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ON ENERGY RELEASE RATES IN AXISYMMETRICAL PROBLEMS

M. A. Astiz, M. Elices and V. Sánchez Galvez*

INTRODUCTION

The problem of calculating energy release rates from a given stress and strain field in an elastic cracked body has been solved by means of the J integral [1]. This integral has been extensively applied to plane stress and plane strain problems. In three dimensional problems such a powerful tool does not exist. Three kinds of energy release rates have been defined [2, 3] corresponding to a movement of translation, rotation or expansion of the crack border, but it seems very difficult to find an energy release rate which will be adequate to describe the real movement of the crack border. In axisymmetrical problems this question seems easier to solve because a crack has only one degree of freedom.

INTEGRAL EXPRESSIONS OF THE ENERGY RELEASE RATE

Statement of the Problem

We will consider an axisymmetrical homogeneous and elastic solid with an internal circular crack, as shown in Figure 1 (the solid has not necessarily to be a cylinder). The problem of an external circular crack can be treated in the same way. The purpose of this paper is to find an integral expression for the energy release rate during crack extension.

To solve this problem we will make use of the integrals [2, 3]:

$$J_k = -\frac{\partial U}{\partial x_k} = \int_S (w\, n_k - T_i\, u_{i,k})\, dS \tag{1}$$

$$M = -l\,\frac{\partial U}{\partial l} = \int_S (w\, x_i\, n_i - T_j\, u_{j,i}\, x_i - \frac{1}{2} T_i\, u_i)\, dS \tag{2}$$

where w is the energy density, S is a closed surface surrounding the crack border, $\underline{n}$ is the outer normal, $\underline{T}$ is the stress vector acting on the outer side of S, $\underline{u}$ is the displacement vector and l is any characteristic length of the crack.

We will now evaluate these integrals in an axisymmetrical problem.

* Escuela de Ingenieros de Caminos, Universidad Politècnica de Madrid, Ciudad Universitaria - Madrid 3, Spain

M. Integral

The surface S will be axisymmetric, as shown in Figure 1. Since the integrand is a scalar quantity, it is independent of the angular cylindrical coordinate, θ. This integrand can be evaluated on the plane $\theta = 0 (x_2 = 0)$. Considering that the element of area is:

$$dS = \rho \, d\theta \, dl \tag{3}$$

where dl is the element of arc length of the curve C, which is the section of S by the plane $\theta = 0$, the M integral can be expressed as:

$$M = 2\pi \int_C \rho (w x_i n_i - T_j u_{j,i} x_i - \frac{1}{2} T_i u_i) \, dl \, . \quad i,j = 1,2,3 \tag{4}$$

But along the curve C we have $x_2 = u_2 = 0$; then the indexes i and j of equation (4) may have only the values 1 and 3.

M is the energy release rate with respect to relative scale change da/a, then:

$$M = - \frac{dU}{\frac{da}{a}} = - \frac{2\pi a^2 \, dU}{2\pi a \, da} = 2\pi a^2 G \, . \tag{5}$$

As a consequence the energy release rate with respect to cracked area becomes:

$$G = \frac{1}{a^2} \int_C \rho (w x_i n_i - T_j u_{j,i} x_i - \frac{1}{2} T_i u_i) \, dl \, . \quad i,j = 1,3 \tag{6}$$

This expression is valid only for linear elastic materials as in equation (2) [4].

J. Integral

The application of the M integral to an axisymmetrical problem has been easy because the symmetric movement of the crack is an expansion equivalent to a relative scale change. It is more difficult to apply the J integral to this problem because this integral is the energy release rate with respect to a translation movement of the crack border.

To circumvent this problem it will be necessary to consider a sector of angular amplitude $d\theta$, as shown in Figure 2, because we are dealing with an axisymmetrical problem and the energy release rate is computed per unit of crack area. The movement of the element of crack border inside this sector can be considered as a translation parallel to the x_1 axis. The hypothesis can be justified by considering that the energy release caused by an increment da of the crack radius will be G a da $d\theta$ and the difference between this energy release and the actual energy release would be G $da^2 \, d\theta$, which can be neglected. The surface of integration is composed of a surface of revolution, S_1, and two bases, S_2 and S_3.

To perform the integration over S_1 it must be taken into account that the components of all the vectorial quantities, $\underline{v}$, of equation (1) ($\underline{v}$ can be the normal, the displacement vector or the stress vector) are:

$$\begin{aligned} v_1 &= v_\rho \cos \alpha \\ v_2 &= v_\rho \sin \alpha \\ v_3 &= v_z \end{aligned} \tag{7}$$

where v_ρ and v_z are the components of $\underline{v}$ in the cylindrical system, and θ is the angular coordinate as shown in Figure 3 and varies between

$$- \frac{d\theta}{2} \text{ and } \frac{d\theta}{2} \, .$$

It can also be shown [5] that the derivatives of the displacement vector are:

$$u_{1,1} = \cos^2 \alpha \frac{\partial u_\rho}{\partial \rho} + \sin^2 \alpha \frac{u_\rho}{\rho} \tag{8}$$

$$u_{2,1} = \left(\frac{\partial u_\rho}{\partial \rho} - \frac{u_\rho}{\rho} \right) \sin\alpha \cos\alpha \tag{9}$$

$$u_{3,1} = \frac{\partial u_z}{\partial \rho} \, . \tag{10}$$

The integral over S_1 can be expressed as a sum of integrals of the type

$$\int_{S_1} f(\rho,z) g(\alpha) \rho \, d\alpha \, dl = \int_{-\frac{d\theta}{2}}^{+\frac{d\theta}{2}} g(\alpha) d\alpha \int_C f(\rho,z) \rho \, dl \, . \tag{11}$$

This integral must be proportional to $d\theta$ because the higher order terms ($d\theta^2$, $d\theta^3$, etc.) would not be significant to the integration in θ. Then only the zero order term will be taken into account in the function $g(\alpha)$. This is equivalent to supposing that:

$$\begin{aligned} \sin \alpha &\simeq 0 \, , \\ \cos \alpha &\simeq 1 \, . \end{aligned} \tag{12}$$

Applying equations (7, 8, 9, 10, 11, 12) to equation (1) we obtain:

$$\int_{S_1} \left(w n_1 - T_i u_{i,1} \right) dS = d\theta \int_C \left(w n_\rho - T_\rho \frac{\partial u_\rho}{\partial \rho} - T_z \frac{\partial u_\rho}{\partial \rho} \right) \rho \, dl \tag{13}$$

Along the surfaces S_2 and S_3 only the component T_θ of the stress vector and the component n_θ of the normal will be non-zero. After having neglected again the terms of higher order in $d\theta$ the J_1 integral over S_2 and S_3 becomes:

$$\int_{S_2} \left(wn_1 - T_i\, u_{i,1}\right) dS = \frac{d\theta}{2} \int_{S_2} \left(-w + T_\theta \frac{u_\rho}{\rho}\right) dS \,, \tag{14}$$

$$\int_{S_3} \left(wn_1 - T_i\, u_{i,1}\right) dS = \frac{d\theta}{2} \int_{S_3} \left(-w + T_\theta \frac{u_\rho}{\rho}\right) dS \,. \tag{15}$$

Then the J_1 integral will be:

$$J_1 = d\theta \int_C \left(w\, n_\rho - T_\rho \frac{\partial u_\rho}{\partial \rho} - T_z \frac{\partial u_z}{\partial \rho}\right)\rho\, dl - \int_{S_2} \left(w - T_\theta \frac{u_\rho}{\rho}\right) dS \,. \tag{16}$$

The energy release rate per unit angle would be $J_1/d\theta$ and, considering the whole solid, the energy release rate with respect to an increment of the crack radius in axisymmetrical problems, J_A, will be:

$$J_A = 2\pi \int_C \left(wn_\rho - T_\rho \frac{\partial u_\rho}{\partial \rho} - T_z \frac{\partial u_z}{\partial \rho}\right)\rho\, dl - 2\pi \int_{S_2} \left(w - T_\theta \frac{u_\rho}{\rho}\right) dS \tag{17}$$

and the energy release rate, G, will become:

$$G = \frac{J_A}{2\pi a} = \frac{1}{a} \int_C \left(wn_\rho - T_\rho \frac{\partial u_\rho}{\partial \rho} - T_z \frac{\partial u_z}{\partial \rho}\right)\rho\, dl - \frac{1}{a} \int_{S_2} \left(w - T_\theta \frac{u_\rho}{\rho}\right) dS \,. \tag{18}$$

A condensed expression for J_A can be obtained by applying Green's theorem:

$$J_A = 2\pi \int_{S_2} \left(\rho \frac{\partial w}{\partial \rho} + T_\theta \frac{u_\rho}{\rho}\right) dS - 2\pi \int_C \left(T_\rho \frac{\partial u_\rho}{\partial \rho} + T_z \frac{\partial u_z}{\partial \rho}\right)\rho\, dl \,. \tag{19}$$

These expressions are valid for nonlinear elastic materials as is equation (1) [4].

The relation between J_A and G can also be found by performing the integration (17) along a circle centred at the tip of the crack with a very small radius (5), and by supposing the stress and strain fields are the same as in a plane strain problem [6].

CONCLUSIONS

Two integral expressions have been derived for energy release rate in axisymmetrical problems. Expression (6) is valid for linear elastic materials and expression (17) is valid for nonlinear elastic materials. Then this expression can be applied to plasticity problems if one can suppose that the deformation theory of plasticity with no unloading gives a good description of the behaviour of this material. Then one can repeat the reasoning of Rice and Rosengren [7] to show that the energy density exhibits a 1/r singularity at the crack tip. For a strain hardening material the stress, strain and displacement field associated with the near-tip dominant singularity must have the form [7, 8]

$$\sigma_{ij} = K_\sigma\, r^{-\frac{1}{n+1}}\, \tilde{\sigma}_{ij}(\theta)$$

$$\varepsilon^p_{ij} = K_\varepsilon\, r^{-\frac{n}{n+1}}\, \tilde{\varepsilon}_{ij}(\theta) \tag{20}$$

$$u_i = K_\varepsilon\, r^{\frac{1}{n+1}}\, \tilde{u}_i(\theta)$$

where n is the strain hardening exponent and $\tilde{\sigma}_{ij}$, $\tilde{\varepsilon}_{ij}$ and $\tilde{u}_i$ are a dimensional functions of θ as in [8].

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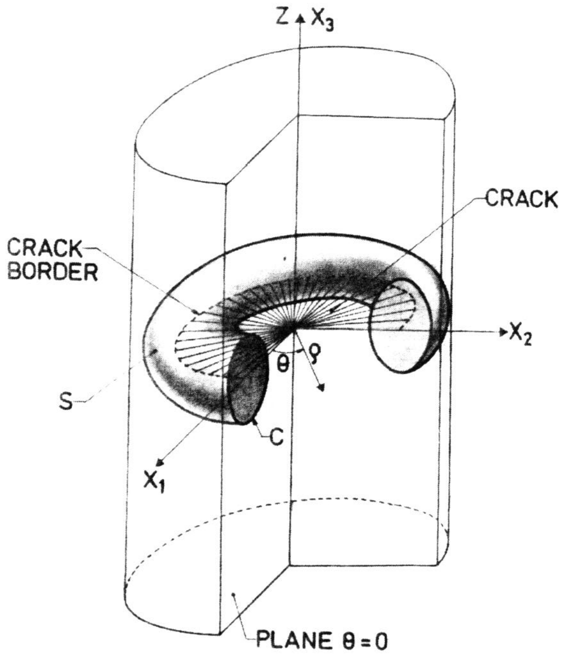

Figure 1

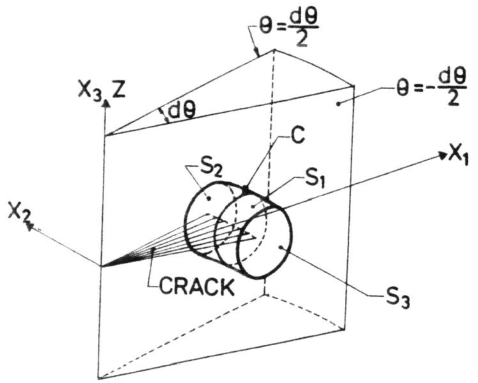

Figure 2

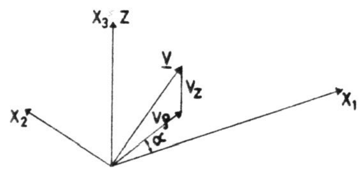

Figure 3