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STATISTICAL CONSIDERATION ON INHOMOGENEITY OF MECHANICAL PROPERTIES OF MATERIALS

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INTRODUCTION

Recently, various theories of continuum mechanics have been published to take the microscopical inhomogeneity of materials into account [1 - 4]. Mechanical quantities appearing in such theories are macroscopic usually and they are considered as averages of the actual quantities distributed inhomogeneously in materials. However, it does not seem that the deriving processes of such average quantities are sufficiently discussed. This paper is concerned with a fundamental consideration of the principle for averaging the inhomogeneous mechanical quantities distributed statistically in materials. A method to obtain mean quantities is introduced by using the internal work done by the inhomogeneous stress acting upon a closed surface of a small region in the material body. Some properties of mechanical quantities derived through the above process are discussed, and it is attempted to argue the relation between the inhomogeneity of stress and the yielding of materials. It is assumed here that the statistical characteristics of distribution of mechanical inhomogeneity are uniform at any place in a material and that the material is considered homogeneous from the macroscopic viewpoint.

AVERAGING OF MECHANICAL QUANTITIES

Stress and strain in continuum mechanics are regarded as mean quantities determined (explicitly or implicitly) through certain averaging processes, as is indicated by their definitions. A mean field quantity is averaged over a certain region in the material body. As far as the mean quantity is expected to be expressed in tensorial form, the averaging process is to be such that it maintains tensorial significance. It is known that this requirement is satisfied when the averaging is carried out isotropically.[1] In this paper, the sphere R with a radius ρ (surface area: A, volume: V) is adopted for the region upon which the mechanical quantities are averaged.

The increment of the internal work per unit volume of R due to the increment of the displacement $D\underline{u}$ is given by

$$Dw = \frac{1}{V} \oint_{\partial R} \underline{tn} \cdot D\underline{u} \, da \qquad (1)^{2}$$

where $\underline{tn}$ is the stress acting on a surface element of ∂R with unit normal

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1. It is considered that averaging of a tensor over open surface which appears in [1, 2] does not fully satisfy this requirement.
2. In this paper the body force is ignored for brevity.

$\underline{n}$. Quantities $\underline{tn}$ and $D\underline{u}$ are generally regarded as inhomogeneous microscopically. In terms of the stress tensor $\underline{\sigma}$ and the increment of the strain tensor (including the rotation) $D\underline{\gamma}$, quantities $\underline{tn}$ and $D\underline{u}$ are given respectively by

$$\underline{tn} = \underline{n} \cdot \underline{\sigma} + \Delta\underline{tn} \tag{2}$$

$$D\underline{u} = \underline{r} \cdot D\underline{\gamma} + \Delta(D\underline{u}) \tag{3}$$

where quantities to which the symbol Δ is attached denote residuals from mean values and $\underline{r}$ is the position vector from the center of R. Now the mean quantities $\underline{\sigma}$ and $D\underline{\gamma}$ are regarded as constant in the following averaging process. It is considered that the expected values of the components of $\underline{\sigma}$ and $D\underline{\gamma}$ are obtained at the vicinity of the center of R when the work done by the residual parts

$$\Delta(Dw) = \frac{1}{V} \oint_{\partial R} \Delta\underline{tn} \cdot \Delta(D\underline{u})da \tag{4}$$

is not dependent on the mean quantities $\underline{\sigma}$ and $D\underline{\gamma}$. This equation is transformed as

$$\Delta(Dw) = \frac{1}{V} \oint_{\partial R} (\underline{tn}-\underline{n} \cdot \underline{\sigma}) \cdot (D\underline{u}-\underline{r} \cdot D\underline{\gamma})da \ , \tag{5}$$

using equations (2) and (3). By differentiating the right hand side of equation (5) with respect to each component of $\underline{\sigma}$ and $D\underline{\gamma}$, and by equating these expressions to zero, we obtain

$$\underline{\sigma} = \frac{1}{V} \oint_{\partial R} \underline{r}\ \underline{tn}\ da \tag{6}$$

$$D\underline{\gamma} = \frac{1}{V} \oint_{\partial R} \underline{n}\ D\underline{u}\ da \tag{7}$$

where products of two vectors without intermediate symbol denote dyad. In the derivation of the above two equations, the following relation is used:

$$\oint_{\partial R} \underline{n}\ \underline{r}\ da = V\underline{I} \tag{8}$$

where $\underline{I}$ is the unit tensor. Equations (6) and (7) are also obtained by the least squares method applied for vectors $\underline{tn}$ and $D\underline{u}$. Namely, $\underline{\sigma}$ and $D\underline{\gamma}$ expressed by these equations give the minimum values of

$$I_t = \oint_{\partial R} \Delta\underline{tn} \cdot \Delta\underline{tn}\ da \tag{9}$$

and

$$I_u = \oint_{\partial R} \Delta(D\underline{u}) \cdot \Delta(D\underline{u})da \tag{10}$$

respectively. Furthermore, for these quantities the equation

$$Dw = \underline{\sigma} \cdot\cdot D\underline{\gamma} + \Delta(Dw) \tag{11}$$

holds, where the symbol $\cdot\cdot$ denotes the double inner product. Equation (11) represents the independence of the work done by the mean quantities and the work done by the residuals.

The magnitude of $\Delta(Dw)$ is affected by the size of R. As it is assumed that statistical characteristics of inhomogeneous distribution of $\underline{tn}$ and $D\underline{u}$ are uniform at any place in material, the value of integral

$$\oint_{\partial R} \Delta\underline{tn} \cdot \Delta(D\underline{u})da = \Delta(Dw)\ V \tag{12}$$

may be regarded as proportional to A in the case where the distributions of the stress and the strain are macroscopically uniform. Thus the magnitude of $\Delta(Dw)$ is inversely proportional to ρ. On the other hand, when the macroscopic fields of stress and strain have gradients in the material body, the errors involved in the approximations for $\underline{tn}$ and $D\underline{u}$ in terms of linear coefficients $\underline{\sigma}$ and $D\underline{\gamma}$ increase as the size of $\overline{R}$ increases. From the above consideration, the most reasonable tensor quantities maybe obtained by use of a sphere with a certain radius and regarded as the macroscopic quantities at the center of the sphere. In the following, ρ is assumed to be constant throughout the material body.

FIELDS OF MECHANICAL QUANTITIES

If a stress field is given by microscopically differentiable continuous stress tensor $\underline{\sigma}^*$ and the equilibrium condition

$$\underline{\nabla} \cdot \underline{\sigma}^* = \underline{0} \tag{13}$$

is fulfilled, equation (6) is transformed as

$$\begin{aligned} \underline{\sigma} &= \frac{1}{V} \oint_{\partial R} \underline{r}(\underline{n} \cdot \underline{\sigma}^*)da \\ &= \frac{1}{V} \int_R \underline{\sigma}^*\ dv + \frac{1}{V} \int_R \underline{r}\ \underline{\nabla} \cdot \underline{\sigma}^*\ dv \\ &= \frac{1}{V} \int_R \underline{\sigma}^*\ dv\ . \end{aligned} \tag{14}$$

Thus it is seen that $\underline{\sigma}$ is a volume average of $\underline{\sigma}^*$. Then $\underline{\sigma}$ becomes a tensor which satisfies the equilibrium condition as shown below.

$$\underline{\nabla} \cdot \underline{\sigma} = \underline{\nabla} \cdot \left(\frac{1}{V} \int_R \underline{\sigma}^*\ dv\right) = \frac{1}{V} \int_R \underline{\nabla} \cdot \underline{\sigma}^*\ dv = \underline{0}\ . \tag{15}$$ [3]

If $D\underline{u}$ is differentiable and continuous, equation (7) is transformed as

$$D\underline{\gamma} = \frac{1}{V} \oint_{\partial R} \underline{n}\ D\underline{u}\ da = \frac{1}{V} \int_R \underline{\nabla}\ D\underline{u}\ dv = \underline{\nabla}\left(\frac{1}{V} \int_R D\underline{u}\ dv\right)\ . \tag{16}$$ [3]

3. The commutability of the nabla $\underline{\nabla}$ and the averaging operator $\frac{1}{V}\int_R dv$ is assured because the region R should be considered to move together with the concerning material point while the shape of R itself is kept constant for all points.

This equation indicates that $\underline{D\gamma}$ is regarded as the strain tensor derived by differentiation of a displacement vector which is obtained as a volume average of $\underline{Du}$. Thus $\underline{D\gamma}$ satisfies the following compatibility equation[4]:

$$\underline{\nabla} \times \underline{D\gamma} = \underline{0} \; . \tag{17}$$

Even if the mean quantities satisfy the macroscopic field equations as stated above, the residual part of work

$$\Delta(Dw) = Dw - \underline{\sigma} \cdot\cdot \underline{D\gamma} \tag{18}$$

takes a finite value generally, and it may be said that the work done by microdeformations defined in the generalized continuum theories [4] is corresponding to this part of internal work.

Now let a plastic deformation takes place in a material body in which magnitude and direction of the stress vector acting on each point of the spherical surface are assumed as constant. Then the following moment is produced:

$$\underline{Dm} = \frac{1}{V} \oint_{\partial R} \underline{tn} \times \underline{Du}\; da$$

$$= \underline{\sigma} \cdot \times \underline{D\gamma} + \Delta(\underline{Dm}) \tag{19}$$

where

$$\Delta(\underline{Dm}) = \frac{1}{V} \oint_{\partial R} \Delta\underline{tn} \times \Delta(\underline{Du}) da \; . \tag{20}$$

As $\underline{Dm}$ is the total moment acting on R, it can be equated to zero and we obtain

$$\underline{\sigma} \cdot \times \underline{D\gamma} = - \Delta(\underline{Dm}) \; . \tag{21}$$

The above equation indicates that the moment related to mean quantities does not vanish unless the residual part of moment equals to zero. This conclusion seems to be significant, because the plastic deformation of an inhomogeneous material needs not to satisfy the St. Venant's assumption stating that principal directions of stress tensor and increment of strain tensor coinside. It may be possible to consider that $\Delta(\underline{Dm})$ is produced by the couple stress [4]. Using such a theoretical model we can derive and eigen equation [6] which is very similar to the equation appearing in the Kondo's theory of yielding [3] in which the yielding is analyzed as analogus phenomenon to the buckling of plates.

It is noted that $\Delta(Dw)$ and $\Delta(\underline{Dm})$ are; given respectively, by the trace and the antisymmetric part of the following tensor

4. When differentiability (and/or continuity) of $\underline{Du}$ is not guaranteed, $\underline{D\gamma}$ does not generally satisfy equation (17), and a material space after such an incompatible deformation becomes a material manifold with the teleparallelism [5].

$$\underline{T} = \frac{1}{V} \oint_{\partial R} \Delta\underline{tn}\; \Delta(\underline{Du}) da \; . \tag{22}$$

This tensor denotes covariances between random variables in the statistical terminology.

INFLUENCE OF INHOMOGENEITY ON YIELD CRITERION

Let us begin with the consideration on von Mises' yield criterion

$$\underline{\sigma}' \cdot\cdot \underline{\sigma}' = k^2 \tag{23}$$

where k^2 is a constant and $\underline{\sigma}'$ is the stress deviation defined in terms of the mean stress p and unit tensor $\underline{I}$ as

$$\underline{\sigma}' = \underline{\sigma} - p\,\underline{I} \; . \tag{24}$$

The expression $\underline{\sigma}' \cdot\cdot \underline{\sigma}'$ is considered as a measure of variation from the mean stress. This measure may be acceptable particularly when the distribution of stress is assumed to be homogeneous. However, when the inhomogeneity of stress distribution is taken into account, the extended yield criterion may be written as

$$\frac{1}{A} \oint_{\partial R} (\underline{tn} - p\underline{n}) \cdot (\underline{tn} - p\underline{n}) da = K^2 \tag{25}$$

where K^2 is a constant. The intensity of stress in yield criteria is usually given in terms of relative values to the initial state where the external forces are absent, and further it is generally considered that the initial stress exists in inhomogeneous materials. Thus $\underline{tn}$ given by equation (2) and the mean stress p are expressed as

$$\underline{tn} = \underline{n} \cdot (\underline{\sigma}_0 + \underline{\sigma}) + \Delta\underline{tn}_0 + \Delta\underline{tn} \tag{26}$$

$$p = \frac{1}{3} \underline{I} \cdot\cdot (\underline{\sigma}_0 + \underline{\sigma}) \tag{27}$$

where quantities to which a suffix o is added represent initial quantities. Substitution of equations (26) and (27) into equation (25) gives

$$K^2 = \frac{1}{3} (\underline{\sigma}'_0 \cdot\cdot \underline{\sigma}'_0 + 2\underline{\sigma}'_0 \cdot\cdot \underline{\sigma}' + \underline{\sigma}' \cdot\cdot \underline{\sigma}') +$$

$$+ \frac{1}{A} \oint_{\partial R} (\Delta\underline{tn}_0 \cdot \Delta\underline{tn}_0 + 2\Delta\underline{tn}_0 \cdot \Delta\underline{tn} + \Delta\underline{tn} \cdot \Delta\underline{tn}) da. \tag{28}$$

Equation (28) is generalized yield criterion in which the initial stress and the microscopical inhomogeneity of stress are taken into account.

Now a simple case where $\underline{\sigma}'_0$ is zero while $\Delta\underline{tn}_0$ is not equal to zero is considered. Further the following two assumptions may be laid down from the physical consideration.

(Assumption 1) Statistically, the residual stress $\Delta \underline{t} n$ acts in such a way that the inhomogeneity of initial stress increases when $p > 0$ (tensile state) and decreases when $p < 0$ (compressive state).

(Assumption 2) Mean value of the inner produce $\Delta \underline{t} n \cdot \Delta \underline{t} n$ increases together with $\sigma \cdot\cdot \sigma$.

For brevity, replacing these assumptions by linear relations as

$$\frac{1}{A} \oint_{\partial R} \Delta \underline{t} n_0 \cdot \Delta \underline{t} n \, da = c_1 p \tag{29}$$

$$\frac{1}{A} \oint_{\partial R} \Delta \underline{t} n \cdot \Delta \underline{t} n \, da = c_2 \, \sigma \cdot\cdot \sigma \, . \tag{30}$$

Then we obtain

$$\sigma' \cdot\cdot \sigma' = C_1 - C_2 \, p - C_3 \, p^2 \tag{31}$$

where

$$\left.\begin{aligned} C_1 &= \frac{3}{1+3c_2} \left(K^2 - \frac{1}{A} \oint_{\partial R} \Delta \underline{t} n_0 \cdot \Delta \underline{t} n_0 \, da \right) \\ C_2 &= \frac{6c_1}{1+3c_2} \, , \qquad C_3 = \frac{c_2}{1+3c_2} \, . \end{aligned}\right\} \tag{32}$$

These coefficients are considered as material constants determined by the initial stress distribution. It is noticed that C_1 takes the smaller value as the amount of inhomogeneous initial stress increases. If we put as $C_2 = C_3 = 0$, we get von Mises' equation (23), and on the other hand, equation (31) reduces to Griffith type equation for two axial stress state in the case where $C_3 = 0$.

CONCLUDING REMARKS

The macroscopic field quantities appearing in ordinary continuum mechanics are considered to be derived through such an averaging process as stated here. In this paper, the examples of measures of deviation from these mean quantities are given in the forms of the right hand sides of equations (9), (10), (22) and (28). As indicated in the preceding sections, such quantities seem particularly important to explain nonelastic behaviours of materials.

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