WEIBULL STATISTICS FOR A BIAXIAL STRENGTH TEST

D. G. Rickerby*

INTRODUCTION

It is accepted that the strength of glass is limited by the distribution of surface flaws. Despite experimental attempts such as the use of an ion-exchange etching technique [1,2], no direct observation of these flaws has been possible. However, by applying the Griffith fracture criterion, one can estimate the distribution of flaw sizes from the results of fracture tests [3,4]. Reasonable estimates are obtained, but a number of approximations have to be made regarding crack shape and orientation.

An alternative approach is the parametric description of the distribution of fracture stresses postulated by Weibull [5]. Using this theory, a number of workers [6,7] have generated Weibull parameters appropriate to various glass surfaces. The test method usually employed is Hertzian indentation, but the theory is perfectly general and may be extended to other loading configurations. Weibull's approach to the problem was based on phenomenological grounds. It was later pointed out [8] that the Weibull theory is in fact a special case of the statistical theory of extreme values.

EXPERIMENTAL METHOD

A hydraulic testing technique [9] was adopted in the present work. A diagram of the apparatus is shown in Figure 1. The specimen, in the form of a 51 mm diameter glass disc, is held against a steel support ring. A neoprene diaphragm transmits the uniform hydrostatic pressure across the entire surface of the disc. Simple plate theory for a freely supported edge condition gives the following results [10] for radial and tangential components of the stress at radius r:

$$\sigma_r = \frac{3Pa^2}{8t^2}\left[(3+\nu)(1-\frac{r^2}{a^2})\right] \qquad (1a)$$

$$\sigma_t = \frac{3Pa^2}{8t^2}\left[(3+\nu) - (1+3\nu)\frac{r^2}{a^2}\right] \qquad (1b)$$

where σ_r and σ_t are the radial and tangential components of surface stress, P is the applied pressure, a is the test area radius and t is the disc thickness.

Calibration of the apparatus was carried out by means of resistance strain gauges mounted on the specimen surface. A comparison of these results with theory is given in Figure 2. Stress components were meas-

*Cavendish Laboratory, University of Cambridge, U.K.

ured at 5 mm intervals across the specimen, and were found to vary linearly with pressure at all gauge locations. In order to allow for slight variations in t, results are normalised by plotting $\sigma t^2/P$ as the ordinate. Note that r_1 = 23.5 mm corresponds to a, whereas r_2 = 25.5 mm corresponds to the actual disc radius.

A series of fracture experiments was carried out on nominally 3 mm thick discs of Pilkington soda-lime silicate sheet glass ($E = 6.9 \times 10^4$ MPa, $\nu = 0.25$). Specimens were tested to failure at a strain rate $\sim 10^{-4} s^{-1}$. A typical specimen is illustrated in Figure 3. Fracture has initiated from point P, approximately 7 mm from the centre, at a central stress, σ_c = 102 MPa, corresponding to an actual fracture stress σ_f = 95 MPa. A polymeric gasket was used to prevent metal and glass contact and thus reduce edge failures due to mechanical pressure. Both central stress at failure and location of the fracture origin were recorded for each disc tested.

WEIBULL STATISTICAL TREATMENT

For a test specimen under the action of a uniform uniaxial tensile stress, the probability of fracture, F, is given by:

$$F = 1 - \exp(-B) \tag{2}$$

B is the "risk of rupture" and is defined as:

$$B = \int n(\sigma)\, d\omega \tag{3}$$

where $d\omega$ represents an elemental solid angle and $n(\sigma)$ is the flaw distribution function, which takes the form:

$$n(\sigma) = \left(\frac{\sigma}{\sigma_o}\right)^m \tag{4}$$

where σ is the applied stress and σ_o is a scaling factor. The quantity m is termed the "Weibull exponent". A more general expression includes the possibility of a threshold stress value, σ_u, below which fracture cannot occur. This being the case, equation (4) becomes:

$$n(\sigma) = \left(\frac{\sigma - \sigma_u}{\sigma_o}\right)^m \tag{5}$$

The "Weibull parameters" - σ_u, σ_o and m - completely define the failure probability of the material under a given stress.

In order to extend the analysis to a non-uniform biaxial stress distribution we require the result [5]:

$$k_b = \frac{2m + 1}{2\pi} k \tag{6}$$

where $k = (1/\sigma_o)^m$ and subscript b indicates the biaxial case. The resultant stress in a given direction at a point at radius r from the centre of the disc is:

$$\sigma = (\cos^2\phi)(\sigma_r \cos^2\psi + \sigma_t \sin^2\psi) \tag{7}$$

and $d\omega = \cos\phi\, d\psi\, d\phi$, where ψ is the angle between the direction of σ and σ_x and ϕ is the angle of orientation of σ with respect to the surface (i.e. $\phi = 0$ at the surface). Initially we assume a two parametric distribution (i.e. $\sigma_u = 0$). Then the "risk of rupture" per unit area for an element surrounding such a point is:

$$\Delta B = 2k_b \int_0^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \cos^{2m+1}\phi\,(\sigma_r \cos^2\psi + \sigma_t \sin^2\psi)^m d\phi\, d\psi \tag{8}$$

Substituting from equations (1a) and (1b) we may rewrite (8) in terms of the central stress, σ_c:

$$\Delta B = \frac{2m+1}{\pi}\left(\frac{\sigma_c}{\sigma_o}\right)^m \int_0^{\frac{\pi}{2}} \cos^{2m+1}\phi d\phi . \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - \frac{1}{2}\frac{r^2}{a^2}(\alpha + \beta\cos 2\psi)\right]^m d\psi \tag{9}$$

where

$$\alpha = 1 + \left(\frac{1 + 3\nu}{3 + \nu}\right) \text{ and } \beta = 1 - \left(\frac{1 + 3\nu}{3 + \nu}\right).$$

For integral values of m, this expression can be evaluated exactly. Experimentally determined values of m are generally non-integral and a numerical method is therefore required to evaluate the integrals.

$$\text{Writing} \quad C(m) = (2m + 1) \int_0^{\frac{\pi}{2}} \cos^{2m+1}\phi d\phi \tag{10}$$

and integrating over the entire specimen to find the total "risk of rupture":

$$B = \int_0^a \Delta B . 2\pi r\, dr \tag{11}$$

which becomes, on substituting for ΔB and putting $r/a = x$:

$$B = 2\pi a^2 \frac{C(m)}{\pi}\left(\frac{\sigma_c}{\sigma_o}\right)^m \int_0^1 \left\{\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \left[1 - \frac{1}{2}x^2(\alpha + \beta\cos 2\psi)\right]^m d\psi\right\} x dx \tag{12}$$

The double integral is evaluated numerically. Its value, I(m), is of form $\pi.D(m)$, so that:

$$B = 2\pi a^2\, C(m)\, D(m)\left(\frac{\sigma_c}{\sigma_o}\right)^m \tag{13}$$

A full three parametric Weibull distribution would be extremely complicated to evalute exactly as there would be variable limits on the integrals which would be dependent on σ_c, since this determines which areas of the specimen are below the threshold stress level. Such areas make no contribution to B and consequently should not be included in the integration. Clearly, as σ_c increases so does the effective area under test. As a first approximation we rewrite equation (13) as:

$$B = 2\pi a^2 C(m) D(m)\left(\frac{\sigma_c - \sigma_u}{\sigma_o}\right)^m \quad (14)$$

This approximation implies that the threshold stress is not a constant, but varies over the specimen with the same functional dependence as the resultant stress component due to σ_r and σ_t, σ_u being its value at the centre of the disc. Thus, the entire surface is included in the integration, so that the approximation improves as σ_c increases and more of the disc is actually under test. Provided that the smaller principle stress component, σ_r, is greater than σ_u the approximation that all areas are under test is valid. The stress components fall off slowly with radius over the first 15 mm, where the majority of fractures initiate, so that for even small values of σ_c most of the surface is above a stress level of σ_u. In addition, the error introduced by assuming σ_u varies across the disc is small by virtue of the slowly varying nature of σ_r and σ_t.

Substituting (14) into (2) we obtain:

$$F = 1 - \exp\left[- 2\pi a^2 C(m) D(m)\left(\frac{\sigma_c - \sigma_u}{\sigma_o}\right)^m\right] \quad (15)$$

Rearranging slightly and taking logarithms twice, this becomes:

$$\ell n\, \ell n \frac{1}{1 - F} = \ell n\, 2\pi a^2 + \ell n\, C(m) + \ell n\, D(m) + m\ell n(\sigma_c - \sigma_u) - m\ell n\, \sigma_o \quad (16)$$

A plot of $\ell n\, \ell n\, 1/1 - F$ against $\ell n(\sigma_c - \sigma_u)$ yields a straight line of gradient m and an intercept from which σ_o may be calculated since C(m) and D(m) may be evaluated numerically once m has been determined. From a set of experimentally obtained fracture data we obtain a series of values of σ_c at failure. The probability of fracture is calculated from [11]:

$$F = n / (N + 1) \quad (17)$$

where n is the number of a given specimen for a total of N specimens ranked in order of increasing strength.

In order to fit the appropriate set of Weibull parameters to the data, a "best fit" is obtained for the experimental points to a line of the form of equation (16). Trial values of σ_u are chosen until a "correlation coefficient" [12] is maximised. The corresponding values of m and σ_o are defined by the slope and the intercept respectively.

RESULTS

The "best fit" Weibull plot is illustrated in Figure 4, and yields the following parameters: σ_u = 48 MPa, σ_o = 7.1 MPa and m = 3.1. Using these values, a curve may be computed from equation (15) illustrating the variation of F with σ_c (Figure 5). This shows excellent agreement with the cumulative histogram of experimental results which is superimposed on the same axes.

In addition to a distribution of fracture stresses, we have also a distribution of fracture origins, representing additional information available from the experimental results. In evaluating I(m) numerically a summation must be carried out across a radius of the specimen. By dividing the cumulative total at a given value of r by the total value of the integral, the cumulative probability of fracture at a given disc radius may be found. Figure 6 illustrates the theoretical prediction together with the experimentally obtained distribution of fracture origins. The discrepancy between the two, which is greater at larger radii, is due to the approximation made in equation (14). The result of this is that the theory overestimates fracture probabilities in the outer region of the disc.

CONCLUSION

The Weibull statistical theory of fracture has been extended to a biaxial non-uniform stress distribution for the two parametric case. The three parametric distribution cannot be easily treated analytically, but an approximation is made so that Weibull parameters may be obtained for this case also. These parameters are used to predict experimental distributions of central stress at failure and of fracture origins. The particular loading configuration described here is a desirable one since edge failures are reduced to 10-20%, compared with laths tested in three or four point bending which usually fail from flaws at the edge of the specimen induced by the cutting process. It has also proved useful in the investigation of reduced strength due to circularly symmetric, induced surface flaws (e.g. Hertzian ring cracks, liquid impact damage) [9].

ACKNOWLEDGEMENTS

The author wishes to thank Dr. J. E. Field and Dr. D. A. Gorham for useful discussions. The work has been supported by the SRC and the Ministry of Defence (Procurement Executive).

REFERENCES

1. ARGON, A. S., Proc. Roy. Soc. A250, 1959, 472.
2. ERNSBERGER, F. M., Proc. Roy. Soc. A257, 1960, 213.
3. ARGON, A. S., Proc. Roy. Soc. A250, 1959, 482.
4. WILSHAW, T. R., J. Phys. D, 4, 1971, 1567.
5. WEIBULL, W., Ingvetenskakad. Handl. No. 151, 1939.
6. OH, H. L. and FINNIE, I., J. Mech. Phys. Solids, 15, 1967, 401.
7. HAMILTON, B. and RAWSON, H., J. Mech. Phys. Solids, 18, 1970, 127.
8. EPSTEIN, B., J. Appl. Phys. 19, 1948, 140.
9. GORHAM, D. A. and RICKERBY, D. G., J. Phys. E. 8, 1975, 794.

10. TIMOSHENKO, S. P., Strength of Materials (Part II, 2nd Ed.), Macmillan, 1941, 140.
11. GUMBEL, E. J., Nat. Bur. Stand., Appl. Math. Series 33, 1954.
12. FREUND, J. E., Mathematical Statistics (2nd Ed.), Prentice-Hall, 1971, 358.

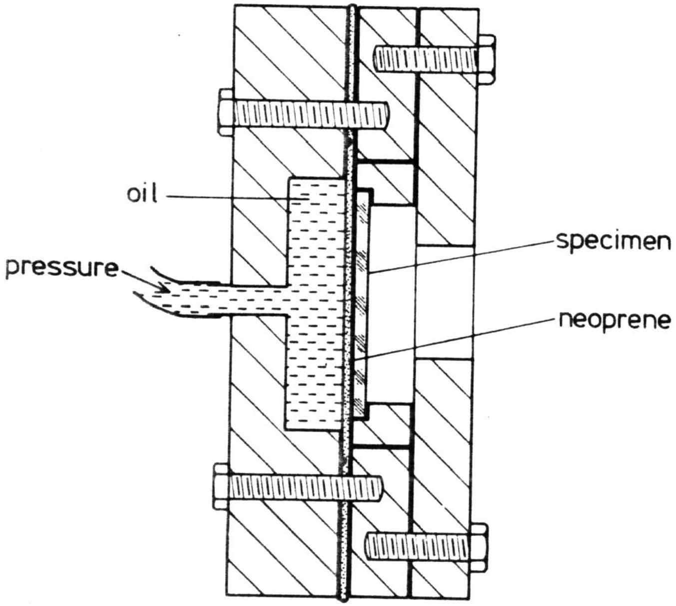

Figure 1 The hydraulic pressure tester

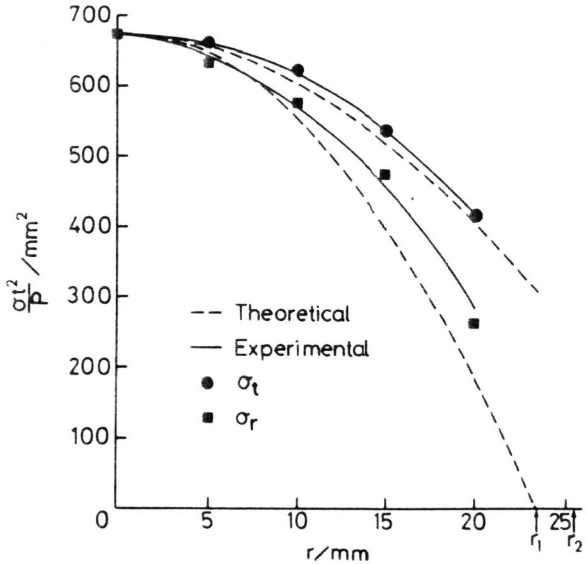

Figure 2 Comparison of calibration data with theory

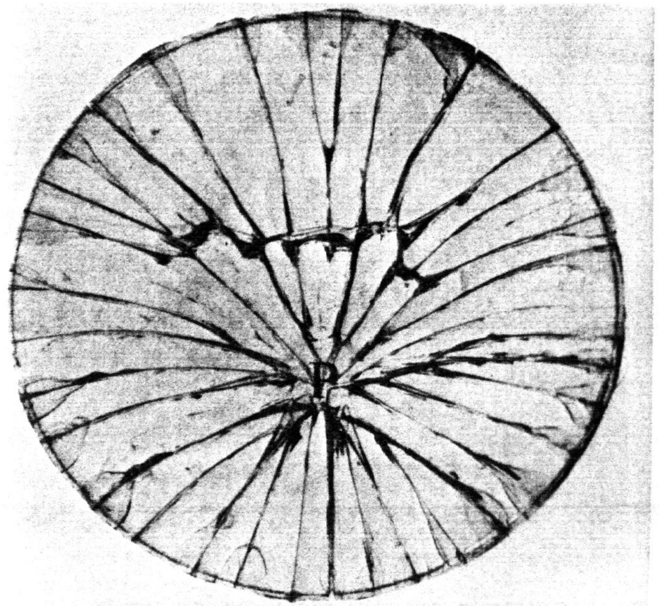

Figure 3 A typical specimen after test

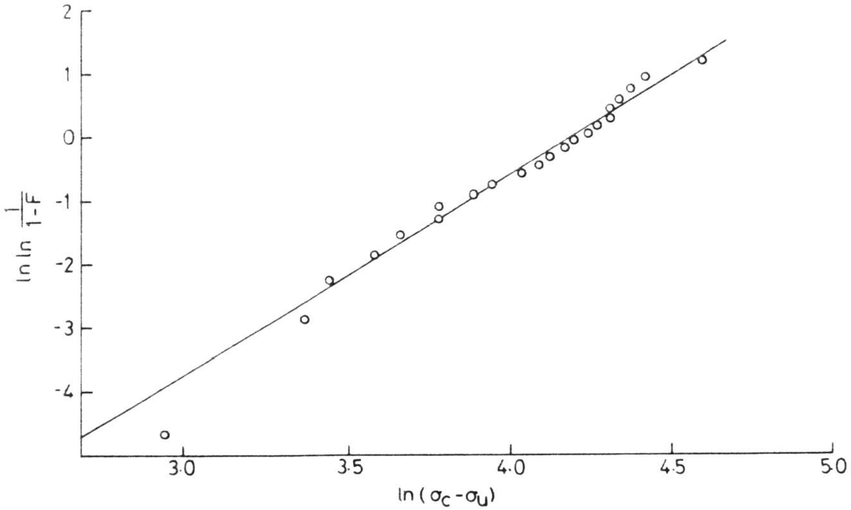

Figure 4 "Best-fit" of Weibull parameters to test data

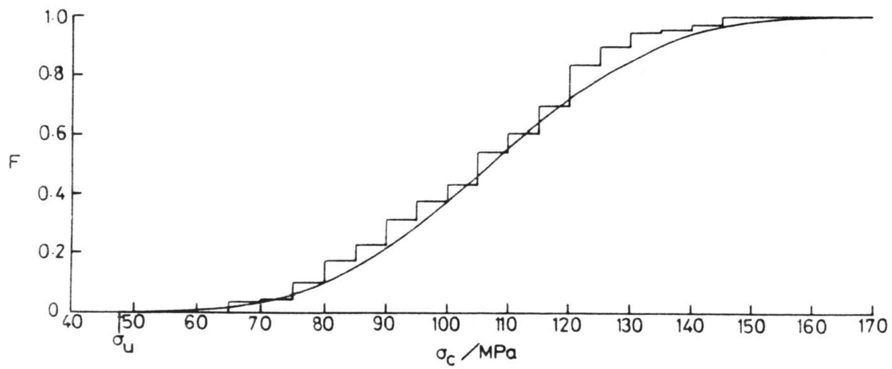

Figure 5 Distribution of central stresses at failure

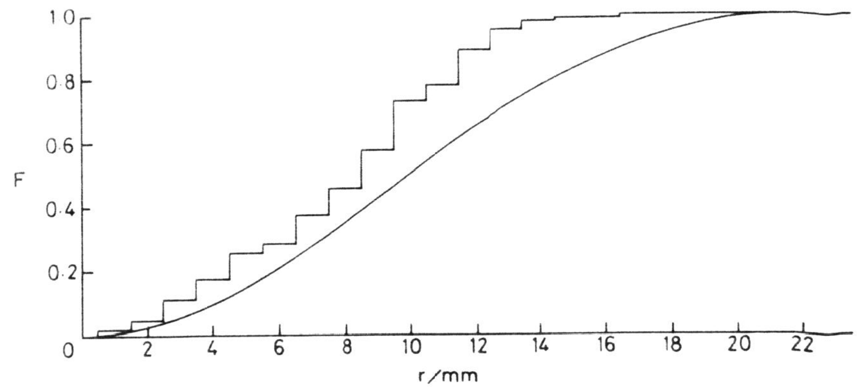

Figure 6 Distribution of fracture origins