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ANALYSIS OF BRANCHED CRACKS UNDER BIAXIAL STRESSES

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INTRODUCTION

Branched cracks are often observed in brittle fracture and also in stress corrosion cracking. The reasons for these crack branching phenomena in brittle fracture have been explained on the basis of dynamic effects [1]. However, there are many characteristics common to both crack branching behaviour in brittle fracture and in stress corrosion cracking. In order to discuss these phenomena, it seems important to analyse the static stress intensity factors for the branched crack model. However, few solutions for such a crack have been obtained. In another report, a general method analysing some kinds of the branched cracks was formulated using a conformal mapping function. The numerical values of stress intensity factors for a branched crack under uniaxial stress were reported in references [2 - 5].

In this paper, taking into consideration that most engineering structures are often subjected to biaxial loading, the numerical solutions for the stress intensity factors of some kinds of branched cracks under biaxial stress are presented. Moreover, using the results obtained, crack extension behaviour under biaxial stresses is discussed.

ANALYSIS

Conformal mapping functions have been applied to the analyses of crack problems by Muskhelishvili [6], Bowie [7] and others. Muskhelishvili and Savin [8] analysed various shaped holes with the method of polynomial mapping approximation. Bowie applied this method to the analysis of a crack emanating from a circular hole. We applied this method, with various improvements, to the analysis of a branched crack.

In the analysis of a branched crack, due to the complicated crack geometry, the polynomial approximation of the mapping function does not converge easily. Moreoever, the first and second mode stress intensity factors coexist and the crack has dual crack tips at which the stress intensity factors are different. This increases the difficulties of the analysis.

The authors have overcome these difficulties and were able to develop a general method of analysis of branched cracks. In other papers [2 - 5], we analysed a branched crack with one or two branches on one side of the main crack in a uniaxial stress field, utilizing the mapping functions which were used by Andersson [9]. In this paper, we analyse a doubly symmetric branched crack, utilizing the mapping function introduced by the Schwartz-Christoffel transformation with several devices in calculation.

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As a result of the high symmetry of the crack geometry, we obtained more accurate solutions than the solutions of the forked crack obtained in another paper. The outline of our method is described below.

Mapping Function

A doubly symmetric branched crack as shown in Figure 1 is considered. A conformal mapping function which maps the crack into a unit circle is given by a Schwartz-Christoffel transformation. As a result of the symmetry, the conformal mapping function is given by equation (1).

$$Z = \omega(\zeta) = A\int^{\zeta} H(\zeta)(1-2\cos 2\beta\, \zeta^{-2}+\zeta^{-4})d\zeta \tag{1}$$

$$H(\zeta) = \left\{1-\frac{1}{\zeta^2}\right\}^{\frac{2\theta}{\pi}-1}\left\{1-\frac{e^{2i\alpha}}{\zeta^2}\right\}^{-\frac{\theta}{\pi}}\left\{1-\frac{e^{2i(\pi-\alpha)}}{\zeta^2}\right\}^{-\frac{\theta}{\pi}} \tag{2}$$

Where the parameters α, β correspond to the branching points and the crack tips respectively as shown in Figure 1 and A is a real constant. The mapping function $\omega(\zeta)$ must be expanded in a series, because of the determination of the parameters α, β and the stress function as stated in the next section. The function $H(\zeta)$ in equation (1) can be expanded in a binomial series as shown in equation (3).

$$H(\zeta) = 1 + \sum_{n=1}^{\infty} h_n\, \zeta^{-2n} \tag{3}$$

Substituting equation (3) into equation (1) and integrating equation (1), we get the following equation:

$$\omega(\zeta) = A\left[\zeta + \sum_{n=1}^{\infty} B_n\, \zeta^{1-2n}\right] \tag{4}$$

For practical reasons the number of terms in equation (4) must be kept finite. If the terms of the polynomial mapping function are truncated at a finite number, the tip of the crack is rounded off. In the analysis of crack problems, the truncation of terms needs some devices to preserve the crack tip geometry without disturbing the overall crack configuration. Referring to Bowie's truncation plan [8], the polynomial mapping function $\omega(\zeta)$ of equation (4) is truncated at a finite term to satisfy the following conditions at the crack tips.

$$\omega'(e^{i\beta}) = 0, \qquad \omega''(e^{i\beta}) = Q \tag{5}$$

Where Q is the exact second derivative of the mapping function of equation (1) at the crack tip and is given by equations (6) and (7).

$$Q = \omega''(e^{i\beta}) = -4A\, H(e^{i\beta})(e^{-2i\beta} - 2\cos 2\beta)e^{-3i\beta} \tag{6}$$

$$H(e^{i\beta}) = \frac{1}{2}\left|\sin\beta\right|^{\frac{2\theta}{\pi}-1}\left|\sin(\alpha+\beta)\,\sin(\alpha-\beta)\right|^{-\frac{\theta}{\pi}}\left\{e^{i\left(\theta+\beta-\frac{\pi}{2}\right)}\right\} \tag{7}$$

To satisfy the above conditions, we add the two correction coefficients to the function $\omega(\zeta)$ which is expanded in series and truncated at the Nth term. Thus we obtain the following mapping function.

$$\omega(\zeta) = A\left[\zeta + \sum_{n=1}^{N+2} B_n^{*}\, \zeta^{1-2n}\right] \tag{8}$$

Next, the parameters α and β must be determined. The parameters follow the relations.

$$0 \leq \beta < \alpha \leq \pi/2 \tag{9}$$

For a given value of β:

$$Z = \omega(\zeta) = 0, \qquad \text{on } e^{i\alpha} < \zeta < e^{i(\pi-\alpha)} \tag{10}$$

The value of α for a given value of β is numerically determined to satisfy the above relation. Thus we obtain the mapping function and its polynomial approximation for the doubly symmetric branched crack.

Stress Function and Stress Intensity Factors

The Muskhelishvili complex stress functions $\phi(\zeta)$, $\psi(\zeta)$ are used in our analysis. Taking the polynomial mapping function $\omega(\zeta)$ obtained by equation (8) into consideration, the stress functions can be defined as follows, when the crack is subjected to uniform uniaxial tension σ in the direction ϕ_0 as shown in Figure 2.

$$\phi(\zeta) = A\sigma\left[\frac{\zeta}{4} + \sum_{n=1}^{N+2} d_n\, \zeta^{1-2n}\right] \tag{11}$$

$$\psi(\zeta) = A\sigma\left[-\frac{e^{2i\phi_0}}{2}\,\zeta + \sum_{n=1}^{N+2} f_n\, \zeta^{1-2n}\right] \tag{12}$$

Where d_n and f_n are the coefficients of the stress functions and they have to be determined so that the stress-free condition is satisfied on the crack edge. This condition is given by equation (13).

$$\overline{\phi}(1/\zeta) + \overline{\omega}(1/\zeta)\phi'(\zeta)/\omega'(\zeta) + \psi(\zeta) = 0, \quad \text{on } |\zeta| = 1 \tag{13}$$

Substituting equations (11), (12) and (8) into equation (13), and equating the coefficients of all positive powers of ζ to zero, a set of linear simultaneous equations with respect to the unknown coefficient d_n can be given. Solving these equations, the stress function $\phi(\zeta)$ to satisfy the boundary conditions is obtained. By means of the function $\phi(\zeta)$ obtained above and equations (6) and (7), the stress intensity factors of the branched crack can be calculated by the following equation.

$$K = K_1 - iK_2 = \frac{2\pi^{1/2}\,\phi'(e^{i\beta})}{[e^{i\theta}\omega''(e^{i\beta})]^{1/2}} \tag{14}$$

Where θ is the branching angle as shown in Figure 1. The value of the stress intensity factors for the branched crack under given biaxial stresses can be obtained by the summation of these kinds of solutions.

NUMERICAL RESULTS

Doubly Symmetric Branched Crack

By means of the method mentioned above, the values of the stress intensity factors of the branched crack were calculated. The accuracy is within 1%. They are normalized by the stress intensity factors of the straight crack length 2c as shown in Figure 3. Figure 3 shows the non-dimensional stress intensity factors (F_1, F_2) of the branched crack, subjected to uniform tension along the y axis and Figure 4 along the x axis, respectively. The solutions for the branched crack under arbitrary biaxial stresses can be calculated by simply superimposing the solutions for the cracks under uni-axial stress. Figure 5 shows the stress intensity factors of the branched crack for $b/a = 0.1$ under various biaxial stresses. λ is the ratio of the tensile stress in the x direction to that in the y direction.

It is well known that in brittle fracture or in stress corrosion cracking, crack branching has often been observed. In such a fracture process, the stable crack branching angle can be considered. Figure 5 shows that for various values of λ, the F_2 (or K_2) value changes sign and becomes zero at a chracteristic branching angle. A branched crack with such a branching angle will grow along the extension line of the branch. If we take a criterion such that each branch can grow only in the direction F_2 (or K_2) = 0 [3], the stable branching angles 2θ for the various value of λ are determined by Figure 5. They are about 25°, 33°, 41°, 52° for $\lambda = -1$, 0, 0.5, 1, respectively. Thus the branching angle increases with increases of the value of λ. For the case of uniaxial stress ($\lambda = 0$), it is well known that the average macro branching angle observed in experiments is almost 30° - 40°. In this discussion, the results for $b/a = 0.1$ are used. If the b/a value increases, the branching angle decreases slightly. This corresponds to the phenomenon that the branched crack changes direction with the growth of the branches. We can obtain stable branching angles under various biaxial stress states. If a branched crack is observed in a structure, it is possible to infer the stress state to which the branched crack was subjected using these results.

Bent Crack

We also analysed a bent crack, as shown in Figure 6, as a limiting kind of branched crack. In this case, the mapping function and the method of analysis are slightly different from those mentioned above. They have been reported in reference [2]. This bent-crack crack-model is very useful when the extension direction of a crack under a mixed mode stress state is discussed [11]. Several authors have tried to analyse the same crack model as used here [12 - 13]. Only numerical results are presented here in order to discuss crack extension behaviour under biaxial stresses. Figure 6 shows the stress intensity factors at the tip of the bent crack with a branch ($b/a = 0.1$) subjected to various biaxial stress states. It is a very difficult problem to consider the stability of crack extension. However, if the model of the bent crack can be used and the stress intensity factors of this crack are known, it seems that the stability of crack extension can be supposed to some extent. We can deduce that for the case of $\lambda = 0$, -1, the crack is very stable because the nondimensional

stress intensity factors at the bent crack tip F_{1B}, F_{2B} vary rapidly at the vicinity of the point $\lambda = 0$. This means that a small deviation in the direction of crack extension from the original direction induces a rapid return to the original direction. On the other hand, for the case of $\lambda = 2$, the crack seems to be unstable and cannot grow in a straight line. Figure 7 shows the directions of crack extension from the tip of a bent crack which are calculated by the theory of maximum circumferential stress proposed by Sih, et al. [10]. For the case of $\lambda = 0$, -1, if the crack is bent for some reason, it will be bent again towards the direction of the extension line of the initial crack. On the other hand, for the case of $\lambda = 2$, the crack grows in the direction of the extension line of the bent branch. Thus the crack has different stability for each biaxial stress state. We suppose that this affects the crack path and crack growth law found in biaxial fatigue tests.

SUMMARY

We constructed a general method for analysis of a branched crack in a given biaxial stress state by means of a conformal mapping function and its series expansion. The calculated values of stress intensity factor of a doubly symmetric branched crack and a bent crack under various bi-axial stresses are presented. On the basis of these results, the crack extension behaviour under biaxial stresses is discussed from our crack morphological view-point. Some of the interesting points are as follows:

1) In a fracture process such that a symmetric branched crack can grow, there is a stable branching angle particular to a given biaxial stress state, which increases with increase of the lateral biaxial tensile stress.
2) Using the solutions for the bent crack, we discussed the stability of the crack under various biaxial stresses. The crack has a particular stability, which may affect a crack extension behaviour.

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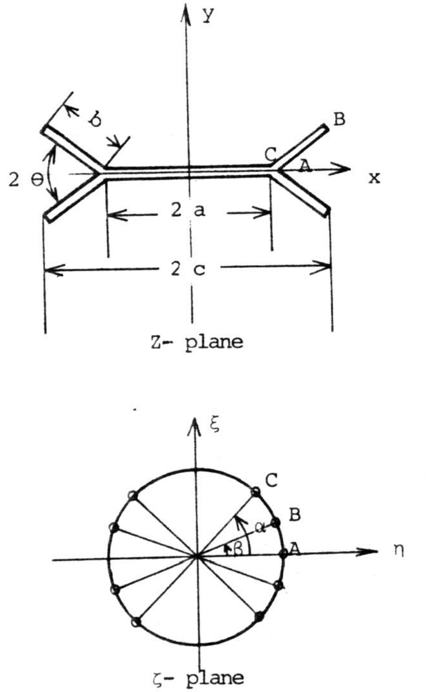

Figure 1 Crack geometry in Z-plane and mapped in mapped ζ-plane

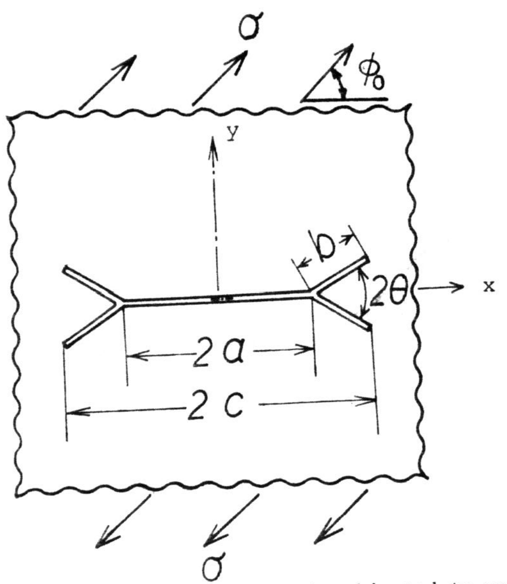

Figure 2 The doubly symmetric branched crack subjected to uniform uniaxial tension

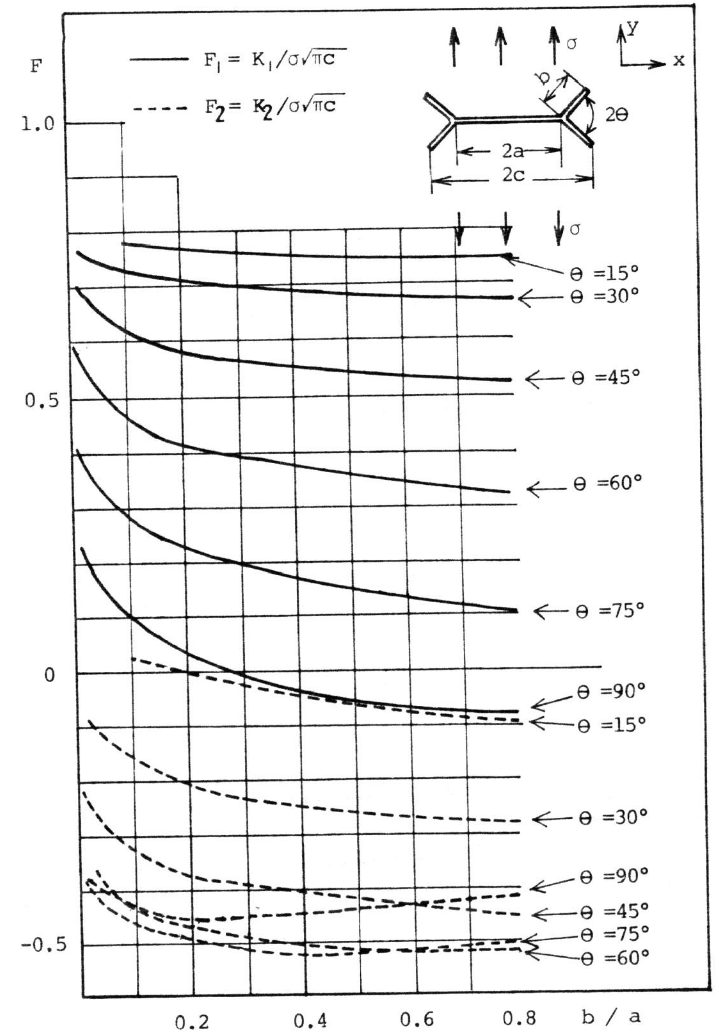

Figure 3 The stress intensity factors of the doubly symmetric branched crack subjected to uniaxial tension in the y-axial direction

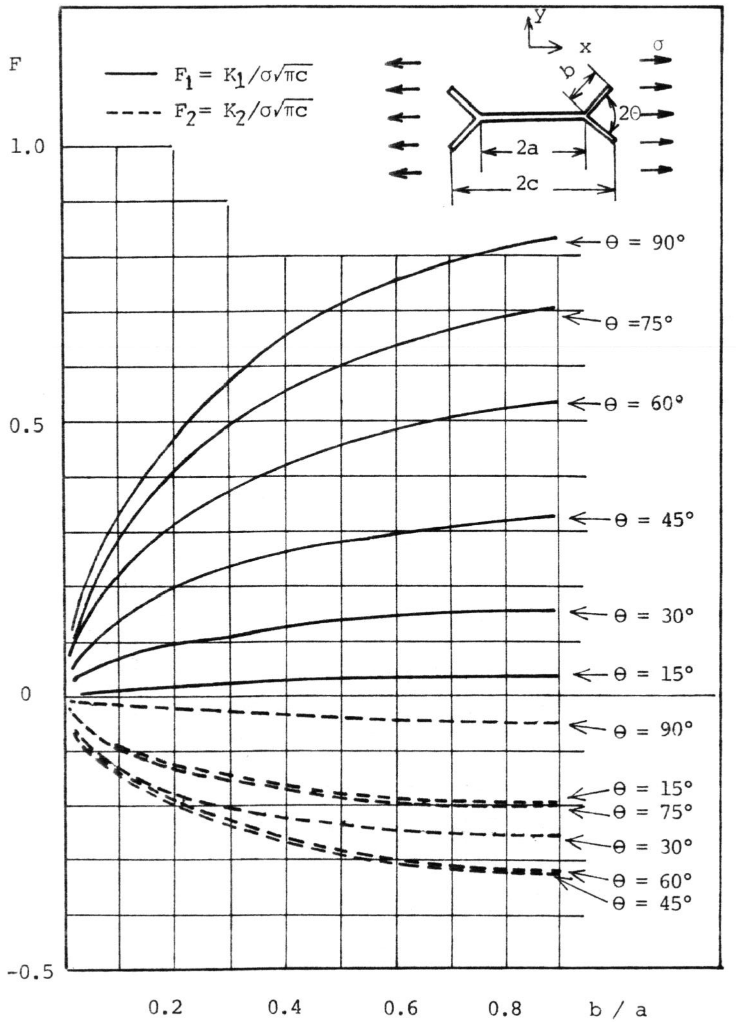

Figure 4 The stress intensity factors of the doubly symmetric branched crack subjected to uniaxial tension in the x-axial direction

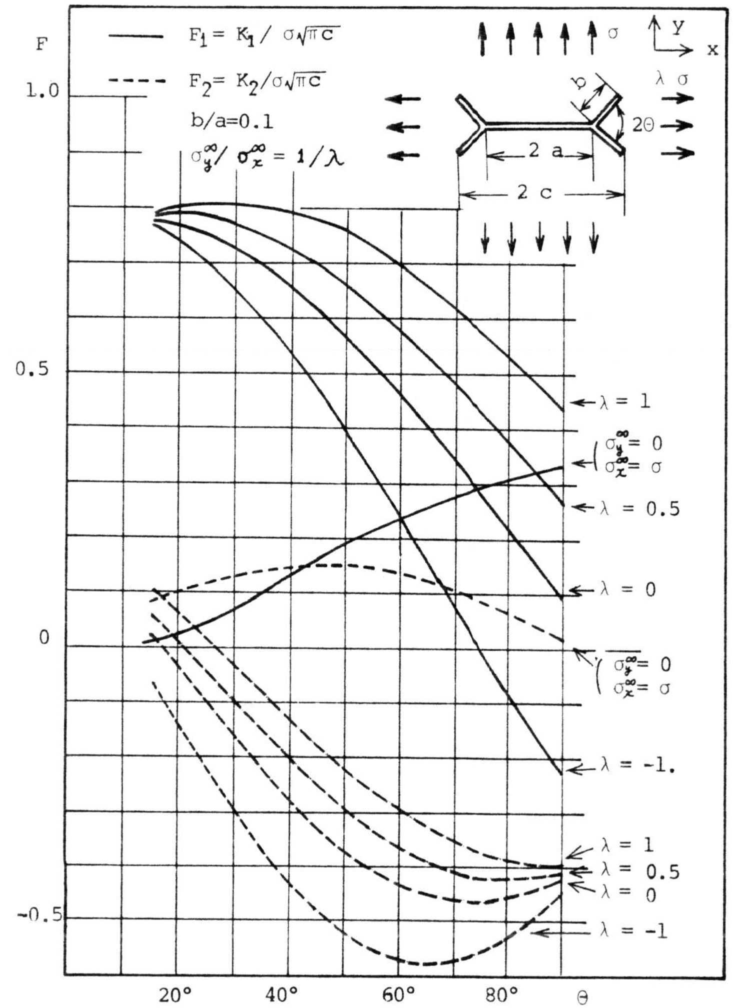

Figure 5 The stress intensity factors of the doubly symmetric branched crack for b/a=0.1 subjected to biaxial stresses

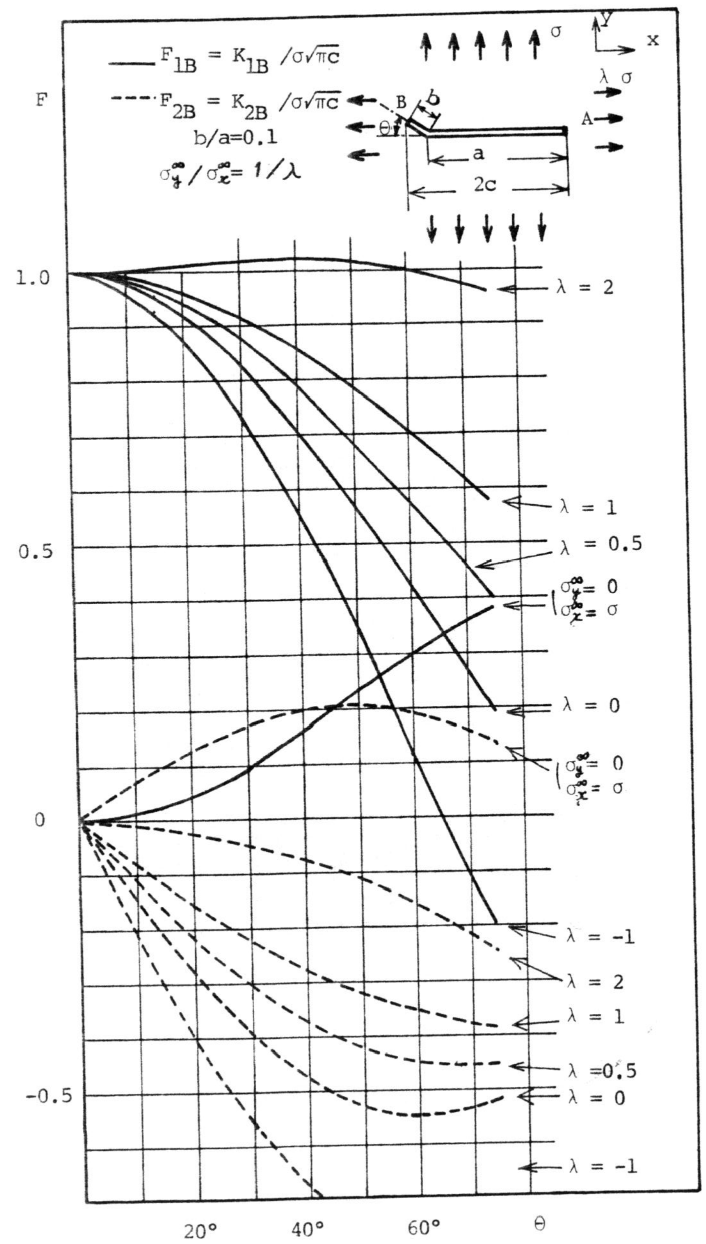

Figure 6 The stress intensity factors of a bent crack for b/a=0.1 subjected to a biaxial stress state

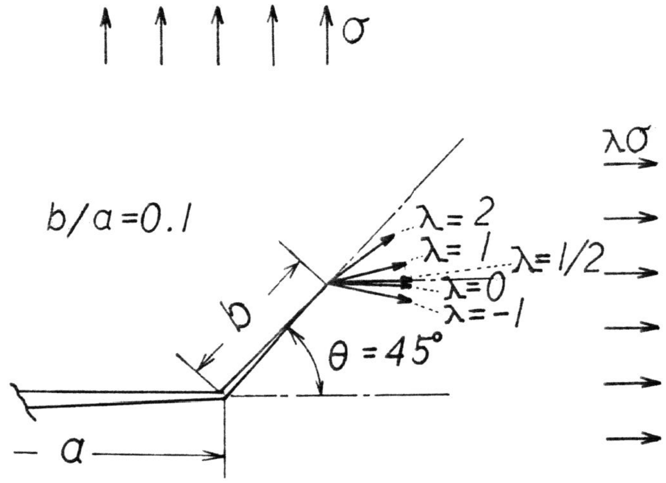

Figure 7 The direction of crack extension from the bent crack tip