THE STRESS INTENSITY FACTORS FOR X-FORMED ARRAYS OF CRACKS

O. Aksogan*

INTRODUCTION

In the present study, the interaction of arbitrary arrays of cracks located along two intersecting infinite straight lines is considered. The method of analysis is similar to the one used by the author for V-formed arrays of cracks [1]. The new feature of a branched crack at the centre is treated by a special procedure during the application of the numerical method.

The analytical method used in this work consists of the joint use of the Mellin transform and the Green's function technique. The system of singular integral equations, thus obtained, is solved by a special application of an effective numerical method [2].

FORMULATION OF THE PROBLEM

In polar coordinates, in the absence of body forces, the stresses and the displacements in plane elasticity can be given as follows:

$$\sigma_r = \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{1}{r}\frac{\partial\Phi}{\partial r},$$

$$\sigma_\theta = \frac{\partial^2\Phi}{\partial r^2}, \qquad \tau_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right),$$

$$2\mu u_r = -\frac{\partial\Phi}{\partial r} + (1-\lambda)r\frac{\partial\psi}{\partial\theta}, \tag{1}$$

$$2\mu u_\theta = -\frac{1}{r}\frac{\partial\Phi}{\partial\theta} + (1-\lambda)r^2\frac{\partial\psi}{\partial r},$$

where the Airy stress function, Φ, and the displacement function, ψ, satisfy the equations

$$\nabla^4\Phi = 0\ , \qquad \nabla^2\psi = 0\ , \qquad \frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial\theta}\right) = \alpha^2\Phi\ . \tag{2}$$

In (1) and (2), $\lambda = \nu/(1+\nu)$ for plane stress and $\lambda = \nu$ for plain strain, μ is the shear modulus and ∇^2 is the harmonic operator.

*Middle East Technical University, Ankara, Turkey.

MS243

The Mellin transform of a function $f(r)$, defined and suitably regular in $(0 < r < \infty)$, and its inverse are defined by

$$\tilde{f}(s) = \int_0^\infty f(r) r^{s-1}\, dr \,, \qquad f(r) = \frac{1}{2\, i} \int_{c-i\infty}^{c+i\infty} \tilde{f}(s) r^{-s}\, ds \,, \tag{3}$$

where c is such that $r^{c-1}f(r)$ is absolutely integrable in $(0,\infty)$. The transforms of the derivatives can be found by using the relation

$$\int_0^\infty r^j \frac{d^j f(r)}{dr^j} r^{s-1}\, dr = (-1)^j \frac{\Gamma(s+j)}{\Gamma(s)} \tilde{f}(s) \,, \tag{4}$$

provided

$$r^{s+\ell-1} \frac{d^{\ell-1} f(r)}{d\, r^{\ell-1}} \to 0 \text{ as } r \to (0,\infty) \,, \quad \ell=1,\dots,j \,. \tag{5}$$

In plane elasticity problems, in which polar coordinates are used, the solution of (2) for each infinite wedge gives:*

$$\tilde{\Phi}(s,\theta)=Z_1\exp(is\theta)+\bar{Z}_1\exp(-is\theta)+Z_2\exp[i(s+2)\theta]+\bar{Z}_2\exp[-i(s+2)\theta],$$

$$(r^2\tilde{R})=2i(s+1)\{Z_1 s\ \exp(is\theta)+Z_2(s+1)\exp[i(s+2)\theta]-\bar{Z}_2\exp[-i(s+2)\theta]\},$$

$$(r^2\tilde{V})= -\frac{s+1}{\mu}\{Z_1 s\ \exp(is\theta)+Z_2(s+1)\exp[i(s+2)\theta]+\kappa\bar{Z}_2\exp[-i(s+2)\theta]\} \tag{6}$$

where

$$R = \tau_{r\theta} + i\sigma_\theta \,, \qquad u = u_r + iu_\theta \,,$$

$$V = \frac{\partial u_r}{\partial r} + i\frac{\partial u_\theta}{\partial r} \,, \qquad \kappa = 3 - 4\lambda \,, \tag{7}$$

and Z_1 and Z_2, with their complex conjugates $\bar{Z}_1$ and $\bar{Z}_2$, are independent of θ.

In the present work, the isotropic homogeneous infinite plane is separated into four infinite wedges along the four lines of cracks (see Figure 1). Let the union of all the straight line segments representing the cracks along one radial line be called L and the remainder L', the former being finite and the latter infinite. The singular part of the solution may be formulated with the following boundary conditions:**

* The complex notation used here is only for convenience.

**The crack surface tractions, considered here, are the reversed self-equilibrating stresses along the crack lines for the medium without the cracks under the actual loading.

$$\begin{aligned}
R_1(r,0) &= R_4(r,2\pi) && \text{on } L_1+L_1' \,, \\
V_1(r,0) &= V_4(r,2\pi) && \text{on } L_1' \,, \\
R_1(r,0) &= w_1(r) && \text{on } L_1 \,, \\
R_{j-1}(r,\theta_j) &= R_j(r,\theta_j) && \text{on } L_j+L_j' \,, \\
V_{j-1}(r,\theta_j) &= V_j(r,\theta_j) && \text{on } L_j' \,, \\
R_j(r,\theta_j) &= w_j(r) && \text{on } L_j \,, \quad j=2,3,4
\end{aligned} \tag{8}$$

where the subscripts show the region or the boundary to which a certain quantity pertains. Needless to say, $w_j(r)$ are the complex tractions on the surfaces of the cracks.

Making use of the Mellin transform and the Green's function technique following the method used in [1], the stress expressions for the wedge-shaped domains are found in the following form:

$$R_\ell(r,\theta)= \sum_{j=1}^{\ell} \int_{L_j} H_{1j}(r,\theta,\rho_j)d\rho_j + \sum_{j=\ell+1}^{4} \int_{L_j} H_{2j}(r,\theta,\rho_j)d\rho_j \,, \ell=1\dots,4 \tag{9}$$

where

$$H_{\alpha j}(r,\theta,\rho_j)= \frac{1}{\pi} \int_{c-i\infty}^{c+i\infty} \frac{\mu ds}{\rho_j(\kappa+1)(\beta-1)} \left(\frac{\rho_j}{r}\right)^{s+2} \Big\{\beta^{\alpha-1}\exp\Big[is(\theta-\theta_j)\Big]\Big[-sg_j(\rho_j)$$

$$+i(s+2)f_j(\rho_j)\Big]+(s+1)\beta^{\alpha-1}\exp\Big[i(s+2)(\theta-\theta_j)\Big]\Big[g_j(\rho_j)-if_j(\rho_j)\Big]$$

$$+\beta^{2-\alpha}\exp\Big[-i(s+2)(\theta-\theta_j)\Big]\Big[g_j(\rho_j)+if_j(\rho_j)\Big]\Big\} \,, \alpha=1,2, \ j=1,\dots,4 \tag{10}$$

in which $\beta = \exp(2is\pi)$ and*

$$g_1(r)+if_1(r) = V_1(r,+0) - V_4(r,2\pi-0) \,, \quad r \text{ on } L_1+L_1' \,,$$

$$g_j(r)+if_j(r) = V_j(r,\theta_j+0) - V_{j-1}(r,\theta_j-0) \,, \ r \text{ on } L_j+L_j' \,, \ j=2,3,4. \tag{11}$$

Applying the stress expressions (9) to the third and sixth of equations (8), the integral equations of the problem are found as

*The unknown functions, f and g, are the densities of the dislocations of opening and edge-sliding modes, respectively.

$$\sum_{j=1}^{\ell} \int_{L_j} H_{1j}(r,\theta_\ell,\rho_j)\,d\rho_j + \sum_{j=\ell+1}^{4} \int_{L_j} H_{2j}(r,\theta_\ell,\rho_j)\,d\rho_j = w_\ell(r),$$

$$r \text{ on } L_\ell, \quad \ell=1,\dots,4. \tag{12}$$

For continuity of the displacements, from the second and fifth of equations (8):

$$\int_{c_{j\ell}}^{d_{j\ell}} \left[g_j(\rho_j)+if_j(\rho_j)\right]d\rho_j = 0\ , \quad j=1,\dots,4\ , \quad \ell=1,\dots,n_j\ , \tag{13}$$

where n_j are the numbers of the cracks on the corresponding radial lines (see Figure 1 for the integration limits).

During the solution of (12), the kernels H_{1j}, H_{2j} are evaluated making use of the residue theory by a special procedure (see reference [1]). After lengthy but straightforward computations, (12) takes the following form, in terms of real variables:

$$\frac{1}{\pi}\sum_{j=1}^{4}\int_{L_j}\left\{\left[\frac{\delta_{\alpha j}}{\rho_j-r} + \frac{1-\delta_{\alpha j}}{2\rho_j}H_1^*\left(\frac{\rho_j}{r}\ ,\ \theta_{\alpha j}\right)\right]g_j(\rho_j) + \frac{1}{2\rho_j}H_2^*\left(\frac{\rho_j}{r}\ ,\ \theta_{\alpha j}\right)f_j(\rho_j)\right\}d\rho_j = \frac{\kappa+1}{2\mu}q_\alpha(r) \tag{14}$$

$$\frac{1}{\pi}\sum_{j=1}^{4}\int_{L_j}\left\{\frac{1}{2\rho_j}H_3^*\left(\frac{\rho_j}{r}\ ,\ \theta_{\alpha j}\right)g_j(\rho_j) + \left[\frac{\delta_{\alpha j}}{\rho_j-r} + \frac{1-\delta_{\alpha j}}{2\rho_j}H_4^*\left(\frac{\rho_j}{r}\ ,\ \theta_{\alpha j}\right)\right]f_j(\rho_j)\right\}d\rho_j = \frac{\kappa+1}{2\mu}p_\alpha(r)$$

$$r \text{ on } L_\alpha, \quad \alpha=1,\dots,4,$$

where

$$w_\alpha(r) = q_\alpha(r) + i\,p_\alpha(r)\ ,$$

$$\theta_{\alpha j} = \theta_j - \theta_\alpha = 2M_{\alpha j}\,\pi/N_{\alpha j}\ , \qquad \alpha,j=1,\dots,4$$

and $H_\ell^*(x,2M\pi/N)$, $\ell=1,\dots,4$, are defined in the Appendix.

NUMERICAL RESULTS

A special application [1] of an effective numerical method [2] is used for the solution of (14) subject to the continuity conditions (13). The nodal points for collocation are chosen at the zeros of the Chebyshev polynomials, in the ranges pertaining to each and every crack in the medium (see reference [1] for details). It must be noted that, when two or more cracks meet at the origin, the continuity conditions (13) do not apply to them separately. In that case, if there is a central symmetry in the crack setting and a central symmetry or antisymmetry in the loading (crack surface tractions), the consequent central symmetry or antisymmetry in the unknown functions renders the application of the numerical technique possible. What needs to be done, in that case, is to choose the collocation points as if each crack, terminating at the origin, is extended to the other side of the origin by a reflection.

Although the numerical solution yields the values for any elasto-mechanic quantity, we will be concerned with the stress intensity factor only. Besides the closed form solutions for two and three collinear cracks, comparisons of the numerical results of the present work were also made with the graphical presentations of Isida [3] for other arrays of isolated cracks. The results matched perfectly. Surprisingly enough, even the cases of parallel cracks have been treated (the results matching with those of [3]), just by taking the angle between the cracks small enough. (It must be noted that, the origin being at infinity, the case of exactly parallel cracks cannot be treated by the present method).

The special case of four symmetrically situated radial cracks under constant internal pressure was also treated and the results were in good agreement with those of Tweed and Rooke [4].

The results of the present work for cross-shaped cracks with two pairs of unequal arms, loaded with unequal constant normal tractions were compared with those given by Sneddon and Das [5]. The results obtained for the case of nonuniform internal pressure for a cross-shaped crack with equal arms were compared with the results of Stallybrass [6]. There was a mismatch of about 0.1 per cent for both cases when 20 points of collocation were taken along each branch.

Some other cases, which cannot be found in the literature, have been considered. Choosing a suitable parameter for each case, the two types of stress intensity factors at all crack tips were presented in graphical form (Figures 2 - 4). For these computations sixteen collocation points were taken along each isolated crack and each branch of the X-shaped crack. The results for the limiting cases of single and two or three collinear cracks were observed to coincide with those in the literature [3]. Because of limited space, loadings which cause partial closures could not be included here, although a number of such cases were treated, making use of the procedure in references [7, 8]. For the same reason an X-formed array of antisymmetry could not be exposed here.

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APPENDIX

$$H_1^*(x, 2M\pi/N) = (x^2-1)C_1 + x\cos(2M\pi/N) ,$$

$$H_2^*(x, 2M\pi/N) = (x^2-1)S_1 - x^2 S_2 + x\sin(2M\pi/N) ,$$

$$H_3^*(x, 2M\pi/N) = (x^2-1)S_1 - S_2 - x\sin(2M\pi/N) ,$$

$$H_4^*(x, 2M\pi/N) = (1-x^2)C_1 + (x^2+1)C_2 - x\cos(2M\pi/N) + 2$$

where

$$\begin{Bmatrix} C_j \\ S_j \end{Bmatrix} = \sum_{\ell=1}^{N} \left[\left(\frac{N}{2(x^N-1)} + \frac{\ell}{2} \right)^{2-j} \frac{2x^{N-\ell}}{x^N-1} \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} (2\ell M\pi/N) \right], \quad j=1,2 .$$

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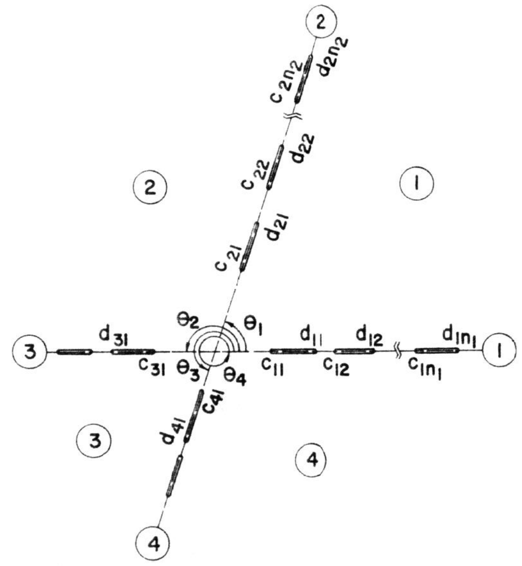
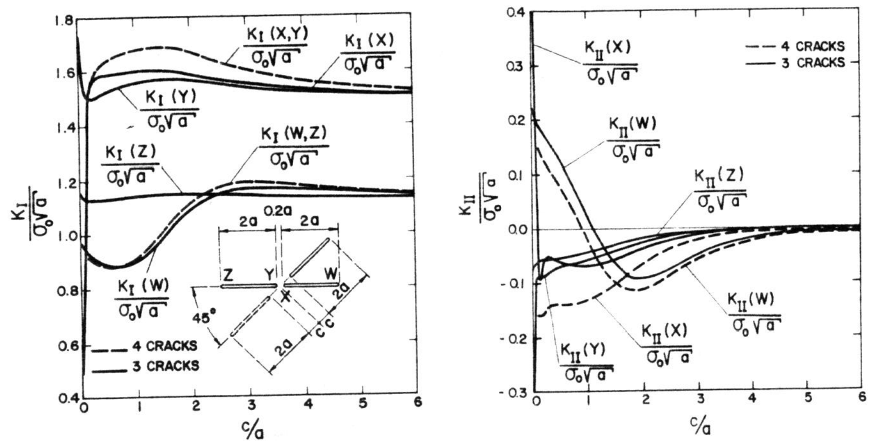


Figure 1 The Geometry and Notation



(a) (b)

Figure 2 (a) Opening Mode Stress Intensity Factors for 3 and 4 Radial Cracks Under Uniform All-Round Tension, σ_0

(b) Edge-Sliding Mode Stress Intensity Factors for 3 and 4 Radial Cracks Under Uniform All-Round Tension, σ_0

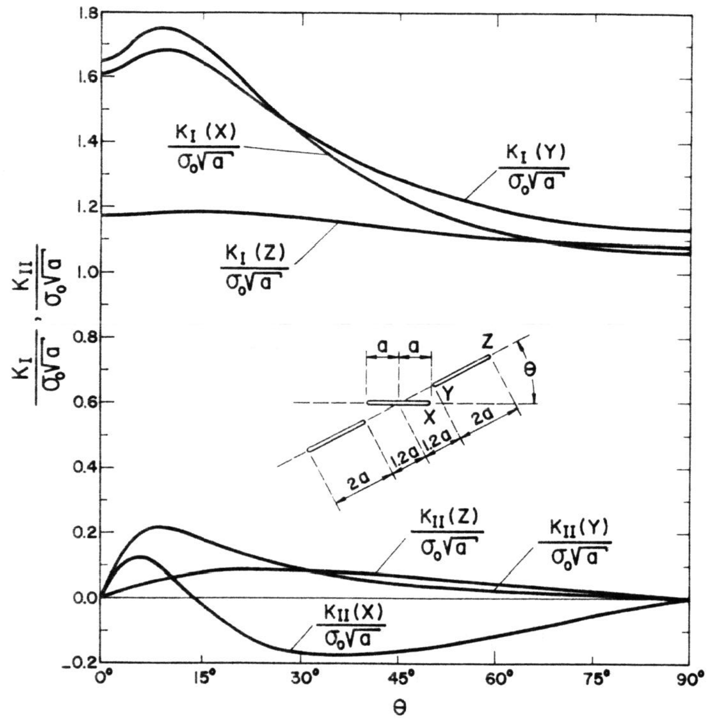


Figure 3 Stress Intensity Factors for an Array of Cracks with Central Symmetry Under Uniform All-Round Tension, σ_0

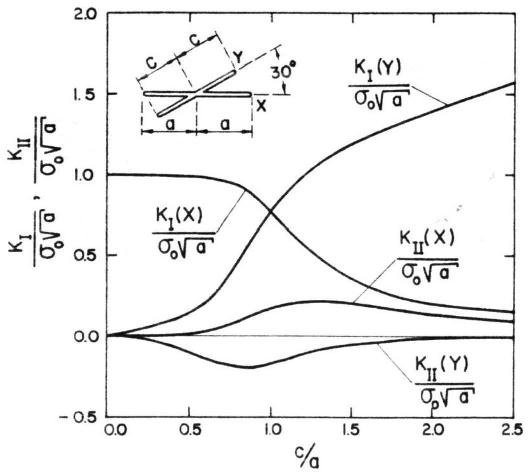


Figure 4 Stress Intensity Factors for an X-Shaped Crack Under Uniform All-Round Tension, σ_0