the inclusion is less rigid than the matrix, $E_2<E_1$, then the stress intensity factor N_{R_2} is found similarly from (5.8) as

$$N_{R_2}^2 \leq \frac{3E_1}{4\pi^2(1-\nu_1^2)R^2} W\left(\frac{\rho}{R}, \frac{E_2}{E_2}, P, \nu_1\right) \tag{5.15}$$

The quantity $W\left(\frac{\rho}{R}, \frac{E_2}{E_2}, P, \nu_1\right)$ can be calculated from the data reported in

*We may mention that Fig. 1 in [8] does not exactly specify the scale for the vertical axis. The corresponding reduction coefficients can easily be established with the help of formulae (3.10), (3.11) and the data listed in Table 1 published in [8].

[9], which gives the numerical results of solution of the problem of a flat annular crack in a homogeneous elastic space. For example, for $\nu_1 = \nu_2 = 0.3$ and $R = 1$, $\rho = 0.7$, according to [9]:

$$N_{R_1}^2 \geq \frac{E_1}{E_2} P^2 0.027 \quad , \quad N_{R_2}^2 \leq \frac{E_1}{E_2} P^2 0.027 \tag{5.16}$$

4. Let the piecewise homogeneous body described in Section 1 be an infinite body with an inclusion in perfect adhesion with the body. A penny-shaped crack is wholly contained within the inclusion. The elastic constants of the inclusion are E_1, ν_1, and of the surrounding medium are E_2, ν_2. If $E_1 > E_2$, then from (5.9) we may estimate the stress intensity factor from below. Using Theorem 1 we may write the following sequence of inequalities:

$$W \geq W_s\left(R, \frac{\rho}{R}, \frac{E_2}{E_1}, \nu_1\right) \geq W_c\left(R, \frac{\rho}{R}, \frac{E_2}{E_1}, \nu_1\right) \geq W_\infty(R,\infty,E_1,\nu_1) \tag{5.17}$$

where $W_s(R,\rho/R,E_2/E_1,\nu_1)$ is the total potential energy of the body with spherical inclusion of radius equal to the diameter of the initial $W_c\left(R, \frac{\rho}{R}, \frac{E_2}{E_1}, \nu_1\right)$ is the potential energy of a body containing a cylindrical inclusion, its generator being parallel to the axis of symmetry of the body and W_∞ is the energy of space with constants E_1, ν_1 and containing the same penny-shaped crack. Substituting (5.17) into (5.9), for $E_1>E_2$ we obtain

$$N^2 \geq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_s}{R^2} \geq \frac{3E_2}{4\pi^2(1-\nu_1^2)} \frac{W_c}{R_c^2} \geq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_\infty}{R^2} = N_\infty^2 \tag{5.18}$$

Here N_∞ is the exact value of stress intensity factor for a penny-shaped crack in a space with constants E_1, ν_1. Similarly, for an inclusion less rigid than the matrix $E_1<E_2$, by virtue of (5.8), we obtain a system of inequalities of opposite sense:

$$N^2 \geq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_s}{R^2} < \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_c}{R^2} \leq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_\infty}{R^2} = N_\infty^2 \tag{5.19}$$

We may mention here one qualitative implication that follows from (5.18) and (5.19):

In a space with a penny-shaped crack, if the rigidity is decreased (increased) outside some axisymmetric domain containing a crack, the stress intensity factor on the crack contour decreases (does not increase).

5. Consider a cylinder containing a penny-shaped crack in perfect adhesion with a rigid medium and under conditions of sliding contact at the boundary. Then

$$N_{c(r,c)}^2 \leq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_{c(r,c)}}{R^2} \leq \frac{3E_1}{4\pi^2(1-\nu_1^2)} \frac{W_{c(s,c)}}{R^2} \tag{5.20}$$

where $N_{c(r,c)}$, $W_{c(r,c)}$ are the stress intensity factor and total energy for the case of perfect adhesion of the cylinder to the medium and $W_{c(s,c)}$ is the total energy for the case of sliding contact at the interface. The values of $N_{c(r,c)}$ are reported in [10], while the values of $W_{c(s,c)}$ in [11]. In particular, for $R/d_1 = 0.6$, $N_{c(r,c)}/N_\infty = 0.87$, and $(N_{c(r,c)}/N_\infty)_B = 1.14$. It is clear that the agreement between the estimated and true values becomes worse as R/d_1 approaches unity. This is natural because in a problem with sliding contact, the value of N will increase without bounds as $R/d_1 \to 1$, while in the case of perfect adhesion there will be no such effect. Nonetheless, it should be borne in mind that the elasticity problems for a piecewise homogeneous body under sliding contact at the boundary are solved in far simpler manner than the corresponding problems under perfect adhesion at the boundaries. Therefore, the bounds of stress intensity factor for $R/d_1 \to 1$ in perfect adhesion problems can be obtained using the energy determined from the solution of the simpler problem with sliding at the boundaries.

5.2 If a crack grows being in the state of limit equilibrium, or at least there exists such a value of stress intensity factor N_* that the crack growth can be disregarded for $N \leq N_*$, then it is possible to derive the necessary conditions for a crack to be non critical directly from the energy bounds. Indeed, by Irwin's formula, to N_* there corresponds some value

$$T_* = \frac{\pi(1-\nu^2)}{E} N_*^2 \tag{5.21}$$

which is the minimum necessary work that has to be done to increase the area of the crack by unity.

Then, evidently, an increase in the crack area from S_o to some value S_1 should call for an energy expenditure not less than $T_*(S_1-S_o)$.

Therefore, if we could demonstrate that the external forces and potential energy of deformation do not provide the necessary amount of energy at some stage of crack growth, that would naturally imply that the initial crack is not critical.

Suppose that the external loads are specified. Then it is natural to construct the corresponding energy diagram for the total energy. Let the initial crack contour Γo of area S_o be given. If the crack is dangerous, then as the crack grows under invariable loads, the area S will increase and the total energy will take the value W(S); $W(S_o) = W_o$ If we could show that

$$\min_{S_o \leq S_{c\infty}} [W(S) - W_o - T_*(S - S_o)] < 0 \tag{5.22}$$

then it would be a sufficient condition for the crack to be non critical. Of course, a sufficient condition is also given by a more approximate inequality in which W(S) is replaced by some majorant, and W_o by some lesser quantity. This gives a means to make use of the energy bounds. In order to estimate the lower bound of the energy W_o, we may either take a more rigid part of the body, or replace the contour Γo by a simpler contour Γ_* enclosed in it, or use both these two approaches.

The concepts of extremal contours can be used in finding the upper bound of W(S). From the already proved inequality $W(S) \leq W^*(S)$, where $W^*(S)$ is the energy of the body with the crack, whose contour coincides with that of an extremal one of area S. If we modify the body, say by decreasing the rigidity of the material at some of its subdomain, then the corresponding function of energy of extremal contours $W_1{}^*(S)$ will majorize the function $W^*_1(S)$:

$$W_1{}^*(S) \geq W^*_1(S) \tag{5.23}$$

Indeed, let Γ and Γ_1 be the extremal contours of area S in the initial and less rigid bodies respectively. Then on decreasing the rigidity of the initial body without altering the contour Γ, we obtain

$$W(\Gamma) = W^*(S) \leq W_1(\Gamma)$$

on the other hand, by the properties of extremal contours in an "unaltered" body, we have

$$W_1(\Gamma) \leq W^*_1{}_1(S)$$

Hence, (5.23) follows from this inequality.

6. Bounds for certain integral characteristics of solutions of contact problems of elasticity

Using Theorem 4.1 and arguments similar to those applied in proving it, we may construct double-sided bounds for the displacement of a die (or) the force which acts on the die when the contact area, die shape, and contact conditions vary. Similar theorems are known for contact problems of perfectly plastic bodies.

Theorem 1: Let a rigid die with sharp edges and flat base Ω be pressed into an elastic body by a force acting normal to the die base so that the die moves in the direction of action of the force. (The die is either in sliding contact or in perfect adhesion with the elastic body.)

If the rigidity is decreased (increased) at some part of the body, for the die displacement to be the same it is necessary that the force be diminished (increased).

Indeed, the force is

$$\underline{P} = \iint_\Omega \vec{\sigma}_n(x)\,d\sigma \tag{6.1}$$

Therefore, by virtue of Corollary 2 of Theorem 4.1 (inequality (4.10)) for $g(x) = h = \text{const}$ we have

$$h\underline{P} \geq h\underline{P}_1 \ , \ \underline{P} > \underline{P}_1 \tag{6.2}$$

Corollary 1

The force driving the die being constant, a decrease (increase) in the rigidity at some part of the body causes an increase (decrease) in the displacement of a die with flat base. (This is an obvious sequel of Theorem 1).

Corollary 2

In order that two smooth or perfectly adhered dies with flat bases Ω and Ω_1 $(\Omega>\Omega_1)$ may have the same displacement, it is necessary that the force applied to the die Ω should be greater than the force applied to the die Ω_1 $(P>P_1)$.

From Theorem 1 it follows that the transition from the die Ω_1 to the die Ω may be regarded as an infintely great increase in the rigidity of the initial body on a part of the boundary Ω/Ω_1.

Corollary 3

For the same driving force, on passing from the die Ω_1 to the die $\Omega>\Omega_1$, the displacement decreases. (This is a trivial consequence of Corollary 2.)

Theorem 1 and its corollaries 1, 2 and 3 give a means for constructing two-sided bounds for the displacement of dies of complicated shapes with flat base Ω from the solution for dies of simpler forms whose bases Ω_1 and Ω_2 are inscribed in or circumscribe Ω : $\Omega_1 < \Omega < \Omega_2$. Examples of the application of such considerations for a smooth die are given in [12-15]. By way of example we shall estimate the displacement of a square die in perfect adhesion with a semi-space on the basis of the exact solution of the problem of a circular die in adhesion with a semi-space [16].

Let 2α be the side of the square. Consider two circular dies Ω_1 and Ω_2 of radii α and $\alpha\sqrt{2}$ respectively. According to [16] :

$$w_1 = \frac{1}{16\theta(3\mu+\lambda)}\frac{P}{\alpha}, \qquad w_1 = \frac{1}{16\theta(3\mu+\lambda)}\frac{P}{\alpha\sqrt{2}} \tag{6.3}$$

where $\theta = \frac{1}{\pi}\ln[(3\mu+\lambda)/(\mu+\lambda)]$, λ and μ is the Lamé coefficient. Therefore

$$w_2 \leq w \leq w_1$$

If the approximate value of w is taken to be $\frac{1}{2}(w_1+w_2)$ then the error in the determination of w by means of (6.4) and (6.5) will not exceed $[(w_1-w_2)/(w_1+w_2)] \leq 17\%$.

Corollary 4

Consider two dies with nonplanar bases of arbitrary configuration with sharp edges, assuming that their contact areas are the same :

$$x_3 = -\psi_1(x_1,x_2) \leq 0, \quad x_3 = -\psi_2(x_1,x_2) \leq 0$$

$$\psi_1(x_1,x_2) \leq \psi(x_1,x_2), \quad \forall (x_1,x_2)\in\Omega \tag{6.5}$$

Then, for the same driving force P, the displacement of the enveloping die ψ_2 is less than that of the enveloped die ψ_1. This is proved by applying Corollary 1 twice : by passing from a die with flat base to the die ψ_2 by successively increasing the rigidity in the region between the flat base and the surface $\psi_1(x_1,x_2)$ and between the surfaces $\psi_2(x_1,x_2)$ and

$\psi_1(x_1,x_2)$.

In particular, for the same driving force P and contact area Ω, the displacement of a die with nonplanar base is less than that of the enveloped die with flat base.

In Corollary 4 the contact areas of dies have been taken to be equal. However, as in the proof of Theorem 6.1, we may show that if the initial die with contact region Ω and surface $\psi(x_1,x_2)$ is replaced by some other die of contact region $\Omega_1<\Omega$ and surface $\psi_1(x_1,x_2) \leq \psi(x_1,x_2)$, $\forall(x_1,x_2)\in\Omega_1$ then the displacement increases for the same driving force.

The same considerations can be used to prove :

Theorem 6.2

Let a die with flat base Ω of arbitrary configuration be pressed by a force P into an elastic body so that one of the following three conditions is satisfied : 1) no friction at the contact region, 2) friction forces act at the contact region, 3) adhesion exists between the die and the elastic space. Then

$$w_1 > w_2 > w_3 \tag{6.6}$$

where w_1, w_2, w_3 are the displacements of the die under contact conditions (1), (2) and (3) respectively.

Proof :

First we shall establish that $w_1 > w_2$. Under contact condition (1), the total energy of the system is

$$W_1(\vec{u}_1) = \int_V A(\vec{u}_1)\, d\tau - w_1P \tag{6.7}$$

where V is the spatial domain occupied by the body.

Choose as the admissible displacement field of a smooth die the field of displacement corresponding to the action of a die under condition (2). Then, by the minimum potential energy principle, we have

$$W_1(\vec{u}_1) \leq W_1(\vec{u}_2) = \int_V A(\vec{u}_2)\, d\tau - w_2P \tag{6.8}$$

On the other hand, when the die acts under friction

$$W_2(\vec{u}_2) = \int_V A(\vec{u}_2)\, d\tau - [w_2P - W_{\psi r}] \tag{6.9}$$

where $W_{\psi r}$ is the work of friction. By virtue of the Clapeyron theorem for equilibrium, for a die under condition (2), we have

$$\int_V A(\vec{u}_2)\, d\tau = \frac{1}{2}w_2P - \frac{1}{2}W_{\psi r} \tag{6.10}$$

and under condition (1)

$$\int_V A(\vec{u}_1)\, d\tau = \frac{1}{2}w_1P \tag{6.11}$$

From (6.7), (6.8), by virtue of (6.10) and (6.11), we obtain

$$-\frac{1}{2}w_2P - \frac{1}{2}W_{\psi r} \geq -\frac{1}{2}w_1P \qquad (6.12)$$

or

$$w_2 \leq w_1 - \frac{1}{2}\frac{W_{\psi r}}{P} \qquad (6.13)$$

Since the work of friction ($-W_{\psi r}$) is negative, Eq. (6.13) shows that the displacement under friction at the contact region is less than the displacement under sliding contact.

The inequality $w_2 > w_3$ is proved in a similar way by taking the field of displacement corresponding to the action of a fully connected die in equilibrium as the admissible displacement field for the action of a die under friction.

APPENDIX A

Consider a three-dimensional elasticity problem for a layer of thickness $H = 2h$ containing an opening-mode crack occupying the domain G in the midplane $x_3 = 0$. Assume that there are no tractions at the layer faces and only normal wedging loads $q(x_1,x_2)$ act on the crack surface. The boundary conditions of the problem are thus of the form :

$$\begin{aligned} &x_3 = 0 \qquad \sigma_{33} = -q(x_1,x_2),\ (x_1,x_2)\varepsilon G,\ \sigma_{13} = \sigma_{23} = 0 \\ &x = \pm h \qquad \sigma_{33} = \sigma_{31} = \sigma_{32} = 0 \end{aligned} \qquad (A.1)$$

We shall reduce this problem to an integral equation for $w(x_1,x_2)$ which is the displacement of points on the crack surface in the direction of x_3. For this purpose, as in the problem of a crack in an infinite space, we shall first calculate the stresses at the layer boundaries $x_3 = \pm h$ assuming that the function $w(x_1,x_2)$, i.e. the shape of the crack, is known. Then we shall write the solution to the problem of a layer (without cracks), whose faces are subjected to the action of tractions equal but opposite to stresses calculated in the first problem. Finally we shall compute the sum of normal stresses which arise in both the problems at the crack and then equate it to the given loads. As a result, we shall obtain the required integral equation.

In expressing the solution of a problem for an infinite space, it is convenient to use the Papkovich-Neuber representations [5,17]. In our case

$$u = -x_3\frac{\partial B_3}{\partial x_1} - \frac{\partial B_o}{\partial x_1}, \qquad v = -x_3\frac{\partial B_3}{\partial x_2} - \frac{\partial B_o}{\partial x_2}$$

$$w = 2(1-\nu)B_3 - x_3\frac{\partial B_3}{\partial x_3} \qquad (A.2)$$

$$\frac{1}{2\mu}\sigma_{13} = -x_3\frac{\partial^2 B_3}{\partial x_1 \partial x_3}, \qquad \frac{1}{2\mu}\sigma_{23} = -x_3\frac{\partial^2 B_3}{\partial x_2 \partial x_3}$$

$$\frac{1}{2}\sigma_{33} = \frac{\partial B_3}{\partial x_3} - x\cdot\frac{\partial^2 B_3}{\partial x_3^2} \qquad (A.3)$$

where $\partial B_o/\partial x_3 = (1-2\nu)B_3$ (by virtue of symmetry relative to the plane $x_3 = 0$ [17]).

The function B_3 can be represented as

$$B_3 = -\frac{1}{4(1-\nu)}\frac{\partial}{\partial x_3}\iint_G \frac{r(\eta_1,\eta_2)\,d\eta_1\,d\eta_2}{\sqrt{(x_1-\eta_1)^2+(x_2-\eta_2)^2+x_3^2}} \qquad (A.4)$$

Then from (A.2) and (A.4) we get

$$w(x_1,x_2,+0) = \begin{cases} r(x_1,x_2) & ,\ (x_1,x_2)\varepsilon G \\ 0 & ,\ (x_1,x_2)\bar{\varepsilon} G \end{cases} \qquad (A.5)$$

i.e. the function $r(x_1,x_2)$ is identical with the displacement of points on the crack surface in the direction of x_3. For a crack in an infinite space, $r(x_1,x_2)$ satisfies the following equation :

$$-\frac{2(1-\nu)}{\mu}q(x_1,x_2) = \Delta_{x_1,x_2}\iint_G \frac{r(\eta_1,\eta_2)d\eta_1 d\eta_2}{\sqrt{(x_1-\ _1)^2+(x_2-\ _2)^2}} \qquad (A.6)$$

It is more convenient to carry out further calculations with the Fourier transform in (x_1,x_2) with the parameter $\xi = (\xi_1,\xi_2)$. It is not difficult to prove (see, for example, [18]) that

$$\tilde{B}_3 = \frac{\pi}{2(1-\nu)}\tilde{r}e^{-x|\xi|} \qquad (A.7)$$

The stresses in the plane $x = h$ are given by the following expressions :

$$\begin{aligned} &\frac{1}{2\mu}\ (x_1,x_2,h) = \frac{\partial}{\partial x_1}F_\xi^{-1}(\tilde{\tau}) = \tau_1 \\ &\frac{1}{2\mu}\ (x_1,x_2,h) = \frac{\partial}{\partial x_2}F_\xi^{-1}(\tilde{\tau}) = \tau_2 \end{aligned} \qquad (A.8)$$

$$\frac{1}{2\mu}\sigma_{33}(x_1,x_2,h) = \frac{\pi}{2(1-\nu)}F_\xi^{-1}\{|\xi|\tilde{r}(1+h|\xi|)e^{-h|\xi|} = \sigma \qquad (A.9)$$

where

$$\tilde{\tau} = \frac{\pi}{2(1-\nu)}h|\xi|\tilde{r}e^{-h|\xi|} \qquad (A.10)$$

Using the formulae given in [17], we shall now write the expressions for Fourier transforms of stresses in a layer under two types of boundary conditions :

$$\text{a)}\quad x_3 = \pm h\ ,\quad \sigma_{33} = -\sigma\ ,\quad \sigma_{31} = \sigma_{32} = 0 \qquad (A.11)$$

b) $x_3 = \pm h$, $\sigma_{33} = 0$, $\sigma_{31} = \pm \tau_1$, $\sigma_{32} = \pm \tau_2$ (A.12)

According to [17], under the conditions (a), we have

$$\tilde{\sigma}_{33}^{(a)} = \{h\ ch(h|\xi|)ch(x_3|\xi|) - x_3 sh(h|\xi|)sh(x_3|\xi|) + |\xi|^{-1} sh(h|\xi|) - ch(x_3|\xi|)\} \frac{2\tilde{\sigma}|\xi|}{\Delta}, \quad \Delta = 2h|\xi| + sh(2h|\xi|)$$

$$\tilde{\sigma}_{31}^{(a)} = -i\xi_1\{h\ ch(h|\xi|sh(x_3|\xi| - x_3 ch(x_3|\xi|)sh(h|\xi|)\} \frac{2\tilde{\sigma}}{\Delta}$$

$$\tilde{\sigma}_{32}^{(a)} = -i\xi_2\{h\ ch(h|\xi|)sh(x_3|\xi|) - x_3 ch(x_3|\xi|)sh(h|\xi|)\} \frac{2\tilde{\sigma}}{\Delta} \qquad (A.13)$$

For tangential tractions of the type (b), according to [17], we have

$$\tilde{\sigma}_{33}^{(b)} = \{x_3 sh(x_3|\xi|)\ ch(h|\xi|) - h\ sh(h|\xi|)ch(x_3|\xi|)\} \frac{2\tilde{\tau}|\xi|^2}{\Delta}$$

$$\tilde{\sigma}_{31}^{(b)} = -i\xi_1 F(x_3,\xi_1,\xi_2,\tilde{\tau})$$

$$\tilde{\sigma}_{32}^{(b)} = -i\xi_2 F(x_3,\xi_1,\xi_2,\tilde{\tau}) \qquad (A.14)$$

where

$$F(x_3,\xi_1,\xi_2,\tilde{\tau}) = \{x_3 ch(x_3|\xi|)ch(h|\xi|) - h\ sh(x_3|\xi|)sh(h|\xi|) + |\xi|^{-1} sh(x_3|\xi|)ch(h|\xi|)\} \frac{2\tilde{\tau}|\xi|}{\Delta}$$

Now, using (A.13) and (A.14), we shall calculate the additional stresses $\tilde{\sigma}_{33}^{(ad)}(x_1,x_2,0)$ which arise in the problem of a layer without any cracks under the joint action of loads (a) and (b) :

$$\frac{2(1-\nu)}{\mu} \tilde{\sigma}_{33}^{(ad)}\Big|_{x_3=0} = \frac{4\pi|\xi|e^{-h|\xi|}\tilde{r}}{\Delta} \{(h|\xi|ch(h|\xi|) + sh(h|\xi|)) \times (1 + h|\xi|) + h^2|\xi|^2 sh(h|\xi|)\} \qquad (A.15)$$

With the help of Fourier transforms (A.6) and (A.15), finally we obtain the following pseudo-differential equation [18] in $r(x_1,x_2)$:

$$2\pi F_{\xi}^{-}\{[-|\xi| + K(h,|\xi|)]\tilde{r}(\xi_1,\xi_2)\} = -\frac{2(1-\nu)}{\mu} q(x_1,x_2)(x_1,x_2)\epsilon G \qquad (A.16)$$

where

$$K(h,|\xi|) = \frac{1}{h} K(h|\xi|)$$

$$K(h|\xi|) = \frac{2h|\xi|(1+2h|\xi|+2h^2|\xi|^2 - e^{-2h|\xi|})}{4h|\xi| + e^{2h|\xi|} - e^{-2h|\xi|}}$$

APPENDIX B

To establish the method of convergence of successive approximations applied to Eq. (2.2), we shall first find the magnitude of $(q_{i+1}-q_i)$. From (2.6) we have

$$q_{i+1} - q_i \le \frac{\mu}{2(1-\nu)} \frac{1}{2\pi} \iint_{-\infty}^{\infty} K(h|\xi|)|\tilde{r}_i - \tilde{r}_{i-1}|d\xi \qquad (B.1)$$

From the definition of $\tilde{r}$ it follows that

$$|\tilde{r}_i - \tilde{r}_{i-1}| \le \iint_G w_i - w_{i-1}\ dS = V_i \qquad (B.2)$$

Thus, with the help of (B.1) and (B.2), we obtain

$$w_{i+1} - w_i | \le R_\infty |q_{i+1} - q_i| \le \frac{\mu R_\infty}{2(1-\nu)} \frac{1}{2\pi} \iint_{-\infty}^{\infty} K(h|\xi|) \iint_G |w_i - w_{i-1}| dS d\xi \qquad (B.3)$$

Integrating over the domain G we get

$$V_{i+1} \le \left[\iint_G h_\infty \cdot 1 dS\right] \frac{\mu}{2(1-\nu)} \frac{V_i}{2\pi} \iint_{-\infty}^{\infty} K(h|\xi|)d\xi \qquad (B.4)$$

After changing to polar coordinates in the plane $\eta = h\xi$ the integral of $K(\eta)$ in (B.4) takes the form

$$T_1 = \frac{1}{2\pi} \iint_{-\infty}^{\infty} K(h|\xi|)d\xi = \frac{2}{h^3} \int_0^{\infty} \frac{r^2(1+2r\pm 2r^2 - e^{-2r})dr}{4r + e^{2r} - e^{-2r}} \qquad (B.5)$$

It is not difficult to calculate the numerical value of the integral which is

$$T = \frac{4.232}{h^3} \qquad (B.6)$$

The integral of $(R_\infty \cdot 1)$ over the domain G is half the volume of a crack occupying the domain G in an infinite space and acted upon by a constant load of intensity 1 on its surface.

If the area of the domain G is S, then as shown in Section 4.2, the volume of crack is not greater than the volume of a penny-shaped crack of the same area, i.e.

$$\iint_G R_\infty \cdot 1 dS \le \frac{4(1-\nu)}{3\mu} \left(\frac{S}{\pi}\right)^{3/2} \qquad (B.7)$$

Then, due to (B.7), from (B.4), we have

$$V_{i+1} \le \frac{1}{2} \frac{S^{3/2}}{h^3} V_i = \epsilon_* V_i \qquad (B.8)$$

The convergence of successive approximations method is guaranteed if $\varepsilon_* < 1$, which holds true when

$$s^{\frac{1}{2}}/h < 1.26 \qquad \text{(B.8')}$$

If only the crack diameter d is known, then we can derive an estimate less accurate than (B.8). In this case we can write

$$\iint_G R_\infty \cdot 1 dS = \frac{2d^3}{\mu}(1 - \nu)\psi(g) \qquad \text{(B.9)}$$

where $\psi(g)$ is half the volume of a crack occupying the domain g of diameter 1 similar to the domain G. According to the theorem proved in Section 1, we can find the upper bound for $\psi(g)$ by calculating half the volume of a penny-shaped crack of diameter 1 enveloping the crack g :

$$\psi(g) \leq \frac{1}{8}\int_0^{2\pi}\int_0^{1}\sqrt{1 - r^2}\, r dr d\theta = \frac{1}{12} \qquad \text{(B.10)}$$

From (B.4) we obtain

$$V_{i+1} \leq (d|h)^3 0.353\, V_i = \varepsilon V_i \qquad \text{(B.11)}$$

Successive approximations converge if $\varepsilon < 1$, i.e., if $(d|h) < 1.42$ or $(d|H) < 0.71$.

The bounds (B.12) and (B.8) are coincident only for a penny-shaped crack, a fact which can be easily verified.

APPENDIX C

We shall first estimate the difference $q'-q_1^o$. We have

$$q'-q_1^o = \frac{\mu}{2(1-\nu)}\frac{1}{2\pi h^3}\iint_{-\infty}^{\infty} K(|\eta|)\left[\tilde{r}\left(\frac{\eta}{h}\right)e^{-i\left(\frac{x_1}{h}\eta_1 + \frac{x_2}{h}\eta_2\right)} - r_o(0)\right]d\eta \qquad \text{(C.1)}$$

Now express the right-hand side as a sum of two integrals by adding and subtracting the expression

$$r(0)e^{-i\left(\frac{x}{h},\eta\right)}$$

in the integrand of (C.1). Thus we get

$$q'-q_1^o = \frac{\mu}{2(1-\nu)}\frac{1}{2\pi h^3}(I_1 + I_2) \qquad \text{(C.2)}$$

where

$$I_1 = \iint_{-\infty}^{\infty} K(|\eta|)\left[\tilde{r}\left(\frac{\eta}{h}\right) - \tilde{r}(0)\right]e^{-i\left(\frac{x}{h},\eta\right)}d\eta \qquad \text{(C.3)}$$

$$I_2 = \iint_{-\infty}^{\infty} K(|\eta|)\left[\tilde{r}(0)e^{-i\left(\frac{x}{h},\eta\right)} - \tilde{r}_o(0)\right]d\eta \qquad \text{(C.4)}$$

Transform the integral I_1 as follows. Since

$$\tilde{r}\left(\frac{\eta}{h}\right) = \iint_{-\infty}^{\infty} w(u_1,u_2)e^{i\left(\frac{u}{h},\eta\right)}du$$

$$\tilde{r}(0) = \iint_{-\infty}^{\infty} w(u_1,u_2)du$$

and changing over to polar coordinates in the plane η ($\eta_1 = \rho\cos\theta$, $\eta_2 = \rho\sin\theta$) we obtain

$$I_1 = \int_0^{2\pi}\int_0^{\infty} K(\rho)\left\{\iint_G w(u_1,u_2)\left[e^{i\rho Z_u} - 1\right]du\right\}e^{-i\rho Z_x}\rho d\rho d\theta =$$

$$= \iint_G w(u_1,u_2)\int_0^{2\pi}\int_0^{\infty} K(\rho)\left[\cos\rho(Z_u - Z_x) - \cos\rho Z_x\right]\rho d\rho d\theta \qquad \text{C.5)}$$

where

$$Z_u = \frac{u_1}{h}\cos\theta + \frac{u_2}{h}\sin\theta\ , \quad Z_x = \frac{x_1}{h}\cos\theta + \frac{x_2}{h}\sin\theta$$

Now we shall estimate I_1 . We have

$$|I_1| = \left|\iint_G w(u_1,u_2)\int_0^{2\pi}\int_0^{\infty} K(\rho)\left[2\sin\left(\rho\frac{Z_u}{2}\right)\sin\rho\left(Z_x - \frac{1}{2}Z_u\right)\right]\rho d\rho d\theta du\right| \leq$$

$$\leq \iint_G w(u_1,u_2)\int_0^{2\pi}\int_0^{\infty} K(\rho)Z_u\left(Z_x - \frac{1}{2}Z_u\right)\rho^3 d\rho d\theta\, |du \qquad \text{(C.6)}$$

Since

$$Z_u\left(Z_x - \frac{1}{2}Z_u\right) = \frac{\sqrt{u_1^2 + u_2^2}}{h}\ \frac{\sqrt{(2x_1 - u_1)^2 + (2x_2 - u_2)^2}}{h}\ \sin(\theta - \psi_1)\sin(\theta - \psi)$$

the inequality (C.6) may be extended

$$|I_1| \leq \iint_G w(u_1,u_2)2\pi\cdot\frac{1}{2}\ \frac{\sqrt{u_1^2 + u_2^2}}{h}\ \frac{\sqrt{2(x_1 - u_1)^2 + (2x_2 - u_2)^2}}{h}\int_o^{\infty} K(\rho)\rho^3 d\rho \leq$$

$$\leq \sqrt{2}\pi d^2\int_0^{\infty} K(\rho)\rho^3 d\rho\iint_G w(u_1,u_2)du \qquad \text{(C.7)}$$

where due account has been taken of the fact that

$$\sqrt{(2x_1-u_1)^2 + (2x_2-u_2)^2} \le 2(3d)^2$$

In (C.7) the integral with respect to ρ is easily calculated and it is equal to

$$\int_0^\infty K(\rho)\rho^3 d\rho = 29.702 \tag{C.8}$$

Further

$$\left|\iint_G w(u_1,u_2)du\right| = \left|\iint_G \left[w_o+(w-w_o)\right]du\right| \le \iint_G w_o du + \iint_G |w-w_o|du \tag{C.9}$$

By definition

$$V_\infty = \iint_G |w - w_o| du \tag{C.10}$$

and according to Appendix B

$$V_\infty \le V_1 + V_2 + \ldots \le \frac{V_1}{1 - \varepsilon} \tag{C.11}$$

where $\varepsilon = 0.353 (d|h)^3$.

Thus, from (C.7), by virtue of (C.9) and (C.11), we get the bound of I_1 :

$$|I_1| \le 3\sqrt{2\pi}\cdot 29.702\left(\frac{d}{h}\right)^2 \left(\iint_G w_o du + \frac{V_1}{1 - \varepsilon}\right) \tag{C.12}$$

Using the expressions $V_1 = V_o\varepsilon$ and V_o, we may rewrite (C.12) as follows :

$$|I_1| \le 3 \sqrt{2\pi} 29.702\left(\frac{d}{h}\right)^2 V_o\left(1 + \frac{\varepsilon}{1 - \varepsilon}\right) \tag{C.13}$$

Now we shall derive the bounds of the integral I_2. After changing over to polar coordinates in the plane η, we obtain :

$$|I_2| = \left|\int_0^{2\pi}\int_0^\infty K(\rho) \iint_G \left[w(u_1,u_2)e^{-i\rho Z_x} - w_o(u_1,u_2)\right]du\rho d\rho d\theta\right|$$

$$= \left|\iint_G \left\{\int_0^{2\pi}\int_0^\infty K(\rho) \left[w(u_1,u_2)\cos(\rho Z_x - w_o(u_1,u_2)\right]\rho d\rho d\theta\right\} du\right.$$

$$= \left|\iint_G \left\{\int_0^{2\pi}\int_0^\infty K(\rho)\left[(w(u_1,u_2)-w_o(u_1,u_2))\cos(\rho Z_x)\right.\right.\right.$$

$$\left. - w_o(u_1,u_2)(1-\cos(\rho Z_x))\right] \times \rho d\rho d\theta\Big\} du \le \{\Lambda_1 + \Lambda_2\} \tag{C.14}$$

$$\Lambda_1 = \left\{\iint_G \left|\int_0^{2\pi}\int_0^\infty K(\rho)[(w(u_1,u_2)-w_o(u_1,u_2))\cos(\rho Z_x)]\rho d\rho d\theta\right| du\right\}$$

$$\Lambda_2 = \left\{\iint_G \left|\int_0^{2\pi}\int_0^\infty K(\rho)w_o(u_1,u_2)(1-\cos(\rho Z_x))\rho d\rho d\theta\right| du\right\} \tag{C.15}$$

In the first integral Λ_1 in (C.15), substituting unity for $\cos\rho Z_x$, and then integrating with respect to θ we get

$$\Lambda_1 \le 2\pi V_\infty \int_0^\infty \rho K(\rho)d\rho \tag{C.16}$$

where

$$V_\infty \le \frac{V_1}{1 - \varepsilon} \tag{C.17}$$

The last inequality follows from (C.11). The integral is

$$\int_0^\infty \rho K(\rho)d\rho = 4.232 \tag{C.18}$$

Therefore

$$\Lambda_1 \le \frac{V_1}{1 - \varepsilon}(2\pi)\cdot 4.232 \tag{C.19}$$

The second integral, Λ_2, after changing the order of integration, takes the form :

$$\Lambda_2 = \iint_G w_o(u_1,u_2)|\Lambda|du \tag{C.20}$$

where

$$\Lambda = \int_0^{2\pi}\int_0^\infty K(\rho)(1-\cos\rho Z_x)\rho d\rho d\theta \tag{C.21}$$

Introduce the function

$$K_1(\rho) = -\int_\rho^\infty rK(r)dr \;, \qquad K_2(\rho) = -\int_\rho^\infty K_1(r)dr \tag{C.22}$$

and integrating (C.21) twice in parts, we find that

$$\Lambda = \int_0^{2\pi} Z_x^2 \int_0^\infty K_2(\rho)\cos(\rho Z_x)\,d\rho\,d\theta = \int_0^\infty (\rho) \int_0^{2\pi} Z_x^2 \cos(\rho Z_x)\,d\theta\,d\rho \qquad (C.23)$$

Since $Z_x = \frac{x_1^2 + x_2^2}{h}\sin^2(\theta-\psi)$, $\psi = \mathrm{arctg}\,\frac{x_1}{x_2}$

(C.23) can be rewritten as

$$\Lambda = \frac{x_1^2+x_2^2}{h^2}\int_0^\infty K_2(\rho)\int_0^{2\pi}\sin^2(\theta-\psi)\cos\frac{\sqrt{x_1^2+x_1^2}}{h}\sin(\theta-\psi)\,d\theta\,d\rho \qquad (C.24)$$

Substituting unity for cosα, we get

$$|\Lambda| \le \frac{x_1^2+x_2^2}{h^2}\,\pi\int_0^\infty K_2(\rho)\,d\rho \le \pi\left(\frac{d}{h}\right)^2\int_0^\infty K_2(\rho)\,d\rho \qquad (C.25)$$

Hence, after integrating twice in parts, we obtain

$$|\Lambda| \le \left(\frac{d}{h}\right)^2\frac{\pi}{2}\int_0^\infty K(\rho)\rho^3\,d\rho = \left(\frac{d}{h}\right)^2\pi\cdot 14.851 \qquad (C.26)$$

Thus,

$$|\Lambda_2| \le \iint_G w_o(u_1,u_2)\,du \left(\frac{d}{h}\right)^2 \pi\cdot 14.851 = V_o\left(\frac{d}{h}\right)^2\pi\cdot 14.851 \qquad (C.27)$$

Finally, by virtue of (C.14), (C.19), and (C.27), we obtain a bound for I_2 :

$$I_2 \le \left\{\frac{\varepsilon V_o}{1-\varepsilon}\,2\pi\cdot 4.232 + V_o\left(\frac{d}{h}\right)^2\pi\cdot 14.851\right\} \qquad (C.28)$$

From (C.2), with due regard for (C.13) and (C.28), we now get

$$|q'-q_1^o|' \le \frac{\mu}{2(1-\nu)^3}\left\{V_o\left(\frac{d}{h}\right)^2\left[3\sqrt{2}\cdot 14.851\,\frac{1}{1-\varepsilon} + (7.425 + (d/h)\cdot\frac{0.353}{1-\varepsilon}\,4.232)\right]\right\} \qquad (C.29)$$

According to (B.11) and (B.12)

$$\frac{1}{h^3}\,\frac{\mu}{2(1-n)}\,V_o \le \frac{1}{12}\left(\frac{d}{h}\right)^3$$

Therefore

$$|q' - q_1^o| \le \left(\frac{d}{h}\right)^5 \frac{85}{12} = m \qquad (C.30)$$

Hence, for $(d/h) < 0.7$ $(m < 1)$ we have

$$q' = q_1^o + (q' - q_1^o) \ge q_1^o - |q' - q_1^o| > 0 \qquad (C.31)$$

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