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FRACTURE

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INTRODUCTION

There are three important processes whereby a condensed phase can be separated into two parts. *Cracking*, in which rows of atoms or molecules are pulled apart normal to their centres of mass; *sliding off*, in which finite rows of them slide over one another until they ultimately part company; and *the removal of individual atoms*, as in vacancy migration or electrochemical attack. These processes, and, in crystals, those of deformation twinning and martensitic transformation also, interact on a microscale during the manufacture, assembly and use of materials to produce each other. Inhomogeneities of material and structure may lead to cracking and voids; cracking is relaxed or blunted by local sliding while sliding and twinning themselves can cause cracking. So our engineering structures generally contain many small cracks and voids, as well as inhomogeneities in material and structure which readily generate them under loads. In fracture we are mostly interested in the conditions under which these small discontinuities can grow and propagate as macroscopic cracks. For this propagation to proceed, two conditions must be satisfied. It is necessary that the decrease of total energy (the elastic energy of the body plus the potential energy of the loading system) be at least equal to the energy required to drive this separation process; and it is necessary that some physical mechanism can take place permitting this separation to occur. It may be convenient to consider the separation process on many different scales. We may look at a catastrophic failure occurring in a massive structure; at a specimen undergoing a fracture toughness test; at a small region near the tip of a larger crack where there is subcritical stable growth or slow extension during fatigue or creep; or at slip occurring on a microscale during the formation of craze nuclei. Whatever the scale however, these two principles govern the extension process.

The engineer must design and build structures and keep them safely in service. So there arises a continual need for practical tests to characterize the properties of materials. As understanding of these properties increases, these tests become more discerning and reliable, but their development must go hand in hand with more fundamental studies. The history of fracture and fracture mechanics is yet another example of how theory and practice interact to their mutual advantage. At the present time, when science is a little unfashionable, it is well to remember that we cannot go against Nature and that our progress will be faster if we learn a little to understand her ways.

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FRACTURE CRITERIA

In a linear elastic material the singular field p_{ij} near the tip of a sharp crack is characterized by the stress intensity factors K_1, K_2, K_3 and has the form

$$p_{ij} = (2\pi r)^{-1/2} K_s f_{sij} \qquad (1)$$

where the f_{sij} depend on θ. If the advance of the crack is governed by the stress field in this region, then we can determine by a test, for example in mode I, the critical value K_{1c} at which the crack will advance catastrophically. Then, if the service environment and other conditions are similar to those of the test, a cracked structure is secured against catastrophic failure until the K_1 for some crack in it reaches K_{1c}. This is the basis of linear elastic fracture mechanics. In the test, the departures from linear elasticity at the crack tip are carefully controlled. Corrections which make the slightly relaxed crack appear a little longer can be applied to extend the approach to small scale yielding. However, difficulties begin to multiply when we recognise that fractures in practice are generally accompanied by considerable departures from linearity, and when we try to make small scale tests on tough materials. The problems are compounded by the facts that fractures in structures occur under combined stresses and by the necessity of making proper allowance for chemical reactions, temperature, and varying stress.

For a sharp crack in an ideally brittle elastic material the critical K criterion embraces both of the fundamental conditions for fracture. The physical process condition is automatically satisfied, in a continuum model by the 'infinite' stress, or in one which is more realistic [1] by the existence of some bond at the crack tip which is always on the point of breaking. Although the situation at the tip of a macroscopic crack is much more complicated than this we have to remember that separation processes of this kind are occurring on a microscale in this region. It is thus important to model them, and particularly to study how the rate at which they occur is affected by temperature and the chemical environment. This is because whether such micro-cracking occurs or not may greatly influence the nature of the whole macroscopic fracture itself.

The energy condition we now formulate in terms of the *energy release rate* or *crack extension force* G, introduced in 1948 by Irwin [2] and shown by him in 1957 [3, 4] to be determined by K. Griffith's energy condition [5, 6] may then be written $G = 2\gamma$, where γ is an effective surface energy for fracture; alternatively we have (in mode I) $G = G_{1c} = K_{1c}^2/2M$, where G_{1c} is a critical value of G and M is an elastic modulus [7]. From this point of view K_{1c} is an indirect way of describing the effective surface energy for fracture. In 1960, G was expressed as a path-independent integral [8], and in 1968 a number of authors [1, 9 - 11] independently gave related expressions for G, one of which is now widely known as the J integral. The generalisation to the dynamic case was also considered at this time [12], and it was shown [1, 12] that the theory of the crack extension force followed naturally from the general theory of forces on elastic singularities developed in 1951 using the *energy-momentum tensor* [13].

Many computations and experiments have been devoted recently to the examination of J and quantities related to it as candidates for fracture criteria in post yield fracture mechanics. This work is not always easy

to follow because of variations in the terminology and interpretation of different authors. Moreover, the confusion is worse confounded by the use of similar symbols for the integrals themselves (which are mathematical entities in their own right, and which can be calculated as numbers without any interpretation if desired), and other quantities. These latter quantities are derived from the experimentally determined load-deflection curves of specimens containing cracks of various lengths, or sometimes, with the help of approximate theories, by other experimental methods. They, and the integrals themselves, are also calculated theoretically entirely from model experiments, by numerical methods using large computers. If the specimens were non-linear elastic these quantities (as well as the integrals themselves) would be crack extension forces, but in the usual practical and model situations they are not (and neither are the integrals). It might be helpful [14] to use some symbols other than J for these pseudo-crack-extension forces, retaining J for the integral defined by Rice [9]. We shall now try to discuss some of the problems arising in this work.

FUNDAMENTAL INTEGRALS

For the linear or non-linear elastic body the quantity

$$F_\ell = \int_S P_{\ell j}\, dS_j \qquad (2)$$

is such that $-F_\ell \delta\xi_\ell$ is the free energy change when all singularities inside a closed surface S drawn in the body are displaced by $\delta\xi_\ell$ [13] (we limit our discussion here to the static case; for the dynamic see [12, 15, 16]). Here

$$P_{\ell j} = W\,\delta_{\ell j} - p_{ij}\, u_{i,\ell} \qquad (3)$$

is the energy-momentum tensor of the elastic field for which the stresses p_{ij} are given by $\partial W/\partial u_{i,j}$ and $+W(u_i, u_{i,j}, X_i)$ is the strain energy density, assumed, for generality, to depend not only on the field quantities, but also explicitly on X_i, the initial coordinates. (We use a notation which makes (2) valid for the finite deformation of a non-linear elastic material; p_{ij} is the (unsymmetrical) nominal or Boussinesq or second Piola-Kirchhoff stress-tensor, the commas denote differentiation with respect to the X_i and S is a surface in the undeformed body. -W is, in the static case, the Lagrangian density function from which the field equations are derived from a variational principle [17 - 21]. The treatment can readily be extended if need be to a material of grade n [19, 20]. Using the field equations, we can show that

$$\frac{\partial P_{\ell j}}{\partial X_j} = \left(\frac{\partial W}{\partial X_\ell}\right)_{exp} \qquad (4)$$

where "exp" denotes the explicit derivative, with $u_i, u_{i,j}$ and $X\ ,j\neq\ell$, held constant. Putting $\ell = 1$, regarding the crack either as a distribution of dislocations [1] or as a singularity in its own right [15], and letting $dS_j = n_j ds$ for $j = 1,2$, we get for the crack extension force,

$$F_1 = \int_\Gamma (W\,\delta_{1j} - p_{ij}\, u_{i,1})\, n_j ds \qquad (5)$$

The divergence of the integrand vanishes if $(\partial W/\partial X_1)_{exp} = 0$; that is, if the material is homogeneous in the direction X_1 of the crack extension. However, it may be inhomogeneous in the X_2 direction; for example, the crack might lie between two different media. The integral J [9] is of the same form as (5), but W may be replaced by W', the density of stress working

$$W'(X_m,t) = \int_0^t p_{ij}(X_m,t')\left\{\partial u_{i,j}(X_m,t')/\partial t'\right\}dt' \qquad (6)$$

where t is a parameter denoting the progress of the deformation. The derivation of F_1 and the proof of its path-dependence involves the assumption of the existence of the function W. J and F_1 are thus identical and independent of Γ in linear and non-linear elasticity. Deformation plasticity, provided there is no unloading, can be regarded as a kind of non-linear elasticity. Thus, if the same strains and displacements are used in both J and F_1, they are again identical and path-independent. In a region of plasticity modelled by the incremental theory, J may be evaluated with du interpreted as the total shape displacement giving the shape change of the solid; that is, $du_i = du_i^E + du_i^P$, where E and P denote the elastic and plastic contributions. No general proof that it is then path-independent has been given, although, as discussed at ICF3 [22], it may be approximately so [23]. It is clear that the arguments leading to the path-independence of F_1 depend on the existence of the function W. Now, if in the actual loading the density of stress working is independent of the stress-strain path, W' is a function only of the current state and not of the strain history. Thus W' can be used for W in F_1; then if u is the total shape displacement, $J\delta$ and F_1 are the same and are independent of Γ, for we cannot tell that the field quantities were not derived from a density function [24]. A steadily moving plastic-elastic crack is an example of this kind [25]. The matter has been discussed recently in terms of the DBCS model [26, 27]. It is emphasized that, in general, J is path-independent in any situation where W' is independent of the stress-strain path by which the current state is reached [27]. It will be evident that by using various combinations of the elastic, plastic (or total) strains and displacements appearing in the two terms of the integrands a considerable number of integrals resembling J and F_1 can be obtained. It would be helpful, when these are evaluated numerically, if the quantities being evaluated were very clearly defined. Studies of the path-dependence of J in incremental plasticity are continuing [14, 28, 29].

If plastic flow has occurred at the crack tip, the integral F_ℓ gives the resultant force on the crack tip and on all the dislocations inside S [13, 26], but it is defined only for paths in the elastic region. However, an integral Q_ℓ may be derived [22, 30] which reduces to F_ℓ in the elastic region and which can be taken through a continuous distribution of dislocations representing the crack tip plasticity (and any micro-cracking there). This integral Q_ℓ was given at ICF3 and is [22]

$$Q_\ell = \int_S \left(W\delta_{\ell j} - p_{ij}\,\beta^E_{\ell i}\right) dS_j \qquad (7)$$

where W is the elastic energy density and $\beta^E_{\ell i}$ the elastic distortion tensor giving the spatial increments of elastic displacement $du_i^E = dx_\ell \beta^E_{\ell i}$ in a continuous distribution of dislocations (the elastic displacement u_i^E does not exist [31, 32, 33]). Q_ℓ reduces to zero when shrunk on to the crack tip [22] for the small scale yielding from an edge slit in anti-plane

strain [34]. As discussed at ICF3 [22], it would not be surprising if a realistic model of crack tip plasticity showed that the crack and its plastic zone were in neutral equilibrium, in the sense that any energy released by crack advance is absorbed by plastic work. This may be shown to be so for the quasi-static DBCS model [1, 35, 36], for the dynamic DBCS model [15, 37], and, more generally, [38], for elastic-plastic materials having a flow stress tending to a constant value at large strains; see also [39] for further discussion. These questions raise problems about the use of quantities like F_1 and J for the characterization of crack extension [22]. The interpretation of F_1 shows that $F_1\delta\xi$ is the energy released when the crack tip and all dislocations representing the plasticity are displaced in the X_1 direction for $\delta\xi$. This is not (necessarily) an equilibrium displacement of the crack and its plasticity, but the significance of this energy release, and how much of it is mopped up by plastic work in an actual movement of the crack, are not clear. Attention has again been focussed on the matter by recent numerical work [40, 41] confirming the result [25, 38] that there is no energy release rate for a growing crack in plastic-elastic material. If $-\Delta W$ is the work of unloading the initially stressed segments of crack face it is suggested that a crack tip energy release rate $G^\Delta = \Delta W/\Delta a$ calculated over a finite crack growth step Δa should be considered [42, 43] (the quantity $G^\Delta \to 0$ as $\Delta a \to 0$).

Disregarding heat fluxes, we can write for an imposed small extension Δa of the crack tip,

$$-\Delta E_{POT} = \Delta E_{EL} + \Delta w + G\Delta a + O(\Delta a^2)$$

Here $-\Delta E_{POT}$ is the work done by the loading system, ΔE_{EL} the increase in stored elastic energy, Δw the work dissipated in plastic flow and $G\Delta a$ the energy released at the crack tip. In linear and non-linear elasticity, when $\Delta w = 0$, it is the essential property of the integrals F_1 and J that they give G directly. For the fracture condition we then put $G\Delta a = 2\gamma'\Delta a$, where $2\gamma'$ is the effective surface energy for fracture. The result that $G = 0$ when plastic flow is allowed really shows that the plastic elastic continuum models considered are too simple. We have to make a more realistic representation of the fracture process allowing for rate effects, micro-cracking and mechanical instabilities in the fracture zone. At the simplest level, we simply lump some of the plastic work into the fracture energy, recognising it as part of the failure process; this is the extension of the Griffith theory originally proposed by Irwin [2] and Orowan [44]. In a semi quantitative way, as we discuss below, the DBCS model can be used to develop this idea. As has been noted [22], its developments to include rate effects [11, 45 - 47] show some of the qualitative features required to describe slow stable growth and the transition to fast fracture.

From any numerical solution for the plastic-elastic crack, we can calculate not only the integrals F_1 and J and the increment $\Delta E_{POT} + \Delta E_{EL} + \Delta w$ (tending to zero, perhaps, as $\Delta a \to 0$), but also by suitable integration over the developing field, the quantities $+E_{POT}$, E_{EL} and w as we load up a crack of fixed length a. We can then find the derivative $-\partial(E_{POT} + E_{EL} + w)/\partial a$ and compare it with F_1 and J. There is evidence [14] that these quantities are not the same. This we should indeed expect, for, as in the corresponding experimental procedure when we load up specimens with cracks of increasing length [48 - 50], we are not dealing with perfect differentials. The states obtained after loading a specimen of crack length a and allowing it to extend to a + Δa or alternatively loading a specimen of crack length a + Δa are different [21, 51]. It is not established even that ΔA, the area between the load extension curves for the cracks of lengths a and a + Δa, is -JΔa, nor is the connection with crack extension at all straightforward [24]. It is thus a matter for experimental study whether these or related methods [52 - 54] will yield a satisfactory characterisation of the onset of fracture.

THE CRITICAL DISPLACEMENT CRITERION

An interesting development using a critical displacement criterion for fracture began with the appearance of the BCS fracture theory [55, 56], which uses a highly simplified model of the crack tip plasticity consisting of a linear array of dislocations. A similar model to remove the elastic crack tip singularity and represent the plasticity was used by Dugdale [57]; a closely related (though not quite equivalent) procedure for eliminating the crack tip singularity is central to Barenblatt's work [58]. The procedure has also been used by Vitvitskii and Leonov (see [59]); a similar idea was employed by Prandtl [60]. The DBCS model has been elaborated in various ways and very widely applied to discuss many aspects of fracture [61 - 89]; recent reviews have discussed some of these developments [1, 22, 51]. An aspect which is currently receiving increasing attention is the BS (Bilby-Swinden [70]) model in which two (or more) dislocation arrays inclined to the crack are used in an attempt to make a slightly more realistic representation of the plasticity [42, 88 - 91].

The DBCS model has also been used in discussions of the COD concept in post-yield fracture mechanics [92 - 95]. There has been a considerable development of this criterion on the engineering side, but, like its rivals, its status as a single characterizing parameter is still a matter for further elucidation and debate.

Nevertheless the use of the criterion in the BCS theory has been very useful in providing a two-parameter model fo the energy expended in the fracture process, and an interpolation between the Griffith theory (or linear elastic fracture mechanics) and failure after considerable yielding or plastic collapse. If ϕ_c is the critical displacement at the crack tip and σ_1 the stress in the relaxed zone, the fracture stress σ_f is related to the crack length c by the equation [65, 66].

$$\sigma_f/\sigma_1 = (2/\pi)\cos^{-1}\{\exp(-c^*/\pi c)\} \qquad (8)$$

where $c^* = M\,\phi_c/4\sigma_1$, M being an elastic modulus. The condition $c = c^*$ defines the crack length at which the material becomes *notch-sensitive* [65]. The equation (8) reduces to the Griffith condition when $c \gg c^*$; we then have a "low-stress" failure with $\sigma_f \ll \sigma_1$. When $c < c^*$, the fracture stress approaches σ_1, the strength of the layer ahead of the crack. This equation is successful in describing the stresses at which failures occur below general yield in large structures and may be used to estimate dangerous notch sizes in them [64 - 66, 70, 83 - 85]. Also, with σ_1 identified with the ultimate tensile strength or the collapse stress, and with an appropriate stress intensity factor for the geometry considered it is remarkably effective in correlating post yield fractures with defect size in a wide class of materials [68, 69, 96] and can be used to estimate K_{1c} values from "invalid" ASTM tests. It has also been suggested as an interpolation between failure by plastic collapse and

linear elastic fracture mechanics, of potential use in assessing critical defect sizes in large structures and in design [97]. The engineer cannot afford to make mistakes and, if he must, he will test his actual structures to destruction. His inclination is frequently for the simplest approach, based on large-scale tests [98]. Although the great simplifications in the BCS theory are obvious, it is not wholly empirical, and so may be of some value in the correlations which he has nevertheless to make.

The theory gives for the fracture energy $2\gamma'$ the expression $\sigma_1\phi_c$. We distinguish two *modes* of fracture [22, 61]; a stable, *non-cumulative* or *non-localised* mode which occurs when $\sigma_1 \sim \sigma_f$; the material is not notch-sensitive. The non-linearity represented by the dislocations spreads through the specimen much faster than the crack. The second mode is *cumulative* or *localised* and is unstable; in this type of failure the material is notch-sensitive for $c \gg c^*$. A similar localised set of dislocations representing the non-linearity moves with the crack as it grows, so that it can advance without the non-linearity spreading through the whole net section ahead of it; the fracture is a low-stress one with $\sigma_f \ll \sigma_1$ and $c \gg r$, the extent of the non-linearity ($r \sim \pi c^* \sim E\phi_c/\sigma_1$). We can see with this classification the mechanical similarity of fractures with very different values of $2\gamma' = \sigma_1\phi_c$. Thus ideal brittle fracture, discontinuous ductile-cleavage, mode I plane stress necking and mode III ductile tearing, and the 45° shear mode in steel plates are all cumulative. Except for the first all involve a mechanical instability because the capacity to harden has been exhausted, non-linear flow has concentrated, and large strains have occurred. These large strains are possible whenever free surfaces allow large geometry changes, on a microscale at blunting crack tips or in the internal necks between cracks and voids, and on a macroscale when the specimen is (relatively) small in one dimension.

OTHER CRITERIA

A number of other proposals for the characterisation of post-yield fractures have been made, some of which are reviewed at this meeting [52 - 54, 99 - 103]. Their use and applicability are still a matter of active current research. However, the concept of the R curve [100, 104 - 106] perhaps deserves special mention. It touches upon the fundamental conditions for fracture referred to at the beginning of this paper, and also discussed in one of the plenary sessions here [107]. The crack will not run until the total free energy of the whole system begins to decrease as it advances. The R-curve gives explicit recognition to the idea that the physical processes for the advance of the crack (cracking and sliding off in combination on a microscale) can occur, but recognises that these processes are, temporarily, self equilibriating. Just as a material work hardens, so the resistance R to crack propagation rises. Many workers have considered this phenomenon [10]. Here we wish only to draw attention to the fact that it again forces us to think in detail about the processes of sliding, blunting and microcracking which are going on at the crack tip in all the materials we consider [107 - 115]. It may well be that we shall not achieve a complete understanding of these processes without considering their sensitivity to the strain-rate and the environment.

OTHER PATH-INDEPENDENT INTEGRALS

There are other path-independent integrals of use in the theory of fracture besides the J and the F_1 we have already discussed. Before doing so we make a few comments on the rather profligate introduction of "new" integrals currently in fashion. This is not the place for the detailed critique of individual proposals, but we believe that those planning to launch a vessel of this kind should first study carefully the background theory and bear the following points in mind.

Firstly, a complicated integral expression may be path-independent because, in any example of interest, it is identically zero. Secondly, we have to distinguish two types of path-independence. If the two-dimensional divergence of the integrand vanishes, the integral will have the same value for two paths each beginning at a point A on the lower crack surface and ending at a point B on the upper crack surface. However the value may be different for a path beginning at another point A_1 on the lower surface and ending at another point B_1 on the upper. Indeed it will be unless the sum of the contributions from the paths A_1A and BB_1 is zero. If this sum is zero for all A_1 and B_1, then the integral has the same value for *all paths beginning at any point on the lower surface and ending at any point on the upper*; we can slide the points A and B along the crack faces in any manner without changing its value. It is this kind of path-independence which is of real value, since we can deduce values for paths close to the tip from those placed far away at our convenience, where the field quantities are easier to find. Of course, we can always make an expression "path-independent" by subtracting from it the contributions from the paths BB_1 and A_1A. But then, if we wish to use the expression, we still have to evaluate these contributions, and this requires a knowledge of the field close to the crack tip; we have made no real progress.

The general theory of path-independent integrals stems from the work of Noether [116]. They arise for any field when the Lagrangian density function from which the field equations are derived is invariant under the operations of a continuous group. The general consequences for elastic singularities and cracks have been discussed in a number of papers by Eshelby [13, 15, 19, 20]. Gunther [117] was the first to apply Noether's theorem systematically to elastostatics. In addition to F_ℓ he found the integrals

$$L_{k\ell} = \int_S (X_k P_{\ell j} - X_\ell P_{kj} + u_k P_{\ell j} - u_\ell P_{kj})\, dS_j \qquad (9)$$

and

$$M = \int_S \left(X_\ell P_{\ell j} - \frac{1}{2} u_\ell P_{\ell j}\right) dS_j \qquad (10)$$

also given by Budiansky and Rice [118]. F_ℓ, $L_{k\ell}$ and M are path independent because a picture of a general elastic field remains one after it has been respectively translated, rotated and enlarged. Consequently [19, 20] F_ℓ is valid for finite deformation and a non-linear material, provided only that it is homogeneous, while for $L_{k\ell}$ the material must in addition be isotropic. For M we must have linearity in the displacement gradients, but we may have anisotropy. There are some special cases in which these requirements may be relaxed [9]. Arguments have been given [119] that F_ℓ, $L_{k\ell}$ and M are the only path-independent integrals of Noether's type and that in plane situations the only new feature is that (10) reduces to

$$M = \int_S X_\ell P_{\ell j}\, dS_j \qquad (11)$$

a transformation which results from Gauss's theorem [19]. However, in two dimensions, several infinite classes of path-independent integrals have been found [19].

It is interesting to calculate the force F_2 given by (2). We may think loosely of F_2 as the force normal to the crack tip, but its interpretation requires some care. If we evaluate F_2 using the singular stresses (1), that is, for a small circuit about the crack tip, we find that $F_2 = -2K_1K_2$ [120, 121]. This is an example where the integral (2) has a different value for a large circuit round the crack tip; that is, the integral is not path-independent in the really useful sense because there are, outside the singular field, non-vanishing contributions along the crack faces. We cannot make a useful path-independent integral simply by subtracting the crack face terms [121], because we still have to know the field along the crack if we wish to use such an integral. It is indeed [24] easy to show that F_2 is the limit of $(\pi/2)p_{11}u_2$ as the tip is approached along either the top or bottom surface of the crack.

Loosely, we expect F_2 to push the crack sideways, and this raises the interesting question of what determines the path of a crack. This problem also arises in considering fracture under combined stresses and in crack forking. It is a subtle one because the crack constantly alters the field as it proceeds. A possible criterion is that the crack moves so as to keep $F_2 = 0$ [24]. This has been used by Kalthoff [122] in the form $K_2 = 0$ to discuss the angle at which a crack forks. See also [123] for an equivalent proposal. There has been considerable interest for some time both in the "angled-crack" problem and in the more general problem of crack initiation under combined stress and a number of theories have been proposed [124 - 129]; for a selection of earlier references, see [51]. A discussion of these problems based on an analysis [130] of a crack under general loading with a small kink at its tip making an angle α with the main crack was given at a recent meeting [51]; several authors have published analyses of this kind [130 - 135]. However, to discuss the *onset* of deviation, or for *initiation* of the kink under combined stress, the most suitable results are those for the limit when the kink is vanishingly small compared with the main crack. Several criteria for the path and the initiation have been examined using results of this kind [51, 130, 135].

The integral $L_{k\ell}$ enables an alternative interpretation of the force F_2 to be given when the crack with a kinked tip is considered. If f_1 and f_2 are the crack extension forces determined by evaluating (2) round the tip of the kink, then it may be shown [20] that

$$f_2 = \left[\frac{df_1}{d\alpha}\right]_{\alpha=0} \qquad (12)$$

That is, if the tip of the main crack deviates through a small angle $d\alpha$, the change in the crack extension force f_1 is $f_2 d\alpha$. Several authors have also recently examined the problem of the forked crack [134 - 135]; the results show some discrepancies [135]. Again, for the initiation of forking, the case when the forks are vanishingly small is of most interest [135]. Using a $k_2 = 0$ criterion, the predicted branching angle does not differ very much from that observed and calculated by Kalthoff [122].

Reference [19] contains the expression for $P_{\ell j}$ for a material of grade 2; there seems to be some uncertainty in its application to crack problems

[20, 136]. Other examples of the application of path-independent integrals to problems of fracture are also given. They include a discussion of Obreimoff's experiments on mica; of the "trouser test" for rubber using (2) with finite deformation; and of the two-dimensional analogue of the "conical crack", and of the edge crack wedged open by concentrated forces, using (11). The static version of equation (59) of [15] furnishes an integral which is path-independent in the presence of certain types of body forces.

Before leaving the topic of path independent integrals we wish to repeat the brief comment we have made [137] about the use of the quantities $\dot{J}$ or C* in creep crack growth; for a selection of references see [138]. If the material is linear, viscous and incompressible and the flow is slow, then we can make the usual analogy with linear elasticity by replacing the displacement by the velocity, the shear modulus by the viscosity and by setting Poisson's ratio equal to one half. Then an integral of the form of F_1 is path-independent. However, what it represents is the following [20]. A body instantaneously contains a crack of length a under some load and a state of viscous flow is established. All the work done by the external forces is being dissipated by the viscosity and there is a certain dissipation rate. Now the crack is lengthened by Δa; then the elastic-viscous analogy shows that $2F_1\Delta a$ is the *increase* in the rate of dissipation when the boundary loading is held fixed, but the decrease in it if the boundary velocities are kept constant [24]. In other contexts, this kind of integral can perhaps be used [20] to select from a class of slow viscous flows depending on parameters a flow which is actually observed, by requiring that the dissipation be stationary (although the principle involved is not easy to justify). It is not, however, clear how relevant the integral is to creep crack growth. Of course, as is often the case, it is not the value of an integral which is usually compared with experiments, but some quantity which would be the G derived from the compliance of a specimen if we were dealing with elasticity. Moreover, the viscosity is non-linear. Nevertheless, if creep crack growth can be satisfactorily characterised in this way, there will clearly be a need for some re-interpretations.

In fact, one must expect a crack in a linear viscous material to elongate in the direction of the stress [139]. A hole is a special case of an inhomogeneity, and there has been some recent progress in the theory of the deformation of ellipsoidal viscous inhomogeneities [140, 141], a process of interest in glass manufacture, geology and in the interpretation of phenomena in inhomogeneous fluids. The growth of voids at crack tips is, of course, one of the phenomena we must understand if we are to improve our model of the processes going on there [142 - 146].

MOVING CRACKS

We refer only briefly to moving cracks; for more detailed accounts see [11, 12, 15, 16, 51, 147 - 151]. In a general dynamic elastic field there is no path-independent integral for the force on a moving crack. The best we can do [12] is to write the elastic field in the form

$$u_i = u_i^0(X_1 - vt, X_2) + u_i'(X_1, X_2, t) \qquad (13)$$

and try to arrange that near the tip $u_i' \ll u_i^0$. Here the crack tip is moving with instantaneous velocity v in the X_1 direction. Then we can write

$$G = \lim_{S\to 0} \int_S H_{\ell j}\, dS_j \qquad (14)$$

where S is a surface moving with the crack tip and

$$H_{\ell j} = (W + T)\,\delta_{ij} - p_{ij}u_{i,\ell} \qquad (15)$$

Here T is the kinetic energy density. It should be noted that $H_{\ell j}$ is not the dynamic 4 x 4 energy-momentum tensor $P_{\ell j}$. The integral of the dynamic $P_{\ell j}$ gives the force on the crack tip, plus the rate of change of "quasi-momentum" inside S [15].

The integral (14) is, in general, path-independent only when $S \to 0$. If the dynamic elastic field is a special one which moves rigidly with the crack tip, then G is independent of S, but special simple fields can be used to show that this independence cannot be true for arbitrary finite S in a general dynamic field [12]. It may be shown [12] that G vanishes at the Rayleigh velocity for the uniformly expanding crack in plane strain [152, 153], and at the shear velocity for a similar crack in anti-plane strain [154].

The equation of motion may be found by allowing the crack tip to move arbitrarily so that at time t its tip is at $x = \xi(t)$, say, and then calculating the field, as was first done by Kostrov [155] and Eshelby [156, 157]; Freund [158] has extended the work to plane strain. We can then calculate G, which turns out to be a function of ξ and $\dot{\xi}$, but not of $\ddot{\xi}$; the crack tip behaves as if it had no inertia [156]. If $2\gamma(\xi,\dot{\xi})$ is the fracture energy as a function of ξ and $\dot{\xi}$, the equation of motion is

$$G(\xi,\dot{\xi}) = 2\gamma(\xi,\dot{\xi}) \qquad (16)$$

We find that the velocity dependences of G and K are different, and that G contains a factor which increases as the velocity falls. We can thus understand how the conservation of energy can be maintained during crack branching. For instance, a lower limit to the velocity of crack branching might be set by requiring that the crack momentarily stops [15, 158]. With the continuing interest in the solution of dynamic crack problems [159 - 165], we can look forward to further progress in our understanding of discontinuous crack propagation, crack paths and crack branching.

DISCUSSION

We have been able to refer to a few only of the many interesting papers at this meeting. About specific materials and their behaviour in composites [107] we have said very little. We have not mentioned the effect of the environment [166 - 167], the phenomenon of fatigue [168, 169], or the growth of cracks in creep [170], all topics in the front line of practical interest. Nor have we touched on the efforts now being made to take a more three-dimensional look at the fracture problem, which stretches our analytical and computing powers and also raises some interesting topological questions [114]. Among the papers also, there are several references to probability and statistics, applied both to flaws and to microstructure [108, 171] and to failure probabilities [172, 173]. These methods will be with us increasingly, and the analysis of reliability will help to identify more quantitatively some of the critical factors to which we should devote our current attention.

In cavitation during high temperature creep [174] we have to consider thermally assisted motion of individual atoms. This provides us with a gentle introduction to the phenomena where mechanics alone will not do and we have to consider the combined effect of stress, strain, temperature and electro-chemical processes. Cracking, sliding and individual atom movements all play their part in contributing to creep damage. Johnson's work [170] reminds us that the damage may be general or local and that we may relieve stresses both by sliding, and by void formation and microcracking, a process occurring also in brittle materials [115]. Let us remember too that although the macroscopic creep rates of interest to the engineer are very slow, these rates may be much faster in local regions where stress is concentrated. Thus we must consider a range of mechanisms, and the deformation and failure map [175] helps us to view things as a whole, and warns of the pitfalls associated with the long extrapolations that have sometimes to be made. These maps remind us too that as the strain rate rises, the problem of creep fracture passes into that of workability [176, 177]. The emphasis on mechanism also recalls that although the macroscopic behaviour may be viscous, in small regions of crystalline materials at least, we have blocks of *elastic* material, in which atoms and dislocations are moving and material is separating.

Particularly challenging are those fractures where we have to think of the transport of impurities across internal and external surfaces, as these are exposed and films on them reform [166]; of the migration of these impurities and the effects produced when they are segregated, adsorbed on surfaces, or associated with defects; and of their trapping and precipitation as condensed phases or bubbles of gas within the solid [174, 178, 179]. In thinking of these phenomena, as of fatigue, which is also influenced by them [168, 169], we are led yet again to focus on the detail of the microstructure and the processes occurring at the crack tip [108, 114, 115, 180 - 183]. In trying to characterize the intrinsic ductility of a material, the delicate balance between microcrack propagation and slip must be studied [184, 185]. In processes on this scale, the true surface energy (or a modified surface energy which is still quite small) is important, and it has been widely argued (for example, [38, 39, 186]) that when this is changed, the whole macroscopic toughness may be affected. The large observed fracture energy thus depends critically on a much smaller surface energy, important for the microprocesses. Adsorption can make a radical change in the true surface energy (see [187, 188], for example), so that in this way we can explain the effect of trace impurities. Our model will not be complete however unless it takes account of the rates at which the various competing processes occur. Despite the preoccupation with macroscopic toughness, the critical experiments and the interpretation necessary to formulate such a rate-sensitive model of these crack tip processes must be continued.

The subject of fracture embraces the full range of the study of condensed matter; it is necessary both to test large engineering structures, and to use the most refined techniques for detecting the presence of individual atoms. We have to consider the deformation and flow of many types of microstructure, and the effect of the environment upon these processes.

The work touches on earthquakes and the failure of rocks and masses of ice [189]; on the integrity of pressure vessels, pipelines, aircraft and electrical generators; and on the structure of our very selves [190]. There are many interesting phenomena to be investigated and some formidable problems to be solved. We look forward to learning more about them at an interesting meeting.

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REFERENCES

1. BILBY, B. A. and ESHELBY, J. D., In: "Fracture", (ed. H. Liebowitz), Academic Press, New York, 1, 1968, 99.
2. IRWIN, G. R., In: "Fracturing of Metals", ASM, Cleveland, 1948, 147.
3. IRWIN, G. R., J. Appl. Mech., 24, 1957, 361.
4. IRWIN, G. R., Handbuch der Physik, VI, 1958, 551.
5. GRIFFITH, A. A., Phil. Trans. Roy. Soc., A221, 1920, 163.
6. GRIFFITH, A. A., Proc. First International Congress on Applied Mechanics, (ed. C. B. Biezeno and J. M. Bergers), Delft, Waltman, 1924, 55.
7. ESHELBY, J. D., In: "Fracture Toughness", ISI Publication 121, 1968, 30.
8. SANDERS, J. L., J. Appl. Mech., 27, 1960, 352.
9. RICE, J. R., J. Appl. Mech., 35, 1968, 379.
10. RICE, J. R., In: "Fracture", (ed. H. Liebowitz), Academic Press, New York, II, 1968, 191.
11. CHEREPANOV, G. P., Int. J. Solids Structures, 4, 1968, 811.
12. ATKINSON, C. and ESHELBY, J. D., Int. J. Fract. Mech., 4, 1968, 3.
13. ESHELBY, J. D., Phil. Trans., A244, 1951, 87.
14. CARLSSON, A. J. and MARKSTROM, K. M., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, 1, 1977.
15. ESHELBY, J. D., In: "Inelastic Behaviour of Solids", (ed. M. F. Kanninen et al.), McGraw Hill, New York, 1970, 77.
16. BILBY, B. A., In: "Amorphous Materials", Proceedings of the Third International Conference on the Physics of Non-Crystalline Solids, Sheffield (ed. R. W. Douglas and B. Ellis), Wiley, New York, 1972, 489.
17. ESHELBY, J. D., In: "Internal Stresses and Fatigue in Metals", (ed. R. M. Rassweiler and W. L. Grube), Elsevier, Amsterdam, 1959, 41.
18. ESHELBY, J. D., Solid State Physics, 3, 1956, 79.
19. ESHELBY. J. D., In: "Prospects of Fracture Mechanics", (ed. G. C. Sih et al.), Noordhoff International, Leyden, 1975, 69.
20. ESHELBY, J. D., Journal of Elasticity, 5, 1975, 321.
21. BILBY, B. A., Advanced Seminar on Fracture Mechanics, Commission of the European Communities, Ispra 1975, Paper ASFM/75, No. 6.
22. BILBY, B. A., Papers Presented to the Third International Congress on Fracture, Munich, Part XI, 1973, 1.
23. HOWARD, I. C., Private communication, 1973: see [22].
24. ESHELBY, J. D., Private communication, 1974.
25. HUTCHINSON, J. W., Report DEAP S-8, Division of Engineering and Applied Physics, Harvard University, 1974.
26. CHELL, G. C. and HEALD, P. T., Int. Journal Fract., 11, 1975, 349.
27. RICE, J. R., Int. Journal Fract., 11, 1975, 352.

28. NEALE, B. K., CEGB Report No. RD/B3253, 1975.
29. ROCHE, R. L., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
30. BILBY, B. A. and ESHELBY, J. D., Unpublished work.
31. BILBY, B. A., Progr. Solid Mech., 1, 1960, 331.
32. BILBY, B. A., IUTAM Symposium Freudenstadt-Stuttgart, Mechanics of Generalized Continua, (ed. E. Kröner), Springer, 1968, 180.
33. KRÖNER, E., "Kontinuumstheorie der Versetzungen und Eigenspannungen", Springer, 1958.
34. HULT, J. A. H. and McCLINTOCK, F. A., Proceedings of the Ninth International Congress on Applied Mechanics, 8, 1957, 51.
35. SWINDEN, K. H., Ph.D. Thesis, University of Sheffield, 1964.
36. YOKOBORI, T. and ICHIKAWA, M., Reports of the Research Institute for Strength and Fracture of Materials, Tohoku University, Sendai, 2, 1966, 21.
37. ATKINSON, C., Ark. Fys., 35, 1967, 469.
38. RICE, J. R., Proceedings First International Congress on Fracture, Sendai, (ed. T. Yokobori et al.), Jap. Soc. Strength and Fracture of Materials, Tokyo, 1, 1966, Paper A-18, 309.
39. RICE, J. R. and DRUCKER, D. C., Int. Journal Fract., 3, 1967, 19.
40. KFOURI, A. P. and MILLER, K. J., Int. J. Pres. Ves. and Piping, 2, 1974, 179.
41. KFOURI, A. P. and MILLER, K. J., "Crack Separation Energy Rates in Elastic-Plastic Fracture Mechanics", to be published in Proc. Inst. Mech. Eng. (London).
42. KFOURI, A. P. and RICE, J. R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
43. KFOURI, A. P. and MILLER, K. J., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
44. OROWAN, E., Reports on Progr. in Physics, 12, 1949, 214.
45. CHEREPANOV, G. P., Prikl. Mat. Mech., 33, 1968, No. 3.
46. WNUK, M. P., Int. Journal Fract. Mech., 7, 1971, 217.
47. KNAUSS, W. G., Appl. Mech. Rev., 26, 1973, 1.
48. BEGLEY, J. A. and LANDES, J. D., ASTM STP 514, 1971, 1.
49. LANDES, J. D. and BEGLEY, J. A., ASTM STP 514, 1971, 24.
50. BUCCI, R. J., PARIS, P. C., LANDES, J. D. and RICE, J. R., ASTM STP 514, 1971, 40.
51. BILBY, B. A., Conference on Mechanics and Physics of Fracture, Cambridge, January 1975, 1/1-1/11.
52. RICE, J. R., PARIS, P. C. and MERKLE, J. G., ASTM STP 536, 1973, 231.
53. ADAMS, N. J. I. and MUNRO, H. G., Engng Fract. Mech., 6, 1974, 119.
54. BEGLEY, J. A. and LANDES, J. D., ASTM STP 536, 1973, 246.
55. COTTRELL, A. H., Symposium on Steels for Reactor Pressure Circuits, 1960, Special Report No. 69, I.S.I., London, 1961, 281.
56. BILBY, B. A., COTTRELL, A. H. and SWINDEN, K. H., Proc. Roy. Soc., A272, 1963, 304.
57. DUGDALE, D. S., Journal Mech. Phys. Solids, 8, 1960, 100.
58. BARENBLATT, G. I., Prikl. Mat. Mech., 23, 1959, 434,706,893; Advan. Appl. Mech., 7, 1962, 55.
59. VITVITSKII, P. M., PANASYUK, V. V. and YAREMA, S. Ya., Engng. Fract. Mech., 7, 1975, 305.
60. PRANDTL, L., Z. f. angew, Math. und Mech., 13, 1933, 129.
61. COTTRELL, A. H., In: "Properties of Reactor Materials and the Effects of Radiation Damage", (ed. D. J. Littler), Butterworths, London, 1962, 5.
62. COTTRELL, A. H., Proc. Roy. Soc., A276, 1963, 1.
63. COTTRELL, A. H., Proc. Roy. Soc., A282, 1964, 2.
64. COTTRELL, A. H., Proc. Roy. Soc., A285, 1965, 10.

65. COTTRELL, A. H., In: "Fracture", Proceedings of the First Tewkesbury Symposium, 1963 (ed. C. J. Osborn), Butterworths, London, 1965, 1.
66. BILBY, B. A., COTTRELL, A. H., SMITH, E. and SWINDEN, K. H., Proc. Roy. Soc., A272, 1963, 304.
67. SMITH, E., In: Proceedings of the First International Congress on Fracture, Sendai (ed. T. Yokobori et al.), Japanese Society for Strength and Fracture of Materials, Tokyo, I, 1965, 133.
68. HEALD, P. T., SPINK, G. M. and WORTHINGTON, P. J., Mat. Sci. Engng., 10, 1972, 129.
69. WORTHINGTON, P. J., SPINK, G. M. and HEALD, P. T., In: Proceedings of the Third International Congress on Fracture, Munich, IX, 1973, Paper 515.
70. BILBY, B. A. and SWINDEN, K. H., Proc. Roy. Soc., A285, 1965, 22.
71. ATKINSON, C. and KAY, T. R., Acta Met., 19, 1971, 679.
72. ATKINSON, C. and CLEMENTS, D. L., Acta Met., 21, 1973, 55.
73. SMITH, E., Proc. Roy. Soc., A299, 1967, 455.
74. SMITH, E., Int. J. Engng Sci., 5, 1967, 791.
75. BILBY, B. A. and HEALD, P. T., Proc. Roy. Soc., A305, 1968, 429.
76. HEALD, P. T. and BILBY, B. A., In: "Fracture Toughness of High Strength Materials", ISI Publication No. 120, 1968, 63.
77. WEERTMAN, J., Int. Journal Fract. Mech., 2, 1966, 460.
78. WEERTMAN, J., Int. Journal Fract. Mech., 5, 1969, 13.
79. FLEWITT, P. E. J. and HEALD, P. T., In: "Fracture Toughness of High Strength Materials", ISI Publication No. 120, 1968, 66.
80. KOSTROV, B. V. and NIKITIN, L .V., Prikl. Mat. Mekh., 31, 1967, 334.
81. ARTHUR, P. F. and BLACKBURN, W. S., In: "Fracture Toughness of High Strength Materials", ISI Publication No. 120.
82. HOWARD, I. C. and OTTER, N. R., J. Mech. Phys. Solids, 23, 1975, 139.
83. SMITH, E., Proc. Roy. Soc., A285, 1965, 46.
84. HAYES, D. J. and WILLIAMS, J. G., Int. Journal Fract. Mech., 8, 1972, 239.
85. FENNER, D. N., Int. Journal Fract., 10, 1974, 71.
86. CHELL, G. C., Int. Journal Fract., 10, 1974, 128.
87. CHELL, G. C., Int. Journal Fract., 12, 1976, 135.
88. RICE, J. R., J. Mech. Phys. Solids, 22, 1974, 17.
89. KOCHENDORFER, A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
90. RIEDEL, H., Journal Mech. Phys. Solids, 24, 1976, 277.
91. VITEK, V., Journal Mech. Phys. Solids, 24, 1976, 263.
92. WELLS, A. A., Proceedings of the Crack Propagation Symposium, Cranfield, 1, 1961, 201.
93. WELLS, A. A., Brit. Welding Journal, 12, 1965, 1.
94. BURDEKIN, F. M., STONE, D. E. W. and WELLS, A. A., In: "Fracture Toughness Testing, ASTM STP 381, 1965, 400.
95. EGAN, G. R., Eng. Fract. Mech., 5, 1973, 167.
96. HEALD, P. T. and EDMONSON, B., In: "Periodic Inspection of Pressurized Components", Institution of Mechanical Engin-ers, London, 1974, 95.
97. DOWLING, A. R. and TOWNLEY, C. H. A., Int. J. Pres. Ves. and Piping, 3, 1975, 77.
98. SOETE, W., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
99. LIEBOWITZ, H., EFTIS, J. and JONES, D. L., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
100. IRWIN, G. R. and PARIS, P. C., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
101. WITT, F. J. and MAGER, J. R., Nucl. Eng. Design, 17, 1971, 91.
102. ÖSTENSSON, B., Eng. Fract. Mech., 6, 1974, 473.

103. MERKLE, J. G., ASTM STP 536, 1973, 264.
104. IRWIN, G. R., Report 5486, U.S. Naval Research Lab., 1960.
105. KRAFFT, J. M., SULLIVAN, A. M. and BOYLE, R. W., Proceedings of the Crack Propagation Symposium, Cranfield, I, 1961, 8.
106. SRAWLEY, J. E. and BROWN, W. F., ASTM STP 381, 1965, 133.
107. COOPER, G. A. and PIGGOTT, M. R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
108. KNOTT, J. F., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
109. RICE, J. R. and JOHNSON, M. A., In: "Inelastic Behaviour of Solids", (ed. M. F. Kanninen et al.), McGraw Hill, New York, 1970, 511.
110. SMITH, E., Proc. Conf. on Physical Basis of Yield and Fracture, Inst. of Phys. and Phys. Soc., 1966, 36.
111. RITCHIE, R. O., KNOTT, J. F. and RICE, J. R., J. Mech. Phys. Solids, 21, 1973, 395.
112. COWLING, M. J. and HANCOCK, J. W., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
113. HANCOCK, J. W. and COWLING, M. J., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
114. ARGON, A. S., HANNOOSH, J. G. and SALAMA, M. M., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
115. EVANS, A. G., HEUER, A. H. and PORTER, D. L., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
116. NOETHER, E., Göttinger Nachrichten (Math.-Phys. Klasse), 1918, 235, (English translation by M. A. Tavel), Transport Theory and Statistical Physics, 1, 1971, 183.
117. GÜNTHER, W., Abh. braunsch. wisch. Ges., 14, 1962, 54.
118. BUDIANSKY, B. and RICE, J. R., J. Appl. Mech., 40, 1973, 201.
119. KNOWLES, J. K. and STERNBERG, E., Arch. rat. Mech. Anal., 44, 1972, 187.
120. CARLSSON, J., In: "Prospects of Fracture Mechanics", (ed. G. C. Sih et al.), Noordhoff International, Leyden.
121. BERGEZ, D., Revue de Phys. Appliquée, 276, 1973, 1425.
122. KALTHOF, J., Proc. Third International Congress on Fracture, Munich, X, 1973, Paper 325.
123. GOL'DSTEIN, R. V. and SALGANIK, R. L., Int. Journal Fract. 10, 1974, 507.
124. SHAW, M. C. and KOMANDURI, R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
125. PISARENKO, G. S. and LEBEDYEV, A. A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
126. MAREK, P., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
127. JAYATILAKA, A de S., JENKINS, I. J. and PRASAD, S. V., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
128. HAHN, G. T., HOAGLAND, R. G. and ROSENFIELD, A. R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
129. KIRCHNER, H. P. and GRUVER, R. M., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada II, 1977.
130. BILBY, B. A., and CARDEW, G. E., Int. Journal Fract., 11, 1975, 708.
131. PALANISWAMY, K. and KNAUSS, W. G., Int. Journal Fract., 8, 1972, 114.
132. COUGHLAN, J. and BARR, B. I. G., Int. Journal Fract., 10, 1974, 590.
133. CHATTERJEE, S. N., Int. Journal Solids Struct., 11, 1975, 521.
134. KITAGAWA, H., YUUKI, R. and OHIRA, T., Engng Fract. Mech., 7, 1975, 521.
134a. KITAGAWA, H. and YUUKI, R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.

134b. CHEREPANOV, G. P. and KULIEV, V. D., Int. Journal Fract., 11, 1975, 29.
134c. THEOCARIS, P. S. and IOAKIMIDIS, N., J. Appl. Math. and Phys., 27, 1976, 801.
134d. MONTULLI, L. T., Ph.D. Thesis, University of California, 1975.
134e. HUSSAIN, M. A., PU, S. L. and UNDERWOOD, J., ASTM STP 560, 1973, 2; also, Graduate Aeronautical Laboratories, California Institute of Technology, Report No. 74-8, to appear in Mechanics Today (ed. S. Nemat-Nasser), Pergamon.
134f. VITEK, V., "Plane Strain Stress Intensity Factors for Branched Cracks", CEGB Report No. RD/L/N210/76.
135. BILBY, B. A., CARDEW, G. E. and HOWARD, I. C., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
136. ATKINSON, C. and LEPPINGTON, F. G., Int. Journal Fract., 10, 1974, 599.
137. BILBY, B. A., Conference on Creep Crack Growth, Sheffield, The Metals Society, January 1976, unpublished.
138. NIKBIN, K. M., WEBSTER, G. A. and TURNER, C. E., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
139. BERG, C. A., Proc. Fourth U.S. National Congress of Appl. Mechanics, Berkeley, (ed. R. M. Rosenberg), ASME, 2, 1962, 885.
140. BILBY, B. A., ESHELBY, J. D. and KUNDU, A. K., Tectonophysics, 28, 1975, 265.
141. BILBY, B. A. and KOLBUSZEWSKI, M. L., Proc. Roy. Soc., 1977, to be published.
142. McCLINTOCK, F. A., J. Appl. Mech., 35, 1968, 363.
143. HANCOCK, J. W. and MACKENZIE, A. C., J. Mech. Phys. Solids, 24, 1976, 147.
144. RICE, J. R. and TRACEY, D. M., J. Mech. Phys. Solids, 17, 1969, 201.
145. NAGPAL. V., McCLINTOCK, F. A., BERG, C. A. and SUBDHI, M., "Foundations of Plasticity", International Symposium, Warsaw, 1972, (ed. A. Sawczuk), Noordhoff, Leyden, 365.
146. PERRA, M. and FINNIE, I., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
147. Advanced Seminar on Fracture Mechanics, Commission of the European Communities, Ispra, 1975. Papers by IRWIN, G. R. (ASFM/10 and 13), GROSS, D. (ASFM/11), TURNER, C. E. (ASFM/12) and KERKHOF, F. (ASFM/18).
148. BARENBLATT, G. I., ENTOV, V. M. and SALGANIK, R. L., In: "Inelastic Behaviour of Solids", (ed. M. F. Kanninen et al.), McGraw-Hill, 1970, 635.
149. ATKINSON, C. and WILLIAMS, M. L., Int. Journal Solids Struct., 9, 1973, 237.
150. ROSE, L. R. F., Int. Journal Fract., 12, 1976, 799.
151. WILLIAMS, J. G. and BIRCH, J. W., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
152. BROBERG, K. B., Ark. Fys., 18, 1960, 159.
153. BROBERG, K. B., J. Appl. Mech., 31, 1964, 546.
154. AUSTWICK, A., M.Sc.(Tech.) Dissertation, Sheffield, 1968.
155. KOSTROV, B. V., Prikl. Mat. Mekh., 30, 1966, 1042.
156. ESHELBY, J. D., J. Mech. Phys. Solids, 17, 1969, 177.
157. ESHELBY, J. D., In: "Physics of Strength and Plasticity", (ed. A. S. Argon), M.I.T. Press, 1969, 263.
158. FREUND, L. B., J. Mech. Phys. Solids, 20, 1972, 129,141; J. Mech. Phys. Solids, 21, 1973, 47.
159. ROSE, L. R. F., Proc. Roy. Soc., A349, 1976, 497.
160. ACHENBACH, J. D., Int. Journal Solids Structures, 11, 1975, 1301.
161. ACHENBACH, J. D. and BAZANT, Z. P., Journal Appl. Mech., 42, 1975, 183.

162. ACHENBACH, J. D., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
163. ACHENBACH, J. D. and VARATHARAJULU, V. K., Quart. Appl. Math., 32, 1974, 123.
164. FREUND, L. B., Int. Journal Engng. Science, 12, 1974, 179.
165. ROSE, L. R. F., Int. Journal Fract., 12, 1976, 829.
166. SCULLY, J. C., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
166a. McMAHON, C. J. Jr., BRIANT, C. L. and BANERJI, S. K., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
167. KAMDAR, H. H., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
168. COFFIN, L. F., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
169. BEEVERS, C. J., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
170. McLEAN, D., DYSON, B. F. and TAPLIN, D. M. R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
171. RICKERBY, D. G., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
172. TETELMAN, A. S. and BESUNER, P., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
173. NEMEC, J., DREXLER, J. and KLESNIL, M., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
174. GREENWOOD, G. W., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
175. ASHBY, M. F., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
176. KUHN, H. A. and DIETER, G. A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
177. EMBURY, J. D. and LeROY, G. H., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
178. GOODS, S. H. and NIX, W. D., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
179. ALLEN-BOOTH, D. M., ATKINSON, C. and BILBY, B. A., Acta Met., 23, 1975, 371.
180. YOKOBORI, T., KONOSU, S. and YOKOBORI, A. T., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
181. SMITH, E., COOK, T. S. and RAU, C. A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
182. KAUSCH, H. H., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
183. PETERLIN, A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.
184. KELLY, A., TYSON, W. and COTTRELL, A. H., Phil. Mag., 15, 1967, 567.
185. RICE, J. R. and THOMSON, R., Phil. Mag., 29, 1974, 73.
186. WILLIAMS, M. L., Int. Journal Fract. Mech., 1, 1965, 292.
187. PETCH, N. J. and STABLES, P., Nature, 169, 1952, 842.
188. PETCH, N. J., Phil. Mag., 1, 1956, 331.
189. SMITH, R. A., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, II, 1977.
190. PIEKARSKI, K. R., "Fracture 1977", (ed. D. M. R. Taplin), University of Waterloo Press, Canada, I, 1977.