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USE OF THE WEIGHT FUNCTION CONCEPT AND THE CRACK CLOSING METHOD FOR CALCULATING STRESS INTENSITY FACTORS IN PLANE OR AXISYMMETRIC PROBLEMS

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INTRODUCTION

It is becoming increasingly necessary to use linear elastic fracture mechanics and stress intensity factor calculations in engineering. This paper explains why the weight function concept and the crack closing method have been selected to perform these calculations. Description is also given of the methods used in the calculations, in cases where the shape of the cracked structure and the loading pertains to plane or axi-symmetric problems. Relevant practical experience is summarized.

DEFINITION OF WEIGHT FUNCTIONS

The weight function concept was introduced by Bueckner [1, 2] in relation to a calculation method based on boundary integral equations. The concept is a specific application of Betti's theorem to the singular stressfield in the crack tip vicinity. Other expressions based on this concept have been given by Rice [3]. Applications of the weight function concept and of the finite element method have been studied by Labbens, Pellissier-Tanon and Heliot [4] and Paris and McMeeking [5].

For a crack subjected to a mode I pressure $\sigma(x)$, the stress intensity factor can be calculated when the weight function $m(x/a, a/B)$ is known for the crack geometry [4]:

$$K = \sqrt{\frac{2}{\pi}} \int_0^a \frac{m(\frac{x}{a}, \frac{a}{B})}{\sqrt{a-x}} \sigma(x) dx \qquad (1)$$

In the case of circumferential crack in a hollow cylinder with an inside radius R_i:

$$K = \sqrt{\frac{2}{\pi}} \int_0^a \frac{m(\frac{x}{a}, \frac{a}{B})}{\sqrt{a-x}} \frac{R_i + x}{R_i + a} \sigma(x) dx \qquad (2)$$

See Figure 1.

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CRACK CLOSING METHOD FOR CALCULATION OF STRESS INTENSITY FACTORS AND WEIGHT FUNCTIONS

A given structure with a crack subjected to a pressure $\sigma(x)$ is represented by a finite element mesh with a rather long crack (Figure 2). The structure is computed by a finite element method, where the pressure $\sigma(x)$ is replaced by a series of forces applied to the crack nodes $F_1, F_2 \ldots F_n$. The finite element calculation yields in particular the crack opening displacement $(U_1, U_2 \ldots U_n)$ at these points. In the finite element crack configuration, the displacements U_i are called $U_i^{(0)}$, and are calculated successively using n loadings corresponding to n unit forces F_i individually considered, in order to obtain a matrix $C_{ij}^{(0)}$ such that:

$$U_i^{(0)} = C_{ij}^{(0)} F_j \tag{3}$$

The structure strain energy is:

$$W^{(0)} = \frac{1}{2} \sum_{i=1}^{n} U_i^{(0)} F_i = \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij}^{(0)} F_i F_j \tag{4}$$

The crack is required to close over a length equal to the length of the last element of the crack. For that purpose, the force F_n is calculated as function of F_i $(i = 1, n - 1)$ by setting $U_n = 0$. The displacements U_i are modified and they become $U_i^{(1)}$ such as:

$$U_i^{(1)} = C_{ij}^{(1)} F_j$$

The matrix $C_{ij}^{(1)}$ can be calculated from $C_{ij}^{(1)}$ by using a Gaussian elimination. A new strain energy is then determined:

$$W^{(1)} = \frac{1}{2} \sum_{i=1}^{n-1} U_i^{(1)} F_i = \frac{1}{2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} C_{ij}^{(1)} F_i F_j$$

An expression for G or K is derived from the above expressions for W:

$$G = \frac{dW}{dA} \simeq \frac{W^{(0)} - W^{(1)}}{dA} \qquad K = \sqrt{\frac{\varepsilon}{1-\nu^2} G} \tag{5}$$

The calculation can be continued by gradually closing the crack until the length is zero and, using the $C_{ij}^{(0)}$ matrix, K can be determined for any crack length and any pressure $\sigma(x)$. The method can be used also when the loading is not applied to the crack, but to the whole cracked structure. In this case, the F_i forces are the forces which must be applied to shut the crack completely. In particular, if the loading is simply a local force $F_r = 1$

$$G = \frac{dW}{dA} = \frac{1}{2} \frac{C_{rr}^{(0)} - C_{rr}^{(1)}}{dA} \tag{6}$$

Hence the weight functions are deduced, as described in [4]. From a single finite element calculation, it is theoretically possible to calculate the $m(x/a, a/B)$ function in a range such as $0 \leq x/a < 1$, $0 < a/B < 1$. The

method was used with 300 nodes and 30 loadings requiring 30 seconds on a large computer, such as the CDC/7600

ACCURACY OF WEIGHT FUNCTION CALCULATION

The calculation of $m(x/a, a/B)$ using the method described above is very accurate when the length $(x-a)$ is not too small, and corresponds at least to the width of two elements of the mesh (when using elements with 12 degrees or more of freedom). Labbens [4] has shown, that knowing the limit value of m which is 1 when x/a tends to 1 [1, 2, 3], m could be readily defined by interpolation or by fitting in the vicinity of $x/a = 1$. Good approximations of m are thus obtained for any x when for example $0.06 \leq a/B \leq 0.94$; m remains to be determined for low values of a/B. The following observation is used for this purpose.

When $a/b \to 0$, the weight functions of any structure are those of a single edge crack in a semi-infinite strip. These weight functions have been calculated by Bueckner [2]:

$$m\left\{\frac{x}{a}, 0\right\} = 1 + 0.6147\left\{1 - \frac{x}{a}\right\} + 0.2502\left\{1 - \frac{x}{a}\right\}^2 \tag{7}$$

So it is possible to accurately compute the weight functions with respect to a given geometry in the range:

$$0 \leq \frac{x}{a} \leq 1 \qquad 0 \leq \frac{a}{B} \leq 0.94$$

A computer programme was set up using the method presented in section 3 in the most general case, and these observations where taken into account to improve the accuracy of the calculations. The comparisons made between the results obtained for a single edge-cracked strip and those of Bueckner [2] show that the maximum deviation is 4%.

CALCULATION OF STRESS INTENSITY FACTORS USING WEIGHT FUNCTIONS

For a given stress distribution $\sigma(x)$ and a given crack length a, the corresponding value of K can be calculated by two different methods:

a) by directly applying formula (1) or (2).

b) the crack closing method can be directly applied to the stress distribution $\sigma(x)$.

Method b) is approximately equivalent to determining K with weight functions, calculated by a finite element programme without correction for the real limit of $m(x/a, a/B)$ when $x \to a$. Method a) uses the corrected weight functions. In practice, both methods yield the same results for relatively long cracks $a/B > 0.3$, but method a) is more accurate for small cracks, which is not surprising. See specific comparison made in Figure 3.

CALCULATION OF K USING A POLYNOMIAL FITTED PRESSURE

In many cases, the $\sigma(x)$ pressure can be approximated by a polynomial, as performed by Buchalet and Bamford [6] or Raoul and Vagner [7].

$$\sigma(x) = \sigma_0 + \left\{\frac{x}{B}\right\}\sigma_1 + \left\{\frac{x}{B}\right\}^2 \sigma_2 + \ldots\ldots\ldots + \left\{\frac{x}{B}\right\}^m \sigma_m \tag{8}$$

with (1): $$K = \sqrt{\pi a}\;\sigma_0\left[i_0\left\{\frac{a}{B}\right\} + \left\{\frac{a}{B}\right\}\sigma_1\, i_1\left\{\frac{a}{B}\right\} + \ldots\ldots\ldots + \left\{\frac{a}{B}\right\}^m \sigma_m\, i_m\left\{\frac{a}{B}\right\}\right] \tag{9}$$

and $$i_j\left\{\frac{a}{B}\right\} = \frac{1}{\pi}\sqrt{\frac{2}{a}}\int_0^a \left\{\frac{x}{a}\right\}^j \frac{m(\frac{x}{a},\frac{a}{B})}{\sqrt{a-x}}\,dx \tag{10}$$

i_j can be accurately calculated by using method a) and (10). For that purpose, a polynomial approximation of m is determined and the value of i_j can be calculated exactly or by using an appropriate numerical method. The limit values of $i_j(a/B)$ when $a/B \to 0$ are the same for all cracks in all structures, as the limit situation, when a becomes small compared to B and to the radius of curvature of the surface of the body, is that of a crack in a semi infinite half plane.

These limit values can be calculated using (10) and (7)

Limit values of i_j when $a/B \to 0$

j	0	1	2	3	4
i_j (0)	1.12	0.687	0.528	0.446	0.389

Curves giving i_0 for several geometrical configurations have been plotted in Figure 3. They indicate that $i_0(a/B)$ is strongly dependent on the geometry. Curves have been plotted in Figures 4 and 5, giving $i_j(a/B)$, ($j = 0,3$) for circumferential and axial cracks in cylinders.

CONCLUSIONS

In two-dimensional problems, the crack closing method permits determination of K for several crack dimensions with only one finite element calculation.

The m weight function can thus be determined for a given geometry by a single finite element calculation. This is inexpensive and yields the weight functions for any crack length. Having the functions m, K can be determined under arbitrary loading by calculating a simple integral expression. The use of fitted weight functions improves the accuracy of the calculation of K.

For practical purposes, when the variation of the normal stress distribution along the path of crack can be expressed as a polynomial, it is convenient to use influence functions calculated by integrating for each unit power term of the polynomial.

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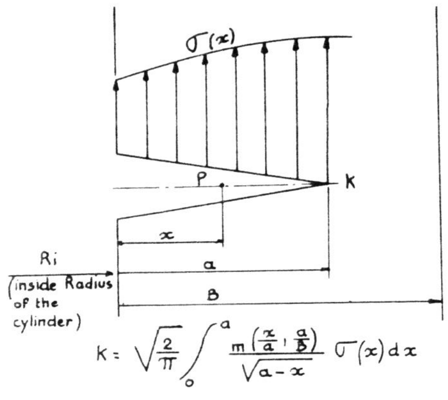

Figure 1 Definition of m

$(\frac{x}{a}, \frac{a}{B})$

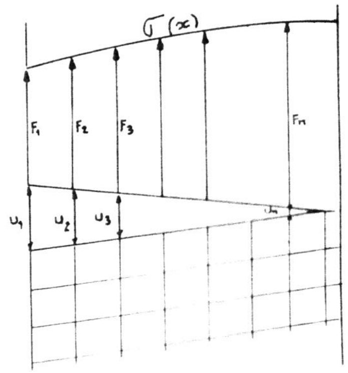

Figure 2 Crack Closing Method

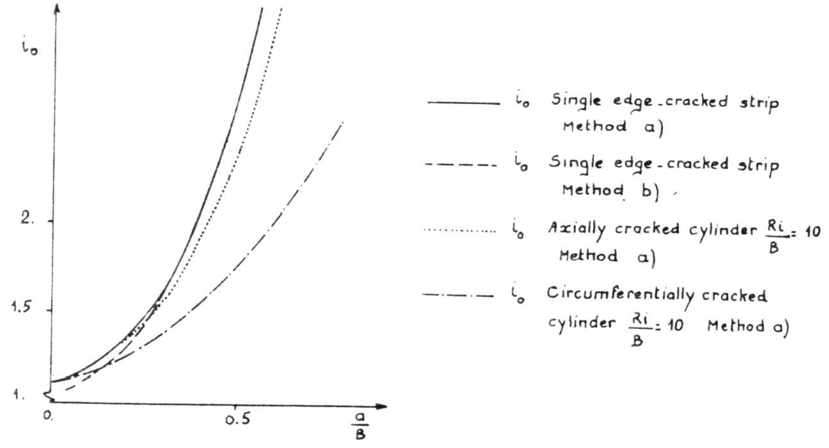

Figure 3 i_o as a Function of $\frac{a}{B}$ in Difference Cases

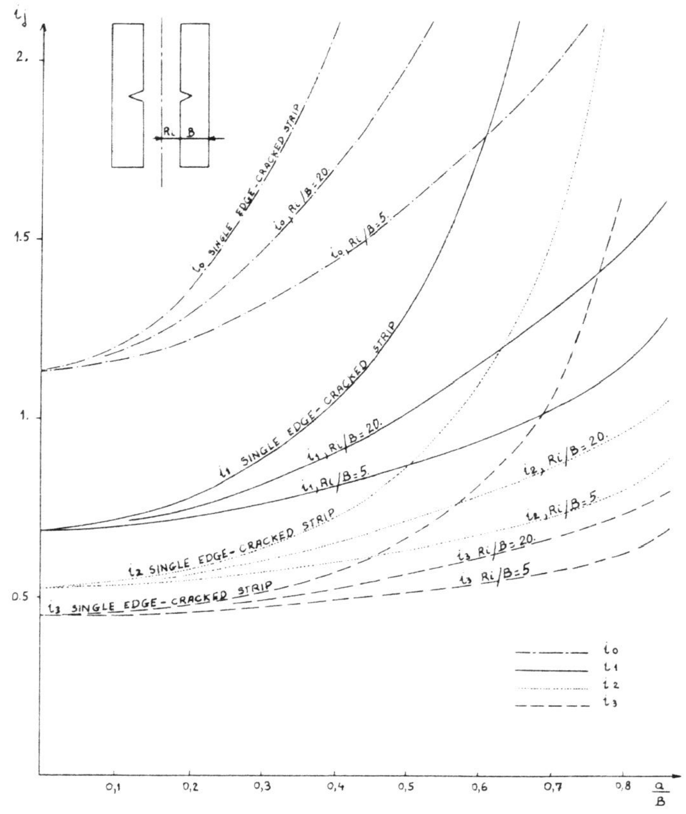

Figure 4 Values of i_j (j = 0.3) Circumferentially Cracked Cylinder and Single Edge-Cracked Strip

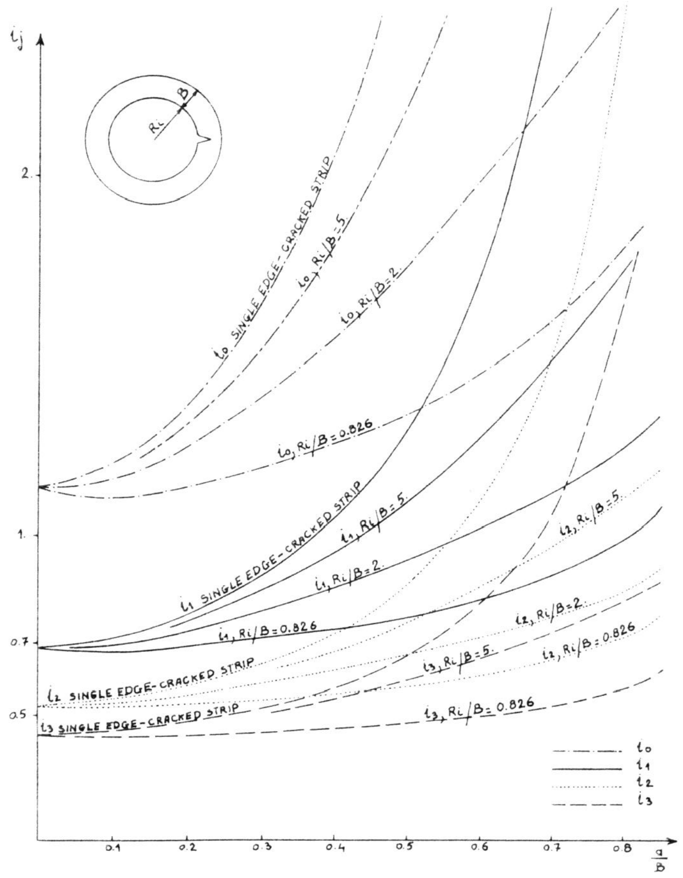

Figure 5 Values of i_j (j = 0.3) Axially Cracked Long Cylinder and Single Edge-Cracked Strip