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FINITE ELEMENT ANALYSIS OF CRACK PROBLEMS IN HIGHLY ELASTIC MATERIALS

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1. INTRODUCTION

It is well recognized [1,2] that the Rivlin-Thomas criterion [3] for the tearing of rubber vulcanizates can be stated in terms of the J integral [4]: A cut (or crack) in a rubber vulcanizate sheet will spread if J reaches a critical value of an energy characteristic of the material. Thus, using the above criterion and a plane stress finite element procedure, the critical load that causes a crack to grow in uniaxial stretching of nicked rubber vulcanizate sheets was calculated in [2] and found to agree closely with existing experimental data. In [5], the tearing energy of two rubber test pieces was computed by using the J integral in conjunction with finite element calculations. The calculated results were again shown to agree with experimental data. It was from the recognition of the equivalence of J with the tearing energy that an experimental technique for measuring the tearing energy of rubber was developed in [6].

The purpose of this paper is to extend the computational procedure used in [2] to calculate J for two other "plane" crack problems. These are the plane strain stretching of a thick strip with an edge cut by a stress σ_o (Figure 1a) and the generalized out-of-plane shear of the same geometry by a shear stress τ_o (Figure 1b). By generalized out-of-plane shear, we mean that while the dominating displacement component may be in the out-of-plane direction, the components in the plane need not be vanishingly small. This is therefore in variance with the so-called anti-plane shear for which the displacement components in the plane are identically zero.

The organization of the paper is as follows: In Section 2, we list the basic equations for a class of two-dimensional finite deformations which contain the plane strain and anti-plane shear as special cases. The materials are assumed to be highly elastic and incompressible. Based on a virtual velocity equation, a finite element procedure is developed. Since formulations of finite element procedures for nonlinear elastic materials have been well documented in the literature [7], only a brief discussion of the present procedure is given. In Section 3, an expression for the J integral pertinent to the problems considered in this paper is derived, while its path independent property is illustrated by direct evaluation. It should be noted that this result is essentially contained in [1,8,9]. In Section 4, we report numerical results obtained for the two crack problems for Mooney materials. The values of J have been computed and plotted in several figures which depict the relationship between J and the applied load (σ_o or τ_o). For the plane strain stretching problem, a comparison is made of the computed J values with those of the corresponding plane stress case. It is found that the J values in the plane strain case normalized by the strain energy density corresponding of σ_o (without

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MS304

a crack) are almost the same as those in the plane stress case. For the generalized out-of-plane shear problem, the J values are compared to the corresponding anti-plane shear solutions.

2. BASIC EQUATIONS FOR COMBINED EXTENSION AND SHEAR DEFORMATIONS

Consider in a cartesian coordinate system (x^i, i = 1,2,3) a body whose cross section at any x^3 occupies a plane domain D which is the same for all x^3 (Figure 2). The class of deformation to be considered is such that the displacement components u_i (i = 1,2,3) satisfy

$$u_i = u_i(x^1,x^2) \ . \tag{2.1}$$

We call (2.1) the combined extension and shear deformations which contain obviously the plane strain ($u_3 = 0$) and the anti-plane shear ($u_1 = u_2 = 0$) as special cases. For deformations characterized by (2.1) the deformed metric tensors G_{ij} and the strain tensors ε_{ij} are defined as follows:

$$G_{ij} = \delta_{ij} + 2\,\varepsilon_{ij} \ , \tag{2.2}$$

and

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}u^k_{,\alpha}\,u_{k,\beta} \ , \tag{2.3}$$

$$\varepsilon_{\alpha 3} = \frac{1}{2}u_{3,\alpha} \ , \quad \varepsilon_{33} = 0 \ ,$$

where δ_{ij} denotes the Kronecker delta and a comma preceding an index denotes partial differentiation. Here we have used the convention that Greek indices range from 1 to 2 and Roman indices, from 1 to 3. The contravariant metric tensors G^{ij} are defined by the usual relations:

$$G^{ij}\,G_{jk} = \delta^i_k \ . \tag{2.4}$$

Let S^{ij} be the Kirchhoff stress tensors on convected coordinates initially coincide with the cartesian coordinates (x^i) and F^i be the surface loadings. Then, the well known virtual velocity equation (e.g., [10]) in its time-derivative form may be written as

$$\int \dot{S}^{ij}\,\delta\dot{\varepsilon}_{ij}\,dV + \int S^{ij}\delta\left[\frac{1}{2}\dot{u}^k_{,i}\,\dot{u}_{k,j}\right]dV = \int \dot{F}^i\delta\,\dot{u}_i\,dS \ , \tag{2.5}$$

where the dot over a symbol denotes the rate of change versus "time", or increment. The volume and surface integrals are referred to the undeformed configuration. (For simplicity body forces were omitted in (2.5)). We assume that the material is incompressible and, hence, the variations of the displacements in (2.5) must be required to satisfy

$$G^{ij}\,\delta\dot{\varepsilon}_{ij} = 0 \ . \tag{2.6}$$

Expressions (2.5) and (2.6) are valid for arbitrary domains. For the configuration shown in Figure 2 and for deformations characterized by (2.1), these expressions can be written as

$$\int_D \left[\dot{S}^{\alpha\beta}\delta\,\dot{\varepsilon}_{\alpha\beta} + 2\dot{S}^{\alpha 3}\delta\,\dot{\varepsilon}_{\alpha 3}\right]dA + \int_D S^{\alpha\beta}\delta\left[\frac{1}{2}\dot{u}^k_{,\alpha}\,\dot{u}_{k,\beta}\right]dA = \int_{\partial D}\dot{F}^i\delta\dot{u}_i\,ds \tag{2.7}$$

$$G^{\alpha\beta}\,\delta\,\dot{\varepsilon}_{\alpha\beta} + 2G^{\alpha 3}\delta\,\dot{\varepsilon}_{\alpha 3} = 0 \ , \tag{2.8}$$

where ∂D is the boundary of the plane domain D.

Finally, we assume that the material of the body is elastic and that there exists a strain energy function W such that the Kirchhoff stresses S^{ij} satisfy

$$S^{ij} = \frac{1}{2}\left[\frac{\partial W}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \varepsilon_{ji}}\right] + pG^{ij} \ , \tag{2.9}$$

where p is a scalar quantity representing the hydrostatic pressure. The stress increments $\dot{S}^{ij}$ can be derived from (2.9) by simply differentiating with respect to "time" to give

$$\dot{S}^{ij} = \frac{1}{2}\frac{\partial}{\partial \varepsilon_{k\ell}}\left[\frac{\partial W}{\partial \varepsilon_{ij}} + \frac{\partial W}{\partial \varepsilon_{ji}}\right]\dot{\varepsilon}_{k\ell} + \dot{p}\,G^{ij} \tag{2.10}$$

$$- 2\,p\,G^{ik}\,G^{j\ell}\,\dot{\varepsilon}_{k\ell} \ .$$

A finite element procedure

Based on the virtual velocity equation (2.7), the incompressibility condition (2.8) and the stress-strain relations (2.9) and (2.10), a finite element procedure has been developed. The details of the procedure appeared elsewhere [11]. Since the general formulations of finite element procedures for large strain and large displacement are well established (e.g., [7,12]), only a brief outline of the present procedure will be given in Appendix A. In the present procedure, the elements are quadrilaterals in the plane domain D. The incompressibility condition (2.8) is satisfied only in an approximate way, namely the sum of the integrand in (2.8) over all integration points is zero. This approximation appears to have alleviated the difficulties encountered when triangular elements were used [13, 14]. Accordingly, elemental hydrostatic pressure and stresses are likewise represented by their average values. By doing so, the quadrilateral element is essentially a constant stress element.

The incremental element-stiffness matrix equations assume the following form:

$$\begin{bmatrix} K_e & C^T \\ C & 0 \end{bmatrix} \begin{Bmatrix} \dot{\underset{\sim}{u}} \\ \dot{p} \end{Bmatrix} = \begin{Bmatrix} \dot{\underset{\sim}{f}} \\ 0 \end{Bmatrix} \ , \tag{2.11}$$

where the matrices K_e, C and the vectors $\dot{\underset{\sim}{u}}$, $\dot{\underset{\sim}{p}}$ and $\dot{\underset{\sim}{f}}$ are defined in Appendix A. We note that the elemental stiffness matrix equations (2.11) are derived for the combined extension and shear deformations (2.1) for which $u_i \neq 0$, (i = 1,2,3). For plane strain deformations ($u_3 = 0$) and for anti-plane shear deformations ($u_1 = u_2 = 0$), the dimensions of the matrix equations (2.11) will accordingly be reduced. In fact, for anti-plane shear deformations, the incompressibility condition (2.8) satisfies automatically so that the matrix C vanishes identically.

For specific problems, the elemental stiffness matrix as given by (2.11) is assembled to form a master stiffness matrix. The area integrals in (A.11) and (A.12) may be computed by using any appropriate numerical quadrature. In the present work we have used a four-point Gaussian quadrature with the four integration points located at s = ± 0.57735 and t = ± 0.57735 in the para-metric s-t domain. The master stiffness matrix equations are then solved by a standard Gaussian elimination technique. For a prescribed load increment, the solution of the equations gives the corresponding incremental displacements and elemental pressure. The master stiffness matrix has the same form as in (2.11), which is apparently not banded. In order to reduce computing time, a reordering of the components of the vectors $\dot{u}$ and $\dot{p}$ has been made so that the resulting master stiffness matrix is banded.

3. THE PATH-INDEPENDENT J INTEGRAL

We now proceed to derive an appropriate expression for the J integral for the combined extension and shear deformations as described in (2.1). We assume that the elastic body has a crack, the crack face being perpendicular to the x^2 direction (see Figure 3). The J integral was defined originally in [4] by

$$J = \int_\Gamma \left[W dx^2 - \underset{\sim}{T} \cdot \frac{\partial \underset{\sim}{u}}{\partial x^1} ds \right] , \tag{3.1}$$

where W denotes the strain energy density, $\underset{\sim}{T}$ and $\underset{\sim}{u}$ denote traction and displacement vectors, respectively. The integral assumes the same value for any path Γ which surrounds the tip of the crack. For deformations described by (2.1), the J integral may be written in the following form:

$$J = \int_\Gamma \left[W dx^2 - S^{\alpha k} (\delta^j_k + u^j_{,k}) \, \nu_\alpha \frac{\partial u_j}{\partial x^1} ds \right] , \tag{3.2}$$

where ν_α denotes the exterior normal of the contour Γ defined in the undeformed geometry. The path-independent property of (3.2) is shown in Appendix B.

4. TWO CRACK PROBLEMS

In this section, we report the numerical results for the two crack problems shown in Figure 1. We assume that the materials are of the Monney kind and the strain energy function can be expressed by

$$W = C_1 \, [(I_1 - 3) + \alpha(I_2 - 3)) \tag{4.1}$$

where C_1, α are material constants and I_1, I_2 are strain invariants defined by

$$I_1 = G_{11} + G_{22} + G_{33} ,$$

$$I_2 = G^{11} + G^{22} + G^{33} . \tag{4.2}$$

The finite element procedure discussed in Section 2 is employed to calculate the stresses and deformation in the strip. Expression (3.2) is used to calculate the J integral.

The finite element grid which is used in both crack problems is shown in Figure 4, where by symmetry only half of the region needs to be analyzed.

The crack geometry is specified by c/b = 0.1. To test the accuracy of the grid, a linear elastic plane strain calculation is first made for a uniform tension in the x^2 direction. The J integral is evaluated along the contour Γ which consists of line segments (shown dotted in Figure 4) that join the mid points of two prescribed sides of an element. Using appropriate values of Young's modulus E and Poisson's ratio ν, the stress intensity factor K is determined from the computed J value and the relation

$$K = [EJ/(1 - \nu^2)]^{1/2} ,$$

and found to be $2.05 \, \sigma_o\sqrt{c}$. This compares well with the known numerical value of $2.15 \, \sigma_o\sqrt{c}$ reported in [15].

We now proceed to discuss our numerical results for the plane-strain stretching problem and for the generalized out-of-plane shear problem.

(a) Plain-strain stretching: For α = 0.0, 0.5 and 1.0, incremental calculations for the stresses and deformation in the strip as functions of the applied load σ_o have been made. As in the case of plane-stress uniaxial stretching stretching [2], it is found that an increment of 0.2 for σ_o/C_1 is satisfactory. Figure 5 shows the computed J values plotted against the nominal extension ratio λ. For comparison purposes, the corresponding plane stress results from [2] have also been plotted in Figure 5. We see that for the same amount of stretching, the J values in plane strain are larger than those in plane stress.

When the J values in Figure 5 are normalized by the quantity $2W_oc$, all curves coincide into one for λ > 1.1 (see Figure 6). Here W_o is the elastic energy density corresponding to σ_o (without the crack) given by

$$W_o = \begin{cases} C_1 \, [(\lambda^2 + \frac{2}{\lambda} - 3) + \alpha(\frac{1}{\lambda^2} + 2\lambda - 3)], & \text{for plane stress} \\ C_1 \, (1 + \alpha) \, (\lambda^2 + \frac{1}{\lambda^2} - 2) , & \text{for plane strain.} \end{cases} \tag{4.3}$$

That the quantity $(J/2W_oc)$, or equivalently (tearing energy/$2W_oc$), is dependent mainly on stretch ratio λ was postulated by Rivlin and Thomas [3] for the stretching of rubber sheets with an edge cut (plane stress). This was verified experimentally by Greensmith [16] who showed that it holds true for c/b < 0.2. Thus for plane stress, results shown in Figure 6 are but a confirmation of results in [3,16]. However, that the quantity $(J/2W_oc)$ in plane strain should also be independent of the material constant α and that it coincides with values in plane stress for λ > 1.1 are new results. Although results shown in Figures 5 and 6 are obtained for a specific crack geometry with ratio c/b = 0.1, they are expected to be valid also for nearby ratios, e.g., 0 < c/b < 0.2, based on similar arguments advanced in [3].

(b) Generalized out-of-plane shear: For α = 0.0, 0.5 and 1.0, the computed J values are plotted against τ_o/C_1 in Figure 7. The dotted curve represents the results for the corresponding anti-plane shear problem ($u_1 = u_2 = 0$). For the latter problem, the governing differential equations are linear (Green and Adkins [10], p. 86) and, hence, the J integral can be explicitly expressed as

$$J = \frac{\pi c \, \tau_o^2}{4C_1 \, (1+\alpha)} \left[\frac{2b}{\pi c} \tan\left(\frac{\pi c}{2b}\right) \right] , \tag{4.4}$$

by using the elastic results in [17].

For $\alpha = 0$, the calculated displacements u_1 and u_2 are found to be identically zero, which means that the assumption underlying the anti-plane shear deformation has precisely been met. Moreover, the calculated pressure is found to remain constant for all stages of incremental loading. This is in complete agreement with the analytical relation

$$[p + 2(1+2\alpha)]/C_1 = -2\alpha\left[\left(\frac{\partial u_3}{\partial x^1}\right)^2 + \left(\frac{\partial u_3}{\partial x^2}\right)^2\right], \qquad (4.5)$$

derived in [10] for anti-plane shear deformations. For $\alpha \neq 0$, u_1 and u_2 do not vanish in general. The anti-plane shear assumption is therefore no longer valid as indicated by the J results in Figure 7. To elucidate this further, we have computed the quantity $[p + 2(1+2\alpha)]/C_1$ for $\alpha = 1$ using the following procedures:

(i) the present finite element procedure without assuming $u_1 = u_2 = 0$;

(ii) the same finite element procedure assuming $u_1 = u_2 = 0$; and

(iii) the analytical solution corresponding to anti-plane shear (using $c/b = \sim 0$ for simplicity).

These results have been plotted in Figure 8 for $\tau_o/C_1 = 4.2$. It can be seen that the results of (ii) agree well with those of (iii) as expected, but are significantly different from those of (i).

Finally, we compute the quantity $J/2W_oc$ where W_o denotes the strain energy density in the strip (without the crack) caused by the shear τ_o at $x^2 = \pm b$. Let λ be the extension ratio of an initial length in the x^2 direction, then the strain energy density W_o can be written as

$$W_o = C_1 (1+\alpha) (\lambda^2 - 1) . \qquad (4.6)$$

The computed values of $(J/2W_oc)$ for $\alpha = 0$, 0.5 and 1.0 are plotted in Figure 9. The dotted line corresponds to the anti-plane shear result which assumes a constant value of 1.58. From this figure it is seen that the dependence of the quantity $J/2W_oc$ on both the material parameter α and the extension ratio λ is rather small. Hence, for practical purposes, it will be sufficient to use the exact anti-plane shear solution (4.4) or, equivalently,

$$\left(\frac{J}{2W_oc}\right) = \frac{b}{c} \tan\left(\frac{\pi c}{2b}\right) \qquad (4.7)$$

to calculate J.

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APPENDIX A

A finite element procedure

Let the domain D shown in Figure 2 be divided into an assemblage of quadrilateral elements. For each element the initial cartesian coordinates (x^1, x^2) and the displacements $\dot{u}$ are mapped bilinearly onto a square s-t domain for $s\varepsilon[-1,1]$ and $t\varepsilon[-1,1]$ by

$$\begin{bmatrix} x^\alpha \\ \dot{u}_i \end{bmatrix} = \sum_n \begin{bmatrix} x^{\alpha(n)} \\ \dot{u}_i^{(n)} \end{bmatrix} q_n (s,t), \qquad (A.1)$$

where

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = (1\mp s)(1-t)/4,$$

$$\begin{pmatrix} q_3 \\ q_4 \end{pmatrix} = (1\pm s)(1+t)/4, \qquad (A.2)$$

and the index n (n = 1,2,3,4) refers to the node number of the four nodes of the quadrilateral.

Denoting

$$\dot{\underset{\sim}{u}}^T = (\dot{u}_1^{(1)}, \dot{u}_2^{(1)}, \dot{u}_3^{(1)}, \dot{u}_1^{(2)}, \ldots, \dot{u}_3^{(4)}) ,$$

$$\dot{\underset{\sim}{E}}^T = (\dot{E}_{11}, \dot{E}_{22}, 2\dot{E}_{12}, 2\dot{E}_{13}, 2\dot{E}_{23}) ,$$

and making use of the strain-displacement relations (2.3), we obtain a matrix representation for the incremental strain-displacement relations,

$$\dot{\underset{\sim}{E}} = (H + \psi B)\,\dot{\underset{\sim}{u}} \ . \tag{A.3}$$

The matrices H, ψ and B in (A.3) are defined as follows:

$$B = \begin{bmatrix} b_1 & 0 & 0 & b_2 & 0 & 0 & b_5 & 0 & 0 & b_6 & 0 & 0 \\ b_3 & 0 & 0 & b_4 & 0 & 0 & b_7 & 0 & 0 & b_8 & 0 & 0 \\ 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_5 & 0 & 0 & b_6 & 0 \\ 0 & b_3 & 0 & 0 & b_4 & 0 & 0 & b_7 & 0 & 0 & b_8 & 0 \\ 0 & 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & b_5 & 0 & 0 & b_6 \\ 0 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 & b_7 & 0 & 0 & b_8 \end{bmatrix} , \tag{A.4}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} , \tag{A.5}$$

and

$$\psi = \begin{bmatrix} \frac{\partial u}{\partial x} & 0 & \frac{\partial v}{\partial x} & 0 & \frac{\partial w}{\partial x} & 0 \\ 0 & \frac{\partial u}{\partial y} & 0 & \frac{\partial v}{\partial y} & 0 & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial y} & \frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial x} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{A.6}$$

Here the coefficients b_i are given by

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = [\mp s_x(1-t) - t_x(1\mp s)]/4,$$

$$\begin{pmatrix} b_3 \\ b_4 \end{pmatrix} = [\mp s_y(1-t) - t_y(1\mp s)]/4,$$

$$\begin{pmatrix} b_5 \\ b_6 \end{pmatrix} = [\pm s_x(1+t) + t_x(1\pm s)]/4,$$

$$\begin{pmatrix} b_7 \\ b_8 \end{pmatrix} = [\pm s_y(1+t) + t_y(1\pm s)]/4,$$

with

$$s_x = \frac{\partial y}{\partial t}/J, \quad s_y = -\frac{\partial x}{\partial t}/J, \quad t_x = -\frac{\partial y}{\partial s}/J, \quad t_y = \frac{\partial x}{\partial s}/J$$

and

$$J = \frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} \ .$$

In the above expressions, notation (x,y) stands for (x^1, x^2). Similarly, notation (u,v,w) stands for (u_1, u_2, u_3).

Making use of the variational equation (2.7) and letting

$$\underset{\sim}{G}^T = (G^{11}, G^{22}, G^{12}, G^{13}, G^{23}),$$

$$\underset{\sim}{S}^T = (S^{11}, S^{22}, S^{12}, S^{13}, S^{23}),$$

$$\Psi = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \gamma & 0 \\ 0 & 0 & \gamma \end{bmatrix}$$

where

$$\gamma = \begin{bmatrix} S^{11} & S^{12} \\ S^{12} & S^{22} \end{bmatrix} ,$$

we obtain the following elemental equilibrium equations:

$$\int (H + \psi B)^T \dot{\underset{\sim}{S}}\, dA + \int B^T \Psi B \dot{\underset{\sim}{u}}\, dA = \dot{\underset{\sim}{f}} , \tag{A.7}$$

where the integral is defined over the initial elemental area and $\dot{\underset{\sim}{f}}$ denotes the vector of incremental nodal forces corresponding to $\dot{\underset{\sim}{u}}$. The displacements $\underset{\sim}{u}$ must also satisfy the incompressibility condition (2.8) which is equivalent to

$$\int G^T (H + \psi B)\, \dot{\underset{\sim}{u}}\, dA = 0 \tag{A.8}$$

by assuming a uniform p inside the element.

We now write the incremental stress-strain relations (2.10) in the matrix form

$$\dot{\underset{\sim}{S}} = Q\,\dot{\underset{\sim}{E}} + \dot{p}\,\underset{\sim}{G} , \tag{A.9}$$

where the dimension of the matrix Q is 5 x 5. Substituting (A.9) into

(A.7) and combining the resulting equations with (A.8) gives the elemental stiffness equations:

$$\begin{bmatrix} K_e & C^T \\ C & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{u}} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \dot{\tilde{f}} \\ 0 \end{bmatrix} , \tag{A.10}$$

where

$$K_e = \int [(H + \psi B)^T \, Q \, (H + \psi B) + B^T \, \Psi \, B] dA \quad , \tag{A.11}$$

$$C = \int G^T \, (H + \psi B) dA \quad . \tag{A.12}$$

Matrix Q for Mooney Materials

For Mooney materials (4.1) and for deformations characterized by (2.1) the stress-strain relations (2.9) are

$$\begin{aligned} S^{11} &= C_1[2 + 4\alpha(1+E_{22}) + p\, G^{11}] \; , \\ S^{22} &= C_1[2 + 4\alpha(1+E_{11}) + p\, G^{22}] \; , \\ S^{12} &= C_1[-4\alpha\, E_{12} + p\, G^{12}] \; , \\ S^{13} &= C_1[-4\alpha\, E_{13} + p\, G^{13}] \; , \\ S^{23} &= C_1[-4\alpha\, E_{23} + p\, G^{23}] \; . \end{aligned} \tag{A.13}$$

Using (2.10), the matrix Q in (A.9) is then given by

$$Q = C_1\alpha \begin{bmatrix} 0 & 4 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & -2 & 0 & 0 \\ & & & -2 & 0 \\ \text{symmetrical} & & & & -2 \end{bmatrix}$$

$$-C_1 p \begin{bmatrix} 2(G^{11})^2 & 2(G^{12})^2 & 2G^{11}G^{12} & 2G^{11}G^{13} & 2G^{12}G^{13} \\ & 2(G^{22})^2 & 2G^{12}G^{22} & 2G^{12}G^{23} & 2G^{22}G^{23} \\ & & (G^{11}G^{22}+G^{12}G^{12}) & (G^{11}G^{23}+G^{12}G^{13}) & (G^{12}G^{23}+G^{13}G^{22}) \\ & & & (G^{11}G^{23}+G^{12}G^{13}) & (G^{12}G^{33}+G^{13}G^{23}) \\ \text{symmetrical} & & & & (G^{22}G^{33}+G^{23}G^{23}) \end{bmatrix} . \tag{A.14}$$

APPENDIX B

To show that (3.2) is path independent, we begin with the volume integral (over the undeformed geometry)

$$\iiint_V \frac{\partial W}{\partial x^1} \, dx^1 dx^2 dx^3 = \iiint_V \frac{1}{2} \left[\frac{\partial W}{\partial \varepsilon_{ik}} + \frac{\partial W}{\partial \varepsilon_{ki}} \right] \frac{\partial \varepsilon_{ik}}{\partial x^1} \, dx^1 dx^2 dx^3 \tag{B.1}$$

where V is an arbitrary volume in the undeformed state.

Adding $\iiint p\, G^{ik} \dfrac{\partial \varepsilon_{ik}}{\partial x^1} \, dx^1 dx^2 dx^3$

which is identically zero for incompressible materials, to the right hand side of (B.1) and making use of (2.3) and (2.9) results in

$$\begin{aligned} \text{R.H.S. of (B.1)} &= \iiint_V \left(S^{ik} (\delta^j_k + u^j_{,k}) \frac{\partial u_j}{\partial x^1} \right)_{,i} dx^1 dx^2 dx^3 \\ &= \iiint_V \frac{\partial}{\partial x^i} \left[S^{ik} (\delta^j_k + u^j_{,k}) \frac{\partial u_j}{\partial x} \right] dx^1 dx^2 dx^3 \\ &\quad - \iiint_V \left[S^{ik} (\delta^j_k + u^j_{,k}) \right]_{,i} \frac{\partial u_j}{\partial x^1} \, dx^1 dx^2 dx^3 \; . \end{aligned}$$

The last term in the above expression vanishes because

$$[S^{ik} \, (\delta^j_k + u^j_{,k})]_{,i} = 0$$

are the equilibrium equations for finite deformation. Applying the divergence theorem to the remaining term and equating to the left hand side of (B.1) gives

$$\iiint_V \frac{\partial W}{\partial x^1} \, dx^1 dx^2 dx^3 = \iint_{\partial V} S^{ik} (\delta^j_k + u^j_{,k}) \, \nu_i \, \frac{\partial u_j}{\partial x^1} \, dx^1 dx^2 dx^3 \; , \tag{B.2}$$

where ∂V denotes the surface which encloses V. The path independent property of (3.2) follows immediately by integrating (B.2) over a volume which is of unit length in the x^3 direction.

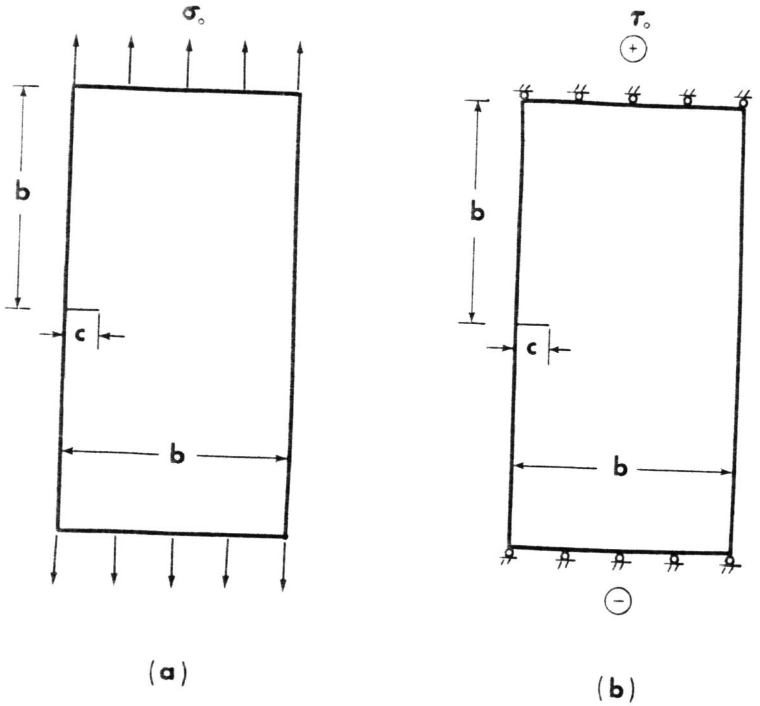

Figure 1 (a) Plane strain stretching and (b) generalized out-of-plane shear of a thick strip with a crack.

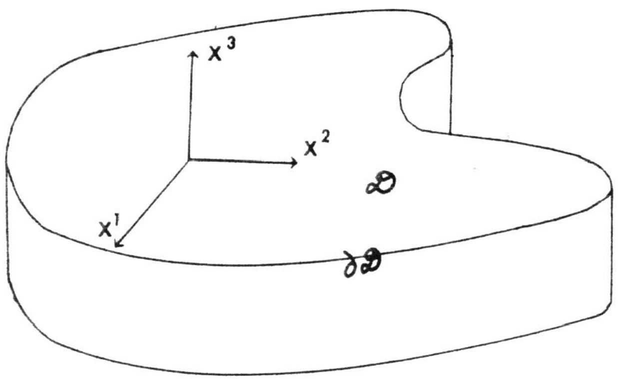

Figure 2 Domain and coordinate system.

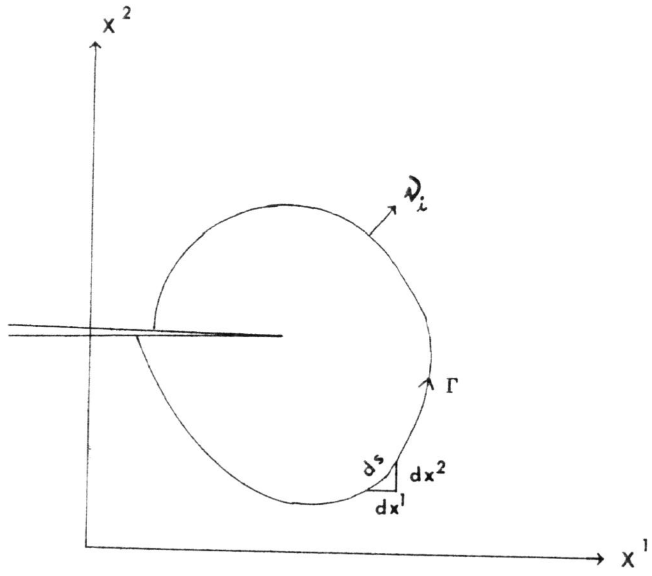

Figure 3 Notation for defining the J integral

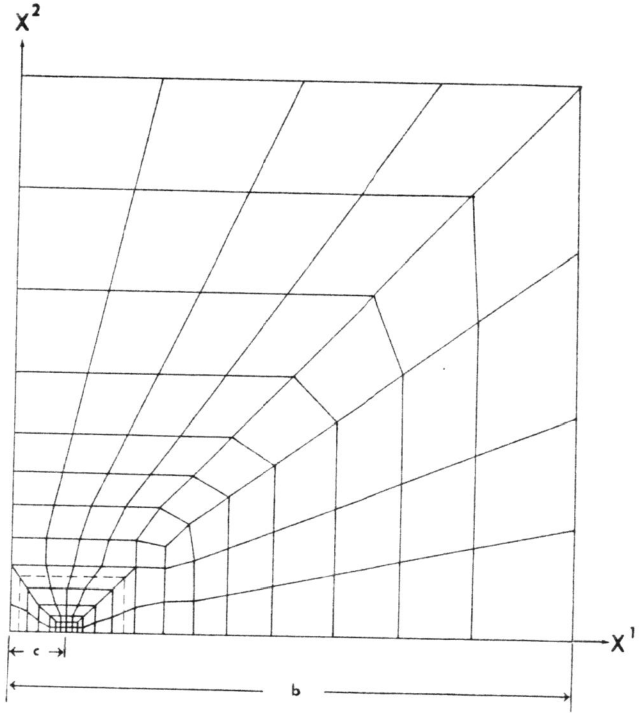

Figure 4 Finite element grid for half of strip for c/b = 0.1. Dotted contour indicates the path for calculating the J integral.

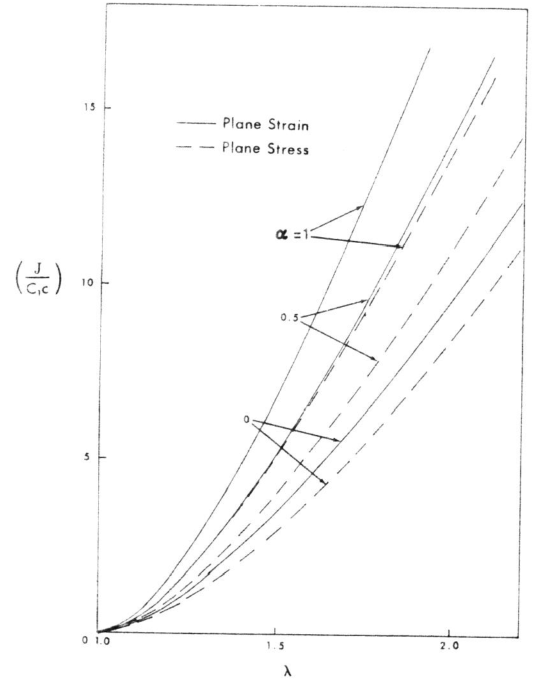

Figure 5 Calculated J values vs. nominal extension ratio λ for plane strain and plane stress stretching.

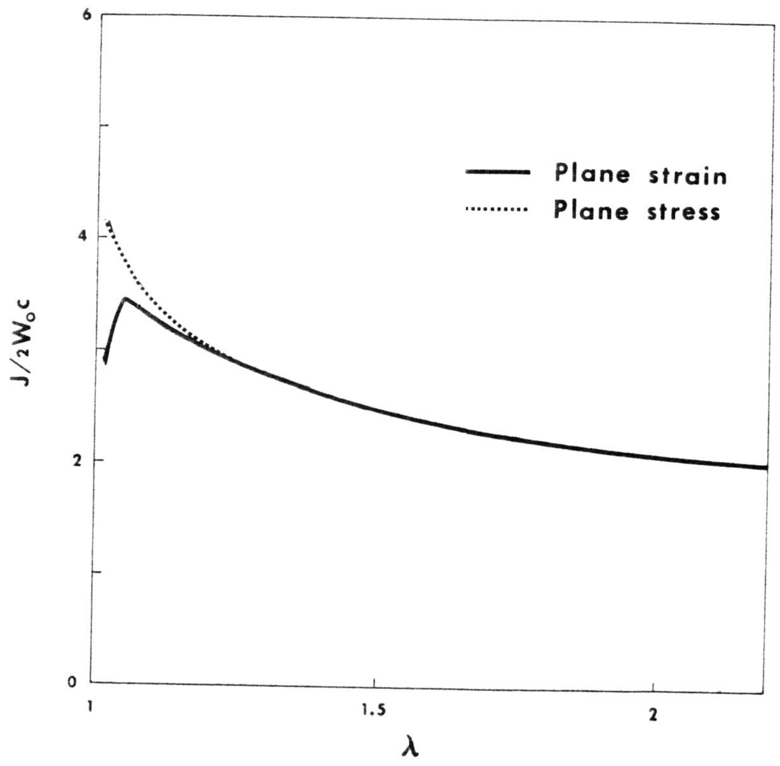

Figure 6 Calculated $J/2W_oc$ values vs. nominal extension ratio λ for plane strain and plane stress stretching.

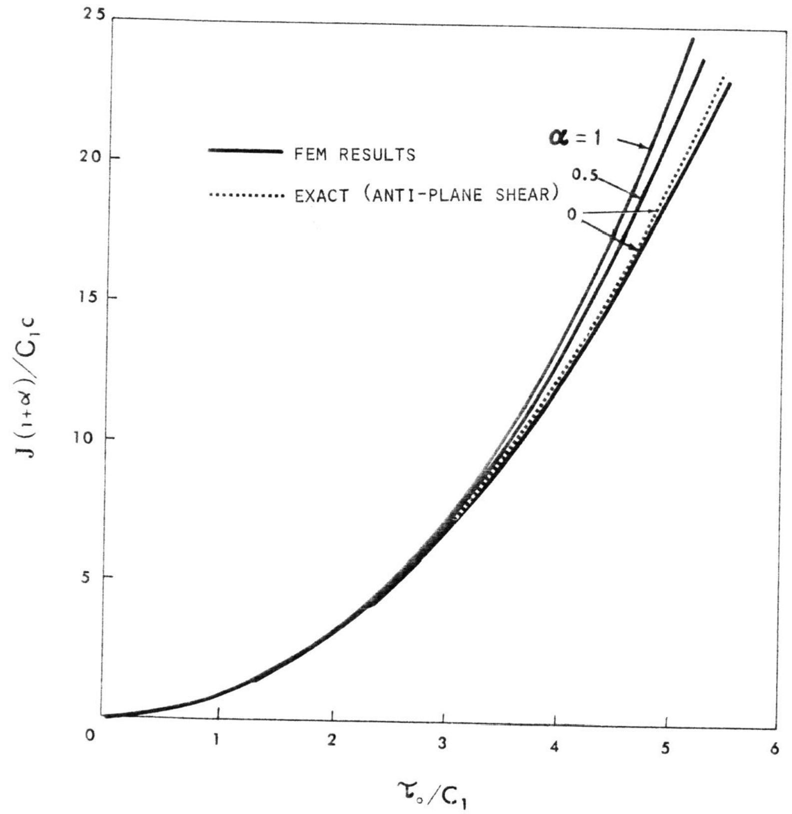

Figure 7 Calculated J values vs. shear stress τ_o for generalized out-of-plane shear.

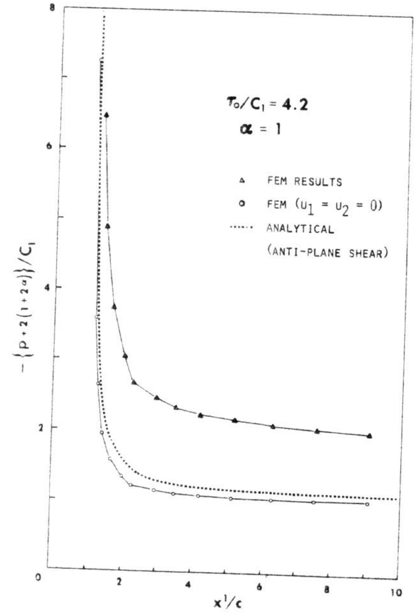

Figure 8 Distributions of normal stresses in the x^3 direction along $x^2 = 0$ calculated by different procedures.

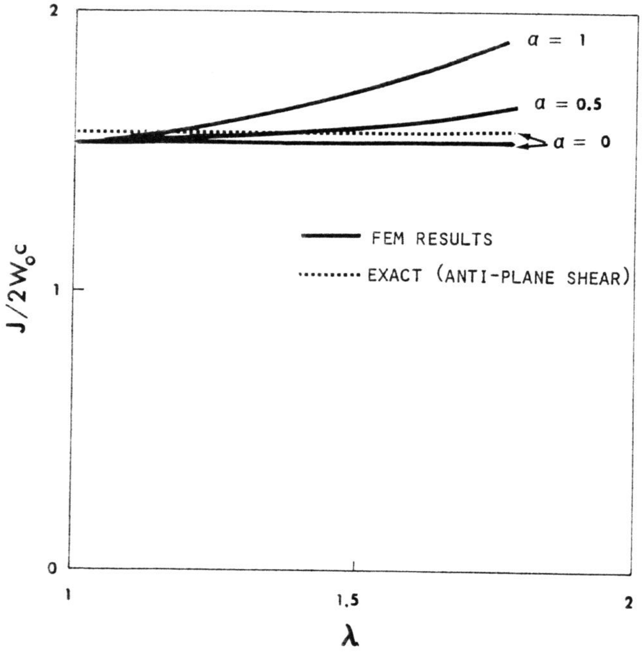

Figure 9 Calculated $J/2W_0c$ vs. nominal extension ratio λ for generalized out-of-plane shear.