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ANALYSIS FOR THE PROBLEM OF MISFITTING INCLUCION AND CIRCULAR ARC CRACKS IN A SHEET

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INTRODUCTION

Tamate [1] combined together the inclusion and straight crack problem and obtained the solution of the problem in series form. An effort of combining together the misfitting inclusion of different materials and circular arc crack problems forms the subject matter of the present paper. Analytical solution of the problem is given in series form and complex variable approach of Muskhelishvili [2] is used throughout.

BASIC EQUATIONS AND STATEMENT OF PROBLEM

Let the region S be the entire plane, cut along the arc L_S(S=1, 2) of a circle of radius R with the centre at the origin of $z=x+iy=r\exp(i\theta)$. The arc L_S is assumed to lie symmetrically on the x-axis and subtend an angle 2α at the centre. By using the complex potentials $\phi(z)$, $\psi(z)$ which are defined in S, we define a new function W(z) in the following manner:

$$W(z) = \bar{\phi}(R^2/z)-(R^2/z)\bar{\phi}'(R^2/z)-(R^2/z^2)\bar{\psi}(R^2/z) \tag{1}$$

Whence $\psi(z)$ can be expressed in terms of $\phi(z)$ and W(z) as

$$\psi(z) = (R^2/z^2)\phi(z)-(R^2/z^2)\bar{W}(R^2/z)-(R^2/z)\phi'(z) \tag{2}$$

where the bar denotes the complex conjugate.

In the absence of body forces, the stress components in polar coordinates σ_r, σ_θ, $\tau_{r\theta}$ and the displacement components U_r, U_θ for the elastic sheet occupying the regions S are expressed in terms of $\phi(z)$ and $\psi(z)$ as

$$\sigma_r+\sigma_\theta = 2[\phi(z)+\overline{\phi(z)}] \tag{3}$$

$$\sigma_r+i\tau_{r\theta} = \phi(z)+\overline{\phi(z)}-\overline{z\phi'(z)}-(\bar{z}/z)\overline{\psi(z)} \tag{4}$$

$$2\mu\frac{\partial}{\partial\theta}\{(U_r+iU_\theta)\exp(i\theta)\} = iz[\kappa\phi(z)-\phi(z)$$
$$+\bar{z}\,\overline{\phi'(z)}+(\bar{z}/z)\,\overline{\psi(z)}] \tag{5}$$

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where μ is the shear modulus and $\kappa=3-4\nu$ for plane deformation, $\kappa=(3-\nu)/(1+\nu)$ for generalised plane stress, ν being Poisson's ratio. The subscript following a comma stands for partial differentiation. Equations (4) and (5) one may write as follows:

$$\sigma_r+i\tau_{r\theta} = \phi(z)+W(R^2/\bar{z})+\bar{z}(\bar{z}/R^2-1/z)\overline{\psi(z)} \tag{6}$$

$$2\mu\frac{\partial}{\partial\theta}\{(U_r+iU_\theta)\exp(i\theta)\} = iz[\kappa\phi(z)-W(R^2/\bar{z})-\bar{z}(\bar{z}/R^2-1/z)\overline{\psi(z)}] \tag{7}$$

Let an infinite elastic plate, isotropic and homogeneous, occupy the aforementioned regions S and be cut out by a circular hole of radius c with its centre at any point M(M may be complex). An inclusion of different elastic material of radius $(c+\eta)$, (η is of the order of displacements admissible in elasticity theory) is supposed to be inserted and bonded to the hole. Further it will be assumed that the stresses vanish at infinity and the edges of the crack are free from external tractions. The crack and the hole do not overlap.

The boundary condition of the problem can be expressed as follows:

(i) At infinity,

$$\phi(z) = 0(z^{-2}),\ \psi(z) = 0(z^{-2}) \tag{8}$$

whence at the origin,

$$W(z) = 0(1) \tag{9}$$

(ii) on the rims of the crack L_s, $\sigma_r^{\pm}+i\tau_{r\theta}^{\pm} = o$;

$$\phi^+(\zeta)+W^-(\zeta) = 0,\ \phi^-(\zeta)+W^+(\zeta) = o \tag{10}$$

where ζ is the coordinate of the point on the cut L_s and superscripts + and - refer to the boundary values of the functions as z approaches from the inside and the outside of the arc L_s respectively.

(iii) On the common circle r=c, when origin is considered as M,

$$(\sigma_r+i\tau_{r\theta}) = (\sigma_r+i\tau_{r\theta})_i \tag{11}$$

$$(U_r+iU_\theta)-(U_r+iU_\theta)_i = \eta \tag{12}$$

where the subscript i refers to the inclusion.

COMPLEX POTENTIALS FOR THE PLATE AND FOR THE INCLUSION

Since the equation (10) are dual homogeneous Hilbert problems for two functions $\phi(z)$ and $W(z)$, which are analytic in the entire plane cut along L_s we can readily construct the complex potentials $\phi(z)$ and $W(z)$ for the infinite plate which satisfy the conditions in (8 - 10) by the use of Muskhelishvili's technique. Taking into account the fact that $\phi(z)$ and $W(z)$ could have poles of various orders at z=M, we can write them as follows:

$$\left.\begin{matrix}\phi(z)\\W(z)\end{matrix}\right\} = \pm 1/2\sum_{j=2}^{\infty} F_{-j}\left\{\frac{1}{z-M}\right\}^j \pm 1/2\sum_{j=o}^{\infty} F_j(z-M)^j$$

$$+1/2X(z)\left[\sum_{j=1}^{\infty} H_{-j}\left\{\frac{1}{z-M}\right\}^j+\sum_{j=o}^{\infty} H_j(z-M)^j\right] \tag{13}$$

where

$$X(z) = [z-R\exp(-i\alpha)]^{-1/2}[z-R\exp(i\alpha)]^{-1/2}[z+R\exp(i\alpha)]^{-1/2}$$
$$[z+R\exp(-i\alpha)]^{-1/2} \tag{14}$$

means the branch, analytic in the entire plane cut along L_s such that $X(z) \to (1/z^2)$ for $|z| \to \infty$. The coefficients $F_{\pm j}$ and $H_{\pm j}$ are to be determined.

The origin of the coordinate system is now shifted to M. The functions $\{\phi(z),\ \psi(z)\}$ transform to new functions $\{\phi_1(z_1),\ \psi_1(z_1)\}$. We drop the suffix 1 for the convenience but remember these are the potentials obtained after shifting the origin to M. By the conditions (8) and (9), we get

$$F_{-j} = \sum_{s=o}^{\infty} L_s H_{-(j+s)}\ (j \geq 2),$$

$$F_j = -\sum_{s=1}^{\infty} L_{-s} H_{j+s}\ (j \geq o), \tag{15}$$

$$\sum_{s=o}^{\infty} L_s H_{-(s+1)} = o,\qquad \sum_{s=1}^{\infty} L_{-s} H_{s-1} = o$$

where the constants $L_{\pm s}$ are known quantities determined from the relation,

$$[z+m-R\exp(-i\alpha)]^{-1/2}[z+M-R\exp(i\alpha)]^{-1/2}[z+M+R\exp(i\alpha)]^{-1/2}$$

$$[z+M+R\exp(-i\alpha)]^{-1/2} = \sum_{j=o}^{\infty} L_j z^j,$$

$$L_{-(2j+1)} = L_{2j+1} = o,\ L_{-(2j+2)} = -L_{2j},\ (j \geq o) \tag{16}$$

The function $\phi(z)$ in (13) and the corresponding function $\psi(z)$, obtained from (2) can be expressed in the following Laurent series in the region $c < |z| < R$:

$$\phi(z) = 1/2\sum_{j=-\infty}^{\infty} C_j z^j,\ \psi(z) = 1/2\sum_{j=-\infty}^{\infty} D_j z^j \tag{17}$$

where,

$$\left.\begin{aligned} C_{-1} &= o, \\ C_{-j} &= F_{-j} + \sum_{s=o}^{\infty} L_s H_{-(j+s)} (j \geq 2), \\ C_j &= F_j + \sum_{s=o}^{\infty} L_s H_{j-s} (j \geq o), \end{aligned}\right\} \quad (18)$$

$$\left.\begin{aligned} D_{-1} &= o \\ D_{-j} &= (j-1)C_{-(j-2)} -{}_2 2 \sum_{s=1}^{\infty} L_{-s} \overline{H}_{j+s-2} (j \geq 2), \\ D_j &= \overline{F}_{-(j+2)} - (j+1)C_{j+2} - \sum_{s=1}^{\infty} L_{-s} \overline{H}_{-(j-s+2)} (j \geq o) \end{aligned}\right\} \quad (19)$$

Thus the form of the function $\phi(z)$ and $\psi(z)$ are determined for the infinite elastic sheet which satisfy the condition at infinity as well as along the rims of the crack L_s.

The potentials $\phi_i(z)$ and $\psi_i(z)$ for inclusion are analytic functions for the region $|z| \leq c$, hence can be written as

$$\phi_i(z) = \sum_{j=o}^{\infty} Y_j z^j, \quad \psi_i(z) = \sum_{j=o}^{\infty} S_j z^j, \quad (20)$$

where the coefficients Y_j and S_j have to be determined by the conditions (11 - 12).

By the conditions (11) and (12), one may obtain

$$\left.\begin{aligned} & C_o + \overline{C}_o - \overline{D}_{-2} c^{-2} = Y_o + \overline{Y}_o, \\ & \left[(\kappa C_o - \overline{C}_o) - \overline{D}_{-2} c^{-2}\right](\mu_i/\mu) = \kappa Y_o - \overline{Y}_o + 4\eta\mu_i/c \end{aligned}\right\} \quad (21)$$

$$\left.\begin{aligned} & C_j c^j + (j+1)\overline{C}_{-j} c^{-j} - \overline{D}_{-j-2} c^{-j-2} = Y_j, \quad (j \geq 1) \\ & \left[\kappa C_j c^j - (j+1) - \overline{C}_{-j} c^{-j} + \overline{D}_{-j-2} c^{-j-2}\right](\mu_i/\mu) = \kappa_i Y_j \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} & C_{-j} c^{-j} - (j-1)\overline{C}_j c^j - \overline{D}_{j-2} c^{j-2} = (1-j)\overline{Y}_j - \overline{S}_{j-2}, \quad (j \geq 2) \\ & \left[\kappa C_{-j} c^{-j} + (j-1)\overline{C}_j c^j + \overline{D}_{j-2} c^{j-2}\right](\mu_i/\mu) = (j-1)\overline{Y}_j + \overline{S}_{j-2} \end{aligned}\right\} \quad (23)$$

When (21 - 23) are solved, the following expressions are obtained

$$Y_j = \frac{\kappa\mu_i/\mu+1}{\kappa_i+1} c^j C_j - \frac{\mu_i/\mu-1}{\kappa_i+1}\left[(j+1)c^{-j} - \overline{C}_{-j} - c^{-j-2}\overline{D}_{-j-2}\right] - \frac{4\mu_i}{(\kappa_i+1)c} \delta_{o,j} \quad (j \geq o), \quad (24a)$$

$$S_j = (j+1)c^{-j-4} D_{-j-4} + c^j D_j - (j+2)^2 c^{-j-2} \overline{C}_{-j-2} (j \geq o), \quad (24b)$$

$$\left.\begin{aligned} C_{-j} &= -\frac{\mu_i/\mu-1}{\kappa\mu_i/\mu+1}\left[(j-1)c^{2j}\,\overline{C}_j + c^{2j-2}\overline{D}_{j-2}\right](j \geq 2), \\ D_{-2} &= \frac{8\mu_i c}{2\mu_i/\mu+\kappa_i-1} - \frac{\kappa\mu_i/\mu-\kappa_i-\mu_i/\mu+1}{2\mu_i/\mu+\kappa_i-1} c^2 (C_o + \overline{C}_o), \\ D_{-j-2} &= (j+1)c^2 C_{-j} - \frac{\kappa\mu_i/\mu-\kappa_i}{\mu_i/\mu+\kappa_i} c^{2j+2}\overline{C}_j (j \geq 1), \end{aligned}\right\} \quad (25)$$

where $\delta_{o,j}$ is Kronecker delta.

Equations (18), (19) and (25) give the following sets of infinite linear equations of H_j and H_{-j}:

$$\sum_{s=o}^{\infty} L_s H_{-(s+1)} = o, \quad (26a)$$

$$\sum_{s=o}^{\infty} L_s H_{-(j+s)} = 1/2\, \frac{\mu_i/\mu-1}{\kappa\mu_i/\mu+1}\left[(j-1)(1-c^2)\left\{\sum_{s=o}^{\infty} L_s \overline{H}_{-(j-s)} - \sum_{s=1}^{\infty} L_{-s} \overline{H}_{j+s}\right\} + \sum_{s=1}^{\infty} L_{-s} H_{-(j-s)} - \sum_{s=o}^{\infty} L_s H_{-(j+s)}\right] c^{2j-2}, \quad (j \geq 2), \quad (26b)$$

$$\sum_{s=1}^{\infty} L_{-s} \overline{H}_{s-1} = o \quad (27a)$$

$$\sum_{s=1}^{\infty} L_{-s} \overline{H}_s = -\frac{4\,c\,\eta\,\mu_i}{2\mu_i/\mu+\kappa_i-1} + 1/2\left\{\sum_{s=o}^{\infty} L_s H_{-s} - \sum_{s=1}^{\infty} L_{-s} H_s\right\} + 1/2\, \frac{\kappa\mu_i/\mu-\kappa_i-\mu_i-\mu_i/\mu+1}{2\mu_i/\mu+\kappa_i-1}\left\{\sum_{s=o}^{\infty} L_s H_{-s} - \sum_{s=1}^{\infty} L_{-s} H_s + \sum_{s=o}^{\infty} L_s \overline{H}_{-s} - \sum_{s=1}^{\infty} L_{-s} \overline{H}_s\right\}, \quad (27b)$$

$$\sum_{s=1}^{\infty} L_{-s}\bar{H}_{j+s} = (j+1)(1-c^2)\sum_{s=o}^{\infty} L_s H_{-(j+s)}$$

$$+ 1/2\,\frac{\kappa\mu_i/\mu-\kappa_i}{\mu_i/\mu+\kappa_i}\left\{\sum_{s=o}^{\infty} L_s \bar{H}_{j-s} - \sum_{s=1}^{\infty} L_{-s}\bar{H}_{j+s}\right\}c^{2j+2}$$

$$(j \geq 1), \qquad (27c)$$

The constants H_i and H_{-j} are determined from the equations (26) and (27) by assigning values to μ_i/μ, κ, K_i, α, c, R. The values of $F_{\pm j}$ are determined from (15) using the values of $H_{\pm j}$ determined earlier. Thus the potentials $\{\phi(z), \psi(z)\}$ are completely known. The coefficients Y_j and S_j are determined by (24) with the help of the values of $H_{\pm j}$ and $F_{\pm j}$ determined previously. Hence $\{\phi_i(z), \psi_i(z)\}$ for inclusion is also known. The stress field for matrix and inclusion can now be determined with the help of (3 - 5).

STRESS INTENSITY FACTORS

By using the definition given by Sih, Paris and Erdogan [3], the stress intensity factors at the crack-tip for the case when the inclusion and crack are concentric can be expressed as $K = (K_1 - iK_2)$.

$$K_1 = \sqrt{\frac{R}{2\sin 2\alpha}}\Bigg[\Big\{H_o^{(i)} + \sum_{j=1}^{\infty}\Big\{H_j^{(r)}-H_{-j}^{(r)}\Big\}\sin j\alpha + \sum_{j=1}^{\infty}\Big\{H_j^{(i)} + H_{-j}^{(i)}\Big\}\cos j\alpha \times \cos\alpha - \Big\{H_o^{(r)} + \sum_{j=1}^{\infty}\Big\{H_j^{(r)}+H_{-j}^{(r)}\Big\}\cos j\alpha - \sum_{j=1}^{\infty}\Big\{H_j^{(i)}-H_{-j}^{(i)}\Big\}\sin j\alpha\Big\}\sin j\alpha\Bigg],$$

$$K_2 = \sqrt{\frac{R}{2\sin 2\alpha}}\Bigg[\Big\{H_o^{(i)} + \sum_{j=1}^{\infty}\Big\{H_j^{(r)}-H_{-j}^{(r)}\Big\}\sin j\alpha + \sum_{j=1}^{\infty}\Big\{H_j^{(i)}+H_{-j}^{(i)}\Big\}\cos j\alpha\Big\}\sin\alpha - \Big\{H_o^{(r)} + \sum_{j=1}^{\infty}\Big\{H_j^{(r)}+H_{-j}^{(r)}\Big\}\cos j\alpha - \sum_{j=1}^{\infty}\Big\{H_j^{(i)}-H_{-j}^{(i)}\Big\}\sin j\alpha\Big\}\cos\alpha\Bigg].$$

Numerical results for the stress intensity factors have been calculated under plane deformation for the following values: $\alpha=10°$, $\mu_i=\mu=5.60\times10^{11}$ gm/cm^2, $\nu_i=\nu=.339$, $c=(1+\epsilon)$, R=1.5(.25)2.5. The results are given in table form.

It is clear from Table 1 that stress intensity factor numerically decreases as radius of the cracks increases.

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Table 1

R = Radius of the Crack	Symmetric Stress Intensity Factor $K_1 = \frac{\text{Symmetric Stress Intensity Factor}}{10^{11} \times \eta}$	Skew Symmetric Stress Intensity Factor $K_2 = \frac{\text{Skew Symmetric Stress Intensity Factor}}{10^{11} \times \eta}$
1.5	-1.00150	-0.04382
1.75	-0.79525	-0.03474
2.00	-0.65105	-0.02843
2.25	-0.54567	-0.02383