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FAILURE PREDICTION OF FINITE FLAWED CERAMIC PLATES UNDER COMBINED STRESSES

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INTRODUCTION

The motivation for this work lies in the necessity to choose appropriate probability based failure theories for brittle materials. Ceramics such as silicon nitride and silicon carbide are currently being used in a number of high stress applications [1]. Since these are brittle materials the usual design procedure for ductile behaviour is not applicable. Therefore it is pertinent to explore methods to determine appropriate failure theories.

Ordinarily the Weibull representation [2] is taken to approximate the strength probability of failure, e.g.

$$P_f = 1 - \exp\left[- K \int_v \left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m dv\right] \qquad (1)$$

where K is related to loading, v is the volume of material. σ and σ_u are fracture and threshold stresses respectively. σ_u is defined as zero for the particular brittle materials considered in this paper. σ_o and m are distribution constants determined from test data. Simple mechanical properties tests are usually conducted to estimate the required data. However, the extensive data required to accurately establish a statistical basis for the analytical model is prohibitive. Furthermore preparation of specimens for combined stress experiments is extremely costly. Therefore relatively little work has been accomplished regarding probability based failure theories for complex stress states. Thus the objective of this analysis was to investigate the feasibility of using plates containing circular holes and subjected to unidirectional and biaxial loadings to establish volume versus strength dependencey and also to discriminate between potential combined stress failure theories. Results of analysis under combinations of tension and compression loadings are reported for a particular hot pressed silicon nitride. Failure distribution is found to be sensitive to geometry and loading conditions and such plate tests offer promise of aid in selecting alternate probability models [3, 4, 5].

STRESS ANALYSIS

The stress analysis reported herein is based on the modified mapping collocation technique [6], which combines the advantages of boundary collocation with that of complex variable methods of Muskhelishvili [7]. A simple form of a mapping function carrying a circle and its exterior in the parameter plane into a circle and its exterior, respectively is used.

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The remaining portion of the boundary in the physical plane corresponds to a directly calculable curve in the auxiliary plane. The continuation arguments of Muskhelishvili are employed to describe stress functions with, for example, traction-free conditions on the circle. Collocation methods can then be introduced to satisfy the conditions on the remaining portions of the boundary. This method is particularly useful for problems involving multi-connected regions where determination of polynomial representation of mapping functions is difficult.

The above analytic approach avoids the inaccuracy and time consuming computations as contrasted to the commonly used, Finite Element Method.

The stresses are expressed in terms of a single-stress function, U(x,y) which satisfies the biharmonic equation,

$$\nabla^4 U = 0 \tag{2}$$

The Airy stress function U(x,y) is defined [7] in terms of two analytic functions of the complex variable z, namely $\phi(z)$ and $\psi(z)$, where

$$U(x,y) = \mathrm{Re}\left[\bar{z}\phi(z) + \int^{z} \psi(z)dz\right] \tag{3}$$

using this representation, the stress and displacement components in rectangular coordinates can be written as [7].

$$\sigma_x + \sigma_y = 2[\phi'(z) + \overline{\phi'(z)}]$$

$$\sigma_y - \sigma_x + 2i\tau_{xy} = 2[\bar{z}\phi''(z) + \psi'(z)] \tag{4}$$

$$2\mu(u+iv) = \chi\phi(z) - z\overline{\phi'(z)} - \overline{\psi(z)}$$

where $\chi = \frac{3-\nu}{1+\nu}$ (Plane Stress) and $\mu = \frac{E}{2(1+\nu)}$

The elastic constants are Poisson's ratio ν and Young's modulus E. If $x_n d_s$ and $y_n d_s$ denote the horizontal and vertical components respectively, of the force acting on an element of arc, ds, then

$$\phi(z) + z\phi'(z) + \psi(z) = i\int (x_n + iy_n)ds \tag{5}$$

Use of the extension concept of Muskhelishvili [3] provides traction free conditions on the surface AB in Figure 1 if $\psi(z)$ is written in the following manner [7]:

$$\phi(z) = -z\overline{\phi'}\left(\frac{1}{z}\right) - \overline{\psi}\left(\frac{1}{z}\right) \qquad z\varepsilon S^- \tag{6}$$

S^- denotes exterior of circle in z planes. This extension definition of ψ provides single stress function $\phi(z)$ representation in addition to traction free conditions. The function $\phi(z)$ can be represented by a Laurent Series of the form

$$\phi(z) = T\sum_{-\infty}^{\infty} \alpha_n \, z^{2n+1} \tag{7}$$

where T is average applied stress on CD in Figure 1 and α_n's are real. Since the traction free surface on AB is satisfied then it is only necessary to consider the outer boundary conditions on CD and DE in Figure 1. These conditions are satisfied by choosing a finite number of boundary stations on the boundaries and prescribing the proper stress and force conditions at these points. The α_n's for the truncated series equation (7) are determined from satisfying the above conditions. The coefficients are determined in least squares sense in that the number of conditions exceeds the degrees of freedom by ratio of two to one.

PROBABILITY THEORY

The probability of failure P_f determinations are made for the ceramic plate with hole using the Weibull model equation (1). P_f is defined as 1 - P_s, where P_s is probability of survival. The model was selected for this initial evaluation of flawed plate testing.

The advantages of the method include, first, the ability to represent variability of mechanical properties by considering distribution flaws as functions of volume, and secondly, P_f for large complex structures can be determined from knowledge of material properties of small simple test specimens. This is an important consideration for ceramic materials which exhibit large variations in strength values. The disadvantage involves using the weakest link hypothesis where failure occurs at weakest point in structures. For certain brittle material this is not a problem as verified by a number of test results. Weibull statistic should be applied where appropriate for the particular material characteristics. The Weibull distribution function for the case of probability of fracture is

$$P_f = 1 - \exp\left[-Kv\left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^m\right] \tag{8}$$

for the small volume of an individual element in which the stress can be assumed constant and the other parameters are defined in equation (1). It should be mentioned that application of the Weibull equation (8) to brittle fracture is based strictly on empirical considerations. The Weibull parameters m and σ_o are determined from the solution of a transcendental equation developed from knowledge of the first three statistical moments of the available strength data. The solution involves applying an iterative scheme using a "digit-place" algorithm. This algorithm eliminates the necessity of drawing graphs and applying interpolation methods, while providing any desired accuracy.

The Weibull distribution is applied to the problem of a ceramic rectangular plate with hole as defined in Figure 2. The stress distribution within the structure is determined for the plane stress conditions using the method described previously. Instead of evaluating individual probability values with finite volume elements of the structure and multiplying these probabilities to obtain P_s for a particular applied load T, the procedure used in this paper involves a numerical integration. This approach provides a more accuate evaluation in addition to reducing computation time. The probability of failure P_f for individual stress components is written in terms of parameters in Figure 2 as

$$P_{f_i} = 1 - \exp\left[KG(\sigma_i)\right] \qquad (9)$$

where σ_1, σ_2 and σ_3 designate principal stress components respectively distributed within the structure and

$$G(\sigma_i) = -t\int_{r_1}^{r_2}\int_{0}^{\pi/2} r\left(\frac{\sigma_i}{\sigma_o}\right)^m d\theta dr \qquad (10)$$

The integration scheme is applied only to the first quadrant due to the symmetry considerations: r_1 is defined as radius of hole, r_2 is determined by location of uniform stress conditions of structure. t is the thickness of the plate. K is defined as unity for the uniform tension case.

Evaluation of P_f for all three stress states may be written as

$$P_f = 1 - \exp K\left[G(\sigma_1) + G(\sigma_2) + G(\sigma_3)\right] \qquad (11)$$

NUMERICAL RESULTS AND CONCLUSIONS

Sheet dimensions in Figure 2 are L = 2.0 and W = 0.5. In order to evaluate the effects of different loading conditions, biaxial, uniaxial and pure shear conditions are considered. Uniaxial loading is initially determined then a superposition argument is used in order to determine the other two loading states. The number of stress function coefficients were set at 23 to 31 depending on the desired accuracy of the stress and force conditions on the rectangular boundary. When the stress function is determined, stresses may be obtained within the structure for any desired points with a simple computation. This avoids the necessity of resolving the problem a number of times which is common for the Finite Element procedure.

In the numerical integration scheme, mesh spacing is varied according to stress gradient; a finer spacing is used in vicinity of higher stresses. The spacing was increased systematically until there was essentially no change in P_s values for the first five significant digits. The scheme required one and three segment evaluation for tangential and radial directions respectively in order to reduce computation and provide the desired accuracy. The number of mesh points selected were 21, 23, 15 and 9 respectively for the tangential and radial directions. Negative stress values were neglected because of their relatively small magnitudes. Although the analysis has the capability of accounting for their effects.

Initially the problem was solved by determining P_s values for the individual volume elements in the structure. These values were multiplied together in order to obtain the total P_s of the entire structure. The accuracy of this method was limited because of the slow convergence of P_s as the number of elements increased. For 1000 elements, changes were still noted in the third significant digit of P_s. This convergence difficulty is also quite common for Finite Element application.

In Figure 3, P_s (probability of survival) is a function of the three stress components where σ_1 is primary governing parameter. Considering the individual stress in relation to P_s calculations σ_2 and σ_3 obtained minimum values of 97% and 99.9% respectively for the largest volume and pure shear loading conditions. Figure 3 describes effects of volume, applied load and variation in stress state within the structures. From Figure 3, it is obvious that loading conditions as well as volumetric factors have important influence on probability of failure determinations of geometry considered. The authors are presently conducting an experimental study of these predictions.

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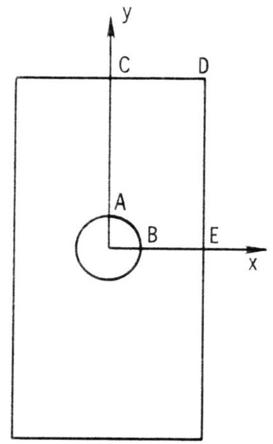

Figure 1 Boundary Definitions for Plate

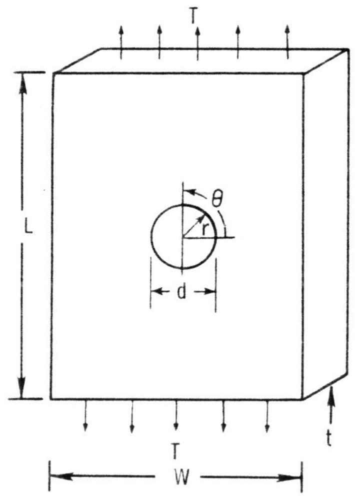

Figure 2 Rectangular Ceramic Plate with Hole

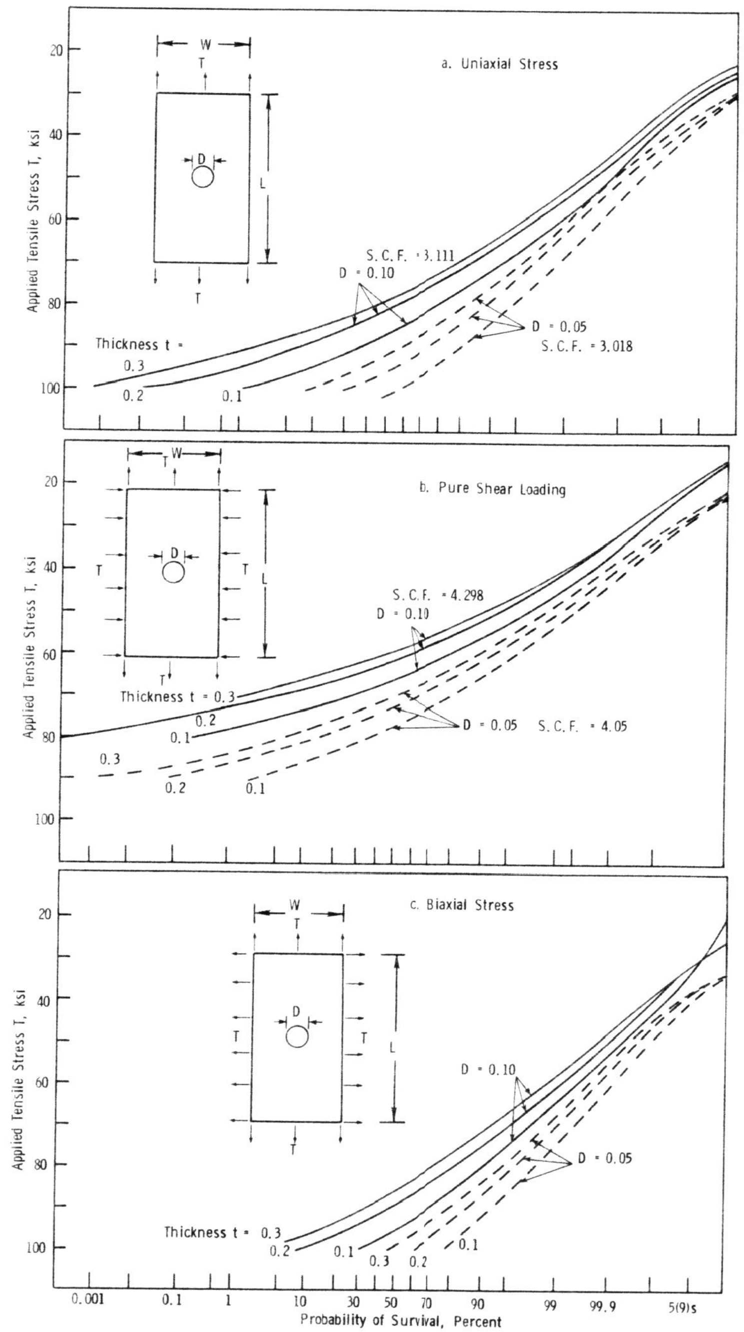

Figure 3 Effect of Geometry on Probability of Survival, σ_o = 96 ksi, m = 11, L = 2, W = 0.5