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DETERMINATION OF CRACK GROWTH IN A MIXED MODE LOADING SYSTEM

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INTRODUCTION

Studies on the propagation of cracks under an applied load are not readily analyzed when the cracks are not lying perpendicular to the applied stress. When a crack is orientated at some angle to the applied field, the mode of fracture is not a simple Mode 1 or Mode 2 but a combination of both.

Griffith [1] considered crack propagation to be primarily an energetic process and that the crack extends in a plane coincident to the plane containing the original crack. In reality the application of a load to a system containing a crack rarely leads to a discrete mode of fracture, usually two or three Modes act simultaneously at the crack front. Erdogan and Sih [2] considered the problem of mixed mode fracture by studying the initial direction of crack growth in a combined stress field. Further studies on mixed mode cracking were made by Sih [3] who showed that the condition for the direction of initial crack growth is given when strain energy density, S, attains a minimum value.

In this paper a new theory is proposed for the direction of crack initiation on the basis of the distortion strain energy. A 'failure curve' for a biaxial loading system is also suggested.

THEORY

To compute the strain energy around a crack tip, consider a sharp slit approximation and an element distance, r, away at an angle θ to the crack. The solutions of the stresses are given below for the combined Modes 1 and 2 in terms of the coordinate system given in Figure 1.

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \tau_{xy}\end{bmatrix} = K_I/(2\pi r)^{1/2}\begin{bmatrix}\cos\theta/_2(1-\sin\theta/_2\sin 3\theta/_2)\\ \cos\theta/_2(1+\sin\theta/_2\sin 3\theta/_2)\\ \sin\theta/_2\cos\theta/_2\cos 3\theta/_2\end{bmatrix} +$$

$$+ K_{II}/(2\pi r)^{1/2}\begin{bmatrix}-\sin\theta/_2(2+\cos\theta/_2\cos 3\theta/_2)\\ \sin\theta/_2\cos\theta/_2\cos 3\theta/_2\\ \cos\theta/_2(1-\sin\theta/_2\sin 3\theta/_2)\end{bmatrix} \quad (1)$$

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$\sigma_z = \nu(\sigma_x + \sigma_y)$ (plane strain)

$\tau_{xz} = \tau_{zy} = 0$

$E = 2(1 + \nu)\mu$, where μ is the shear modulus.

The total elastic strain energy, dW_T, stored in an element $dV = dx.dy.dz$ under three dimensional stress system is given by

$$dW_T = \left[\frac{1}{2E}(\sigma_x^2+\sigma_y^2+\sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) + \frac{1}{2\mu}(\tau_{xy}^2+\tau_{xz}^2+\tau_{zy}^2)\right]dV \quad (2)$$

The total strain energy, dW_T, consists of the sum of the strain energy due to change in volume, dW_V, and the strain energy due to distortion, dW_d.

$$dW_T = dW_V + dW_d \ . \quad (3)$$

The strain energies due to volumetric change (hydrostatic component) and due to distortion are given below.

$$dW_V = \frac{1-2\nu}{6E}\left[\sigma_x^2+\sigma_y^2+\sigma_z^2+2(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x)\right]dV \quad (4)$$

$$dW_d = \frac{1+\nu}{3E}\left[\sigma_x^2+\sigma_y^2+\sigma_z^2-(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x)+3(\tau_{xy}^2+\tau_{xz}^2+\tau_{zy}^2)\right]dV. \quad (5)$$

Upon the substitution of equation (1) into equations (2, 4, 5), the total, hydrostatic and distortion strain energy densities may be written in the form given below.

$$\frac{dW_T}{dV} = \frac{1}{r}S, \quad \frac{dW_V}{dV} = \frac{1}{r}S_V, \quad \frac{dW_d}{dV} = \frac{1}{r}S_d \ . \quad (6)$$

Here S, S_V and S_d are independent of r but depend on the elastic constants, K_I, K_{II} and the angle, θ.

It should be noted that the strain energy densities tend to infinity as r tends to zero. As a result, a cut off point of $r > 0$ has to be introduced to allow for the discontinuity of strain energy density at the crack tip.

THE DIRECTION OF CRACK INITIATION AND PROPAGATION

In a mixed mode system, the strain energy density comprises of that due to a volume change and that due to a distortion effect as shown in Figure 2. A crack in an elastic material in pure hydrostatic tension will extend along a line coincident with the axis, A A', of the crack as shown in Figure 3. It appears that the direction of crack growth takes place along the direction where the distortion strain energy is minimum.

$$\frac{dS_d}{d\theta} = 0 \ \text{at} \ \theta = \theta_o \ \text{and} \ \frac{d^2S_d}{d\theta^2} > 0 \ . \quad (7)$$

The resistance to crack growth is determined by the total strain energy density which reaches a critical value, S'_{crit}, at $\theta = \theta_o$,

$$S'_{crit} = S_V + S_d, \quad \text{at } \theta = \theta_o \ . \quad (8)$$

UNIAXIAL TENSION

For a crack extending in a mixed mode system, K_I and K_{II} are given by the following expressions.

$$K_I = \sigma\sqrt{\pi a}\sin^2\beta \quad , \quad K_{II} = \sigma\sqrt{\pi a}\sin\beta\cos\beta \ . \quad (9)$$

Figure 4 shows the variation of θ_o as a function of β. Table 1 is a comparison of theoretical results of θ_o due to Sih [3], the authors and the experimental results of Erdogan and Sih [2]. The experimental work was performed on plexiglass of dimensions 9" x 18" x 3/16" with a central crack of 2 in. The values of θ_o at $\nu = 0.0$ and 0.33 have been considered to give a meaningful comparison of experimental data with the theoretical values. The reason for this is that the dimensions of the samples tested would not give rise to a pure plane strain behaviour but a combination of plane strain and plane stress. When the normalized strain energy density term is plotted for different values of β, the resulting curves are similar to that of Sih. Hence these curves are omitted in this paper.

UNIAXIAL COMPRESSION

In this case, the values of θ_o for varying angles of β are shown in Figure 5. To determine the failure strength as a function of β, in compression, experiments have been carried out by the authors on "perspex" specimens of 100 x 100 x 6 mm dimensions. Cracks of 35 mm long were introduced using a circular saw of 0.2 mm thickness. These preslotted specimens were tested in uniaxial compression using an Instron machine at a cross head speed of 10 mm/min. Buckling of the samples was avoided by using a special jig. The results given in Figure 6 show an excellent agreement with the theory.

PURE SHEAR

Figure 7 shows the case of plane shear. K_I and K_{II} for this system are given by the following expressions.

$$K_I = 0 \ , \quad K_{II} = \tau(\pi a)^{1/2} \ . \quad (10)$$

The values of θ_o are computed as for previous cases and are given in Table 2 along with those due to Sih [3]. The classical theory due to Griffith postulates that a crack subjected to a pure shear stress is assumed to extend along the plane of the axis of the crack. As evident from the values of Sih and those due to authors, the crack extends in a line which is not coincident with the axis of the initial crack.

BIAXIAL LOADING SYSTEM

The theory described for the determination of crack growth in a uniaxial loading system can also be extended to biaxial loading systems. Such a system is shown in Figure 8 where the applied stress can be either both compressive or tensile or a combination of both. For this system K_I and K_{II} are given by the following expressions.

$$K_I = \sigma_2(\pi a)^{0.5}\left[\frac{\sigma_1}{\sigma_2}\sin^2\beta + \cos^2\beta\right] \qquad (11)$$

$$K_{II} = \sigma_2(\pi a)^{0.5}\left[\frac{\sigma_1}{\sigma_2} - 1\right]\sin\beta\cos\beta\ . \qquad (12)$$

For a known ratio of σ_1/σ_2, the minimum value of $\sigma_{2,crit}$ may be obtained similarly to the previous cases. Figure 9 shows a 'failure diagram' which is drawn on the assumption that the critical flaw size in a material lies in the direction in which $\sigma_{2,crit}$ is minimum.

DISCUSSION

A method for the determination of crack initiation has been put forward on the basis that the direction of initiation is given when S_d reaches a minimum value. It is evident from the results that the proposed theory is different to that of Sih. His work showed that θ_0 takes lower values for lower values of ν when compared with the authors. However, the resistance to crack growth which determines the failure stress shows a good agreement with the work of Sih.

An important outcome of this work and also that due to Sih's studies is that it confirms that the use of the expression, given below, for the strain energy release rate, G, for mixed mode systems, needs to be revised.

$$G = K_I^{\ 2}(1-\nu^2)/E + K_{II}^{\ 2}(1-\nu^2)/E + K_{III}^{\ 2}(1+\nu)/E \qquad (13)$$

(plane strain)

It is important to note that the ratio of compression to tensile strength is not easily evaluated for a real material, due to the random distribution of flaw sizes and their orientations within a material. This argument may also be applied to the biaxial system. The application of statistical methods to determine the strength of brittle materials for both uniaxial and biaxial systems is under progress.

ACKNOWLEDGEMENTS

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Table 1 A Comparison of Theoretical and Experimental Values of θ_0

β	30	40	50	60	70	80	ν
3	-48.0	-39.0	-32	-23	-15	-10	0.0
	-63.5	-56.7	-49.5	-41.5	-31.8	-18.5	0.333
2	-62.4±2.1	-55.6±1.2	-51.6±0.8	-43.1±1.0	-30.7±0.8	-17.3±0.7	0.333
Authors	-64	-56	-49	-42	-32	-19	0.00
	-69.5	-62	-53.5	-45.0	-33	-19	0.333

Table 2 Theoretical Values of $|\theta_0|$ for Pure Shear

ν	0	0.1	0.2	0.3	0.4	0.5	
θ_0	84°	86°	88°	89°	90°	-	Authors
θ_0	70.5	75.6	79.3	83.3	87.2	90.0	3

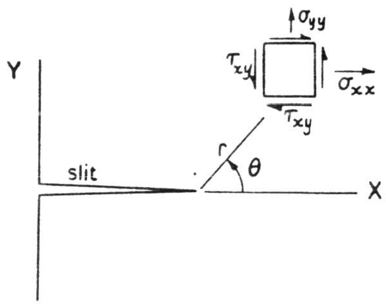
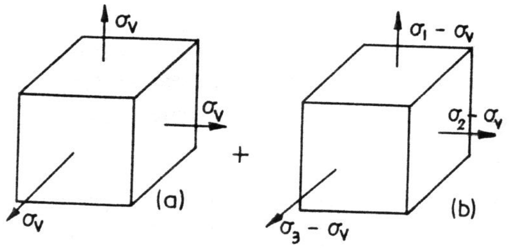


Figure 1 Crack Tip Stresses, Showing Rectangular Components



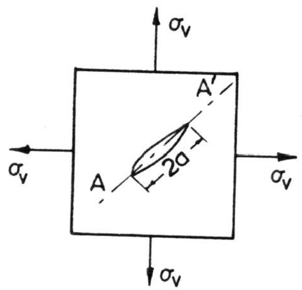


Figure 2 The Stress System for (a) Hydrostatic, (b) Distortion Effects. $\sigma_V = 1/3(\sigma_1+\sigma_2+\sigma_3)$ where σ_1, σ_2, σ_3, are the Principal Stresses

Figure 3 The Direction of Crack Extension in Hydrostatic Loading

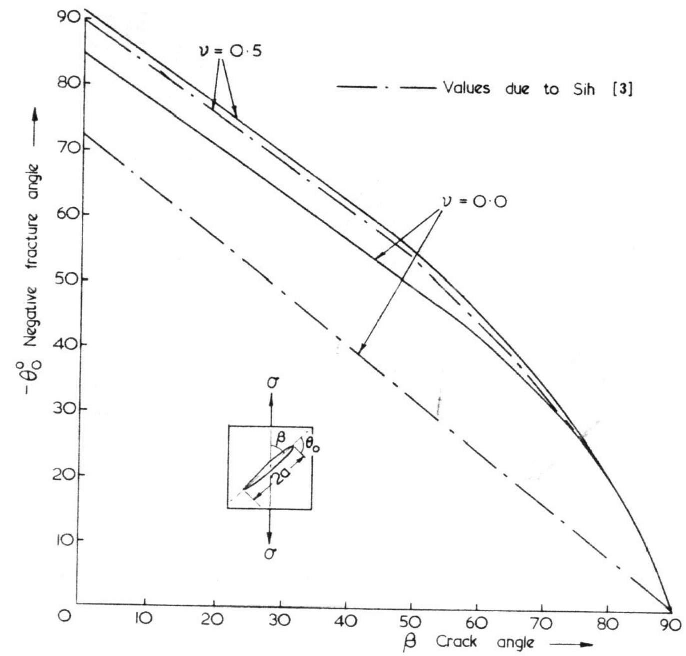


Figure 4 Crack Angle Versus Fracture Angle for the Tension Case

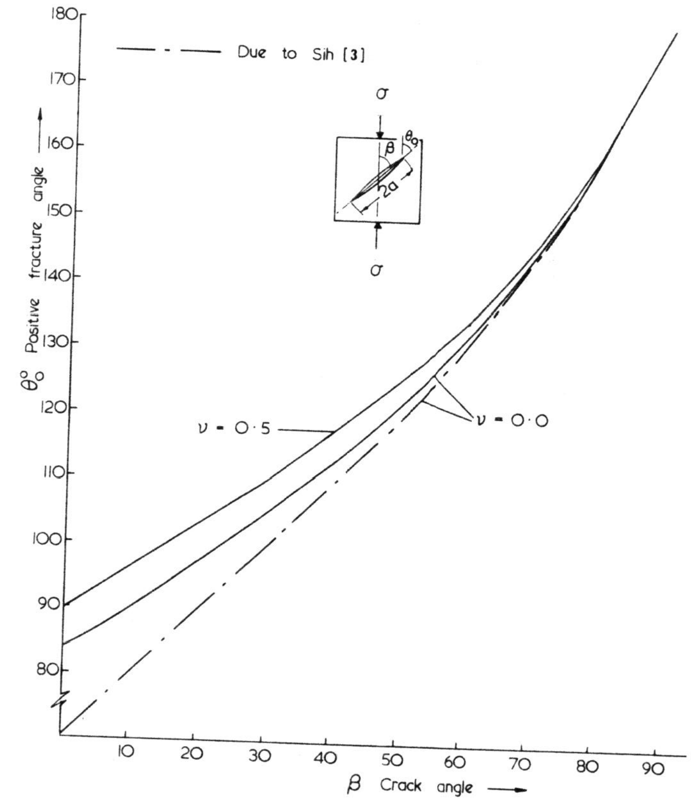


Figure 5 Crack Angle Versus Fracture Angle for the Compression Case

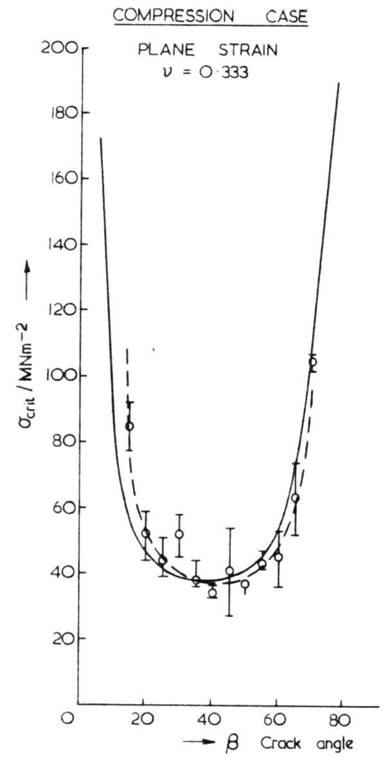


Figure 6 Stress to Failure Versus Crack Angle for Plane Strain Conditions in Compressive Loading. The Theoretical Curve was Drawn with its Minimum Point Having the Same Value of σ_{crit} of the Experimental Curve at the Same Angle. Solid Line Indicates the Theoretical Curve

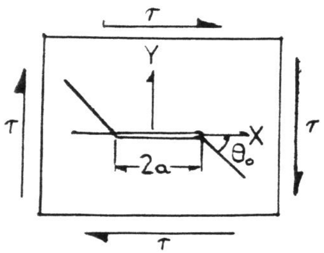


Figure 7 A Line Crack in Shear

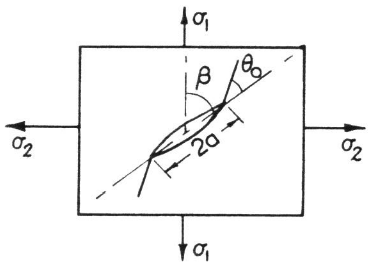


Figure 8 An Inclined Crack in a Biaxial Stress System

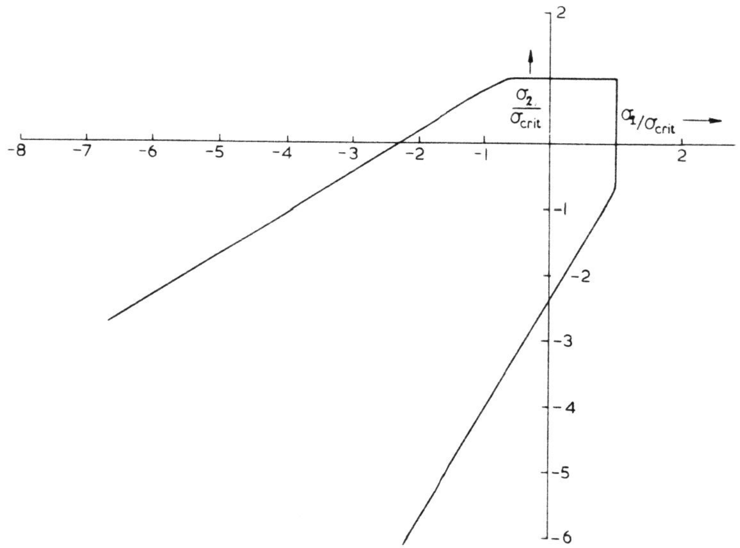


Figure 9 A Biaxial Failure Diagram, Where σ_{crit} is the Stress to Failure for Mode 1. 'ν = 0.3'