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VARIATIONAL BOUNDS AND QUALITATIVE METHODS IN FRACTURE MECHANICS

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INTRODUCTION

According to modern fracture mechanics, to determine the conditions of subsequent growth of a crack of given geometry, it is necessary to know the stress intensity factor in the points of the initial crack contour as well as in the points of all the subsequent positions of the crack contour. This is of minor importance in plane and axisymmetrical problems but gives rise to great difficulties in three-dimensional problems such as the problem of growth of an opening mode crack in the plane of symmetry of an elastic body.

The paper is concerned with some methods of determination of conditions sufficient for a crack to be dangerous or safe in the principal three-dimensional case mentioned above. The key to the problem is the notion of *positivity* which is introduced here. The problem is said to be positive if the application of arbitrary positive (wedging) tractions to crack surfaces gives rise to positive normal displacements of the surfaces and positive normal stresses in the plane of symmetry outside the crack. For positive problems there is the following *comparison principle:* the stress intensity factor at a given point of the crack contour grows as the crack extends outside some arbitrary small region around the point. The stress intensity factor grows also if the additional wedging forces are applied to the crack surfaces. It follows that for positive problems a given crack is more dangerous (i.e., gives rise to fracture during a shorter period of time) than any crack it contains and is less dangerous than any crack that contains it. This makes it possible to take into consideration the conditions of growth of cracks with comparatively simple ("standard") contours only.

DISCUSSION

The principle of comparison makes it possible to construct a two-sided estimate of the stress intensity factor at a given point on the arbitrary smooth contour of a crack considering contours of simpler form enveloping the given crack and enveloped by it and having a common tangent in the point under consideration.

It has been shown [1] that the problem of a plane crack in an infinite three-dimensional body is positive. In [1] it was demonstrated for the example of an elliptical crack in an infinite elastic body that the comparison principle is a highly efficient means of construction of two-sided estimates of stress-intensity factors. The theorem may be extended to include the case of an opening mode crack in a bounded body as soon as

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the boundary is sufficiently far from the crack. It has been proved that the problem of an opening - mode crack situated in the central plane of an elastic layer with traction-free faces is positive as soon as $(d/h) < 0.7$, d - being the diameter of the plane domain occupied by the crack, 2h - the thickness of the layer. The crack opening and stress intensity factor at any point of the crack contour diminish as the thickness of the layer grows, the crack goemetry and tractions on its surfaces being fixed.

The stress intensity factor is a local differential quantity, so that its evaluation involves enormous difficulties. Nevertheless, the load distribution being prescribed there exists a family of specific so-called "extremal" contours with the following two properties: 1) The stress intensity factor is constant along the crack contour. 2) The elastic energy of a body with a crack of given area attains its maximum value for cracks with contours which belong to the family of extremal contours. The respective values of energy and stress intensity factor are functions of area bounded by extremal contours, so the stress intensity factor may be expressed through the derivative of the elastic energy on the crack area. The extremal contours may be used as "barriers" for crack growth in the sense of the introduction. As a result it becomes possible to express the notion of a dangerous or a safe crack in terms of integral quantities such as the crack area and elastic energy of the body with a crack. If the comparison principle is valid for the body under consideration and there is an extremal contour such that the corresponding stress intensity factor is equal to its critical value, then all the cracks contained inside the contour are safe and all the cracks containing the contour are dangerous. Consider an extremal contour Γ which bounds a domain G of area S. It is assumed that there is a supporting domain G_o of area S_o, $G_o < G$. Then the extremal contour generally has two parts one of which $\Gamma'' = \Gamma \cap \partial G_o$ and the second Γ' is free, i.e. lies outside G_o. The stress intensity factor on the free part of Γ is expressed through the corresponding values of elastic energy W as a function of area S: $W = W(S)$ by the formula

$$N^2/_{\Gamma'} = \frac{\mu}{(1-\nu)\pi}\frac{dW}{dS} \tag{1}$$

The true displacements in the points of the surface of a crack with a contour of given form minimize the elastic energy W of the cracked body. So through the definition of an extremal contour is devised a solution of a "maximin" problem:

$$W_o = \max_{mes\, G = S} \min_{G} W \tag{2}$$

Consider an example of an extremal contour. Let a crack occupy a domain G_o in the plane $x_3 = 0$ in elastic space with surfaces act normal tractions acting on its surfaces:

$$\sigma_{33} = -p(1+\varepsilon x_1^2)\ , \ p=const\ , \ \varepsilon=const \tag{3}$$

The constant ε is assumed to be small. If $\varepsilon = 0$ the extremal contour is evidently a circle containing G_o. For small ε and with symmetry assumed it seems reasonable to seek the extremal contour as an elliptical contour of form:

$$x_1 = a_1\cos\theta,\ x_2 = b_1\cos\theta,\ a_1 = a(1+\delta_1),\ b_1 = a(1+\delta_2),\ \delta_{1,2} \ll 1 \tag{4}$$

The unknown $\delta_{1,2}$ may be determined using the condition N=const along the contour. Some algebra gives:

$$\delta_1 = \varepsilon a^2 \frac{29}{345}\ , \quad \delta_2 = -\varepsilon a^2 \frac{3}{115} \tag{5}$$

Now the equations (4), (5) give a family of extremal contours corresponding to load of form (2), a being a parameter. The family may be used to estimate the conditions of limiting equilibrium of plane cracks of arbitrary geometry under loading of form (2) as outlined before.

To apply the approach presented here we must have some effective solutions of three-dimensional crack problems for cracks bounded by etalon contours. Such solutions may be constructed by straightforward use of variational and variational-difference methods. Some numerical results are presented. The results may be obtained using medium range digital computers.

The conditions for a crack to be dangerous or safe having been expressed in terms of an energy criterion there is a possibility of further simplification of the elasticity problem under consideration. The simplification is based on the following statement. Let the external loads $\vec{\sigma}_n = \vec{f}(\vec{x})$ be prescribed on a part S' of the surface S of an elastic body D, on the rest of S being prescribed the displacements, $\vec{u} = \vec{g}(\vec{x})$. The elastic constants of the material are considered to be functions of coordinates: $\lambda = \lambda(\vec{x})$, $\mu = \mu(\vec{x})$. The quantity Q:

$$Q = \iint_{S'} \vec{f}\,\vec{u}\, d\sigma - \iint_{S''} \vec{\sigma}_n \vec{g}\, d\sigma \tag{6}$$

may be considered as a functional of λ and μ; $Q = Q(\lambda,\mu)$. It may be shown that the functional is *monotonic*: $\lambda'(\vec{x}) \geq \lambda(\vec{x})$, $\mu'(\vec{x}) \geq \mu(\vec{x})$, $\forall\, \vec{x} \in D$, then $Q' = Q(\lambda',\mu') \leq Q(\lambda,\mu)$. In particular, for $\vec{g}\equiv 0$ (the part S'' of S is clamped) Q is equal to the work done by the external loads. So it follows that the work increases as the material rigidity in some subdomain of S decreases and vice versa. Thus it is possible to estimate strain energy for a given cracked body in terms of the energies of bodies of simpler geometry with cracks bounded by extremal (at the prescribed crack area) contours. Consider, for example, infinite space with a crack under uniform tension normal to the crack plane. Now the free extremal contours are, evidently, circles; the elastic energy is equal to half the crack volume multiplied by the applied stress. This assumes that the volume V of a plane crack of arbitrary geometry and area S, under internal pressure P is not greater than the volume of a penny-shaped crack of the same area, so:

$$V \leq \frac{P}{E}\,\frac{16(1-\nu^2)}{3\pi^{3/2}}\, S^{3/2} \tag{7}$$

The inequality is an analogue of a well-known inequality for the capacity of a plane domain [2].

In certain cases the energetic bounds may be applied directly to estimate the stress intensity factors. For example, consider a circular crack of radius ℓ around a spherical cavity of radius ρ. The crack and the cavity are under an internal pressure P. The potential energy is, by using linearity and dimensional arguments:

$$W = \frac{P^2\ell^3}{\mu}\,\phi\left(\frac{\rho}{\ell}\right) \tag{8}$$

ϕ being a dimensionless function. Through the statements of this section we have $\phi' \geq 0$. Thus:

$$\frac{\partial W}{\partial \ell} = \frac{3P^2\ell^2}{\mu}\phi\left(\frac{\rho}{\ell}\right) - \frac{P^2\ell\rho}{\mu}\phi'\left(\frac{\rho}{\ell}\right) \leq \frac{3P^2\ell^2}{\mu}\phi\left(\frac{\rho}{\ell}\right) = \frac{3W(\rho,\ell)}{\ell} \leq \frac{3W(\ell,\ell)}{\ell} \tag{9}$$

It implies, using Irwin's formula:

$$N^2 \leq N_o^2 = \frac{3P^2\ell}{4\pi(1-\nu)} \tag{10}$$

For $\nu = 0.25$, $N \leq 0.56\ P\sqrt{\ell}$. For a penny-shaped crack of radius

$$r_o\ ,\ N = \frac{P\sqrt{2r_o}}{\pi} \approx 0.45\ P\sqrt{r_o}.$$

So a crack of radius ℓ surrounding a spherical cavity of radius ρ is less dangerous than a penny-shaped crack of radius

$$r_{eff} = \frac{\pi\ell}{8(1-\nu)} \quad (\text{for } \nu = 0.25,\ r_{eff} = \frac{\pi\ell}{2}).$$

In elastic contact problems this makes it possible to construct bounds for the displacement of a die and/or the force acting on the die, with the complex geometries of contact areas and elastic body and various types of contact (sliding contact, frictional contact, etc.), on the basis of a solution of the respective problems for dies and/or bodies of simple geometries.

REFERENCES

1. GOLDSTEIN, R. V. and ENTOV, V. M., Int. J. Fracture, 6, 1975, 11.
2. POLYA, G. and SZEGO, G., Isoperimetric inequalities in mathematical physics, Princeton University Press, 1951.