Fracture 1977, Volume 3, ICF4, Waterloo, Canada, June 19 - 24, 1977

THE MODIFIED WESTERGAARD EQUATIONS*

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INTRODUCTION

Sih [1] and Eftis and Liebowitz [2] have pointed out that the Westergaard method, which applies to a certain class of plane problems in linear elasticity and is most frequently used in fracture mechanics [3], suffers from a restriction. The restriction is essentially the following: In symmetric problems (mode I), the Westergaard function allows only a hydrostatic tension for the remote state of stress. This means, for example, that the simple case of uniaxial tension is excluded from this formulation (with or without cracks).

The above authors corrected this shortcoming by appending constant terms to Westergaard's stress equations. They did this by appealing to the Goursat-Kolosov and MacGregor complex formulations of the problem. The present note shows how those additions to the Westergaard functions can be made in a more straightforward way without reference to the more sophisticated representations. It is done by simply adding the real part of a term in z^2 to the Airy stress function of Westergaard.

WESTERGAARD'S FUNCTION

The classical problem of plane isotropic elasticity is set up in terms of the Airy stress function Φ, which satisfies the biharmonic equation [3, 4]

$$\nabla^2(\nabla^2\Phi) = 0 \tag{1}$$

and from which the stresses can be derived as follows

$$\sigma_{xx} = \frac{\partial^2\Phi}{\partial y^2}\ , \qquad \sigma_{yy} = \frac{\partial^2\Phi}{\partial x^2}\ , \qquad \sigma_{xy} = -\frac{\partial^2\Phi}{\partial x\partial y}\ . \tag{2}$$

For mode I Westergaard then gave the Airy stress function in terms of the analytic function $Z_I(z)$ of the complex variable $z = x+iy$ as follows:

$$\Phi = \mathrm{Re}\ \bar{\bar{Z}}_I + y\ \mathrm{Im}\ \bar{Z}_I\ , \tag{3}$$

where $d\bar{\bar{Z}}/dz = \bar{Z}$ and $d\bar{Z}/dz = Z$. For mode II he gave it in terms of another analytic function $Z_{II}(z)$ as follows:

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$$\Phi = - y \operatorname{Re} \overline{Z}_{II} \ . \tag{4}$$

Most of the classical results in fracture mechanics have been derived from these equations by suitable choices of Westergaard's analytic functions Z.

THE MODIFICATION

Muskhelishvili [4] has shown that it is possible and also convenient to write the Airy stress function as the real part of a complex but not necessarily analytic function, as follows:

$$\Phi = \operatorname{Re} \left\{ z^* \phi(z) + \chi(z) \right\} , \tag{5}$$

where the Goursat functions $\phi(z)$ and $\chi(z)$ are analytic. Now, if the function $\chi(z)$ includes a term in the analytic function z^2, then we see from (2) that this would add at most constant terms to the stresses. Hence, this provides a very simple way of making the correction proposed by Sih and by Eftis and Liebowitz.

If we write the Goursat functions in terms of the Westergaard functions as follows:

$$\phi = \frac{1}{2}\left(\overline{Z}_I - i \ \overline{Z}_{II}\right) , \tag{6a}$$

$$\chi = \overline{\overline{Z}}_I - \frac{1}{2}\left(\overline{Z}_I - i \ \overline{Z}_{II}\right) z - \frac{1}{2} (A + iB) z^2 , \tag{6b}$$

then the resulting Airy stress function is by (5)

$$\Phi = \operatorname{Re}\overline{\overline{Z}}_I + y\operatorname{Im}\overline{Z}_I - y\operatorname{Re}\overline{Z}_{II} - \frac{1}{2} A(x^2-y^2)+Bxy \ . \tag{7}$$

The first two terms in this expression are the same as (3), the third is (4), and the fifth and sixth are the correction terms. The stresses follow from (2)

$$\sigma_{xx} = \operatorname{Re}Z_I - y\operatorname{Im}Z_I' + 2\operatorname{Im}Z_{II} + y\operatorname{Re}Z_{II}' + A , \tag{8a}$$

$$\sigma_{yy} = \operatorname{Re}Z_I + y\operatorname{Im}Z_I' - y\operatorname{Re}Z_{II}' - A , \tag{8b}$$

$$\sigma_{xy} = -y\operatorname{Re}Z_I' + \operatorname{Re}Z_{II} - y\operatorname{Im}Z_{II}' - B \ . \tag{8c}$$

The displacements in plane-strain can be shown to be

$$2\mu u = (1-2\nu) \operatorname{Re}\overline{Z}_I - y\operatorname{Im}Z_I + Ax + 2(1-\nu) \operatorname{Im}\overline{Z}_{II} + y\operatorname{Re}Z_{II} - By , \tag{9a}$$

$$2\mu v = 2(1-\nu) \operatorname{Im}\overline{Z}_I - y\operatorname{Re}Z_I - Ay - (1-2\nu) \operatorname{Re}\overline{Z}_{II} - y\operatorname{Im}Z_{II} - Bx , \tag{9b}$$

where μ is the shear modulus and ν Poisson's ratio. Equations (8) and (9) are the basic results of Sih and Eftis and Liebowitz.

The rotation of the medium is given by

$$\omega = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) . \tag{10}$$

By (9) the rotation is then easily expressed in terms of the Westergaard functions as follows

$$\mu\omega = (1-\nu) \ (\operatorname{Im}Z_I - \operatorname{Re}Z_{II}) \ . \tag{11}$$

APPLICATIONS

(a) Constant Stress Without Cracks

We obtain case of a constant stress field in the medium by taking constant (complex) values for the Westergaard functions

$$Z_I = C + iD , \tag{12a}$$

$$Z_{II} = E + iH \ . \tag{12b}$$

The values of the constants C, D, E and H are determined by the boundary conditions. Then the stresses are from (8):

$$\sigma_{xx} = C + 2H + A , \tag{13a}$$

$$\sigma_{yy} = C - A , \tag{13b}$$

$$\sigma_{xy} = E - B \ . \tag{13c}$$

We see that there are now enough constants available to obtain any arbitrary set of stresses.

However, for mode I, $E = H = 0$, without the correction terms, $A = B = 0$, only hydrostatic tension is possible, $\sigma_{xx} = \sigma_{yy} = C$, $\sigma_{xy} = 0$. This is the nature of the restriction on Westergaard's original equations, as we mentioned in the Introduction. The correction term A makes it possible in mode I to have σ_{xx} differ from σ_{yy}. The simple case of uniaxial tension is then given by A = - C.

We note, however, that the correction terms are not strictly necessary for a completely arbitrary choice of stresses, if modes I and II are combined, for the term H in (13a), which comes from mode II in (12b), also allows us to choose σ_{xx} different from σ_{yy} in an arbitrary way.

The constant D in (12a) does not play a role in the stresses (13). It is related to the rotation of the medium, together with the constant E, as can easily be deduced by substituting (12) into (11):

$$\mu\omega = (1-\nu)\ (D-E)\ . \tag{14}$$

(b) Small Crack

This is the classical case of an infinite medium with a central crack of length 2a along the x-axis. The solution to this problem is given by the Westergaard functions

$$Z_I = \sigma(1-a^2/z^2)^{-1/2} + C + iD\ , \tag{15a}$$

$$Z_{II} = \tau(1-a^2/z^2)^{-1/2} + E + iH\ , \tag{15b}$$

where the values of the constants σ, τ, C, D, E and H are determined by the boundary conditions.

One set of boundary conditions is that the crack surface is stress free

$$\sigma_{yy} = \sigma_{xy} = 0 \qquad \text{for} \qquad y = 0,\ |x| < a\ . \tag{16}$$

From (15) and (8) this leads to the relations

$$C = A, \qquad E = B\ . \tag{17}$$

Another set of boundary conditions is given by the asymptotic behaviour of the stresses at remote distances from the crack. From (17), (15) and (8) we find at $|z| = \infty$

$$\sigma_{xx} = \sigma + 2(A+H)\ , \tag{18a}$$

$$\sigma_{yy} = \sigma\ , \tag{18b}$$

$$\sigma_{xy} = \tau\ . \tag{18c}$$

We see therefore that σ and τ in (15) represent the remote normal tensile and shear stress applied to the cracked medium. From (18a), we see that the remote tensile stress parallel to the crack can be arbitrarily adjusted to any desired value, either by fixing C (= A) in mode I or H in mode II. This does not affect the first terms in equations (15), which contain the essence of the crack field.

The constants E (= B) and D play no role in the stress field of the cracked medium. From (11) it can easily be shown that they are related to the rotation of the medium. It fact, for $|z| = \infty$ we find

$$\mu\omega = (1-\nu)\ [D-E-\tau]\ . \tag{19}$$

Our result that B does not appear in (18) contradicts a conclusion by Sih, who stated that B cannot vanish for a non-trivial solution. The reason for this is that Sih made the unnecessary assumption that E = - τ in his equation (14), which corresponds to our equation (15b).

CONCLUSION

We have shown that a correction, proposed by Sih and by Eftis and Liebowitz, to remove a restriction on Westergaard's equations, can be made in a very simple way by adding an elementary term to the Airy stress function.

The result is illustrated by two very simple examples. However, applications to more complex problems, such as those discussed by Eftis and Liebowitz, can of course be made in a similar straightforward manner.

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