POROUS RIGID-PLASTIC MATERIALS CONTAINING RIGID INCLUSIONS -
YIELD FUNCTION, PLASTIC POTENTIAL, AND VOID NUCLEATION

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INTRODUCTION

Development of theories of ductile fracture require constitutive relations which show details of material behaviour. Engineering materials frequently are aggregates of several phases of widely varying properties, making the constitutive behaviour of an aggregate more complex than that of any of the individual phases. In this paper, the effect on constitutive behaviour of the presence of rigid particles, embedded in and bonded to a rigid-plastic porous matrix, is examined. It is shown that the yield function is altered, and that the familiar concept of the yield function as a plastic potential must be used more carefully. The results also show how a void nucleation mechanism could destabilize, causing rapid bulk softening and failure [1,3].

YIELD FUNCTION AND PLASTIC POTENTIAL

Strictly speaking, a yield function defines a locus of points in stress space for which a body, at a certain state of plastic deformation, will attain plastic yield. A plastic potential is a function of stress (at yield) which gives the direction of plastic flow via normality. In the conventional theory of rigid-plastic flow, the yield function Φ is used as a plastic potential [2]:

$$\underline{\dot{\varepsilon}} = \Lambda \frac{\partial \Phi}{\partial \underline{\sigma}}, \tag{1}$$

where $\underline{\sigma}$ is the microscopic (pointwise) stress tensor, $\underline{\varepsilon}$ is the microscopic deformation rate tensor, and Λ is a scalar multiplier determined from boundary conditions or hardening data. It can be shown [3,4] that this concept carries over to macroscopic measures of stress ($\underline{\Sigma}$) and deformation rate ($\underline{\dot{E}}$) in a wide class of porous rigid-plastic materials:

$$\underline{\dot{E}} = \Lambda \frac{\partial}{\partial \underline{\Sigma}} \left[\Phi \left(\sigma_y, \, \underline{\Sigma}, \, f \right) \right], \tag{2}$$

where σ_y, the equivalent tensile flow stress of the matrix, is assumed (for simplicity) to be uniform. The theory which led to equation (2) will be extended here to include the effects of void nucleation at rigid particles embedded in the matrix. Special attention is paid to a modification of the stress-dependent nucleation criterion developed by Argon, et al. [6].

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Consider the general case of a rigid, work hardening ductile matrix containing both voids and rigid particles, the latter debonding from the matrix when some critical stress is reached at the particle-matrix interface. One can then write the general expression

$$df = df(nucl.) + df(growth) \equiv b_{ij}d\Sigma_{ij} + b_3^f d\Lambda \qquad (3)$$

f is void volume fraction; d indicates an increment, and b_{ij} and b_3^f are differential coefficients. $df(nucl.)$ is the part of df due to void nucleation at rigid particles; $df(growth)$ is the part of df due to void growth, and is related to macroscopic plastic dilatancy. (See equation (22)). Consider also a yield function for this material, approximated as dependent on f, σ_y, and the first two invariants of $\underline{\Sigma}$:

$$\Phi = \Phi\left(\Sigma_{eqv}, \Sigma_H, \sigma_y, f\right) = 0 \qquad (4)$$

$$\Sigma_H = \frac{1}{3}\Sigma_{kk}, \quad \Sigma'_{ij} = \Sigma_{ij} - \Sigma_H\delta_{ij}, \quad \Sigma_{eqv} = \left(\frac{3}{2}\Sigma'_{ij}\Sigma'_{ij}\right)^{1/2}$$

(Figure 1 contains a schematic of this type of yield function; specific examples were developed in references [3,4]. One is shown in equation (11).) It is a reasonable approximation to limit the stress dependence of df to the first two invariants of stress also:

$$df = b_1^f d\Sigma_{eqv} + b_2^f d\Sigma_H + b_3^f d\Lambda, \qquad (5)$$

where b_i^f, $i = 1, 3$ are differential coefficients. For a work hardening material, one may write

$$d\sigma_y = \frac{d\sigma_y}{d\Lambda} d\Lambda \qquad (6)$$

Now, using equations (4) and (5) and applying consistency to the yield function (commas denote partial differentiation):

$$\delta\Phi = 0 = \left(\Phi,_{\Sigma_{eqv}} + \Phi,_f b_1^f\right)d\Sigma_{eqv} + \left(\Phi,_{\Sigma_H} + \Phi,_f b_2^f\right)d\Sigma_H$$
$$+ \left(\Phi,_{\sigma_y}\sigma_{y,\Lambda} + \Phi,_f b_3^f\right)d\Lambda \qquad (7)$$

When moving along the yield function of a work hardening material, no plastic flow takes place, so $d\Lambda = 0$. Equation (7) then gives the tangent to the yield function as

$$\frac{d\Sigma_{eqv}}{d\Sigma_H} = -\frac{\left(\Phi,_{\Sigma_H} + \Phi,_f b_2^f\right)}{\left(\Phi,_{\Sigma_{eqv}} + \Phi,_f b_1^f\right)} \qquad (8)$$

Using the derivation of a flow rule in references [3,4], and assuming that void nucleation affects only the rate of change of $\dot{\varepsilon}$ and not $\dot{\varepsilon}$ itself, it can be shown that the plastic potential is equivalent to the yield function with nucleation ignored. The tangent to the plastic potential is

thus

$$\frac{d\Sigma_{eqv}}{d\Sigma_H} = -\frac{\Phi,_{\Sigma_H}}{\Phi,_{\Sigma_{eqv}}} \qquad (9)$$

Thus, with nucleation as a mechanism for increasing f, the yield function is no longer strictly equivalent to a plastic potential.

In a specific form of equation (5) (developed in [3] and discussed below),

$$b_1^f = 0, \ b_2^f = 0 \text{ if } d\Sigma_H < 0, \ b_2^f > 0 \text{ if } d\Sigma_H > 0 \qquad (10)$$

Nucleation (the breaking of a matrix-particle bond) is considered irreversible, and cannot take place when the increment of hydrostatic stress is negative.

In [3,4], specific forms of the yield function were developed which did not take nucleation into account. These can be used to calculate the derivatives in equations (8) and (9). An example is given below; it is an upper bound yield function for a ductile matrix containing spherical voids, with the assumptions of a fully plastic flow field in the matrix (upper bound calculation), and uniform σ_y:

$$\Phi = \frac{\Sigma_{eqv}^2}{\sigma_y^2} + 2f\cosh\left(\frac{3}{2}\frac{\Sigma_H}{\sigma_y}\right) - 1 - f^2 = 0 \qquad (11)$$

The following are true for equation (11), and are reasonable to expect from any other forms of Φ:

$$\Phi,_f > 0, \ \Phi,_{\Sigma_{eqv}} > 0, \text{ and } \Phi,_{\Sigma_H} > 0 \text{ for } \Sigma_H > 0 \qquad (12)$$

Using equations (10) and (12) in equation (8), the slope of the yield function is shown to be equal to ($d\Sigma_H < 0$) or more negative than ($d\Sigma_H > 0$) that of the plastic potential. Thus, given a state of yield in stress space, the yield surface will lie on or inside the plastic potential. (See Figure 1.) Only the portion of the yield function for which $d\Sigma_H < 0$ can be used as a plastic potential.

NUCLEATION CRITERIA

Two specific void nucleation criteria were investigated in [3], and will be outlined here. The first, based on the experimental work of Gurland [5] on particle cracking in spheroidized 1.05% carbon steel, favours a criterion based on the strain increment. When a cracked particle is presumed to behave like a void of equal size, and f_{up} is the volume fraction of unbroken particles, the criterion is

$$df(nucl.) \cong C_1^* f_{up} d\bar{E}, \text{ where } \bar{E} = \left(\frac{2}{3}E'_{ij}E'_{ij}\right)^{1/2} \qquad (13)$$

C_1^* is a material constant calculated from the experimental data; Gurland's data gave $C_1^* = 0.29$.

The second criterion is based on the work of Argon et al. [6], who favour a critical normal interfacial (matrix-particle) stress condition for nucleation. Their calculations show that for a single rigid particle, embedded in an infinite matrix undergoing plastic flow under pure shear stress, the normal interfacial stress is very close to σ_y in the matrix. (Note: here, σ_y is the current value, increased by work hardening.) They also study the stress amplifying effect of a second particle in close proximity to the first. Based on their work, the following form is suggested [3]:

$$\sigma_i = M(c)\ \sigma_y + \frac{\Sigma_H}{1-f} \qquad (14)$$

$M(c)$ is a stress amplification factor (a function of the local particle concentration c), and σ_i is the normal interfacial stress. The final term represents, in an approximate way, the microscopic tensile stress at the interface due to Σ_H. Matrix-particle separation (void nucleation) results when σ_i reaches $\sigma_i{}^c$, the critical value. (Note: reference [7] shows that $\sigma_y{}^* \leq \sigma_i{}^c \leq 6\ \sigma_y{}^*$, where $\sigma_y{}^*$ is the initial tensile yield stress of the matrix, is a reasonable range of values for engineering materials.) $M(c)$ is calculated as the ratio of upper bound values of the interfacial stress for interacting particles with local concentration c to the upper bound stress when no interaction takes place. Calculations of the upper bound stresses were based on [6], as modified in [3]. (In [3], the plastic drag (k_s) at a displacement incompatibility is a work hardening quantity; in [6], it is a material constant.) Curves of $M(c)$ versus c are shown in Figure 2 for various values of n, the matrix hardening exponent.

Define c* as the value of c at which $\sigma_i{}^c$ is reached (equation (14)). A statistical calculation is done in [6], resulting in the fraction of randomly distributed particles which are in local concentration of c* or more. Assuming that a separated particle behaves like a void of equal volume, the following incremental form results (via Leibnitz's rule for differentiation of definite integrals).

$$df(\text{nucl.}) = -\ \frac{0.4412}{\Gamma\left(\dfrac{c^*}{\bar{c}} + 1\right)}\ dc^*, \qquad (15)$$

where Γ is the Gamma function and $\bar{c}$ is the volume average of c. c* decreases as σ_y, the average hardening state of the matrix, increases. (See equation (14) with $\sigma_i = \sigma_i{}^c$, and Figure 2.) Increments in σ_y can be estimated from the macroscopic behaviour of the material, the current state, and the matrix properties:

$$d\bar{\varepsilon} = \frac{\Sigma_{ij}\ d\Sigma_{ij}}{\sigma_y(1-f)},\ \ d\sigma_y = \frac{d}{d\bar{\varepsilon}}\Big(\sigma_y(\bar{\varepsilon})\Big)^* d\bar{\varepsilon},\ \ \bar{\varepsilon} = \left(\frac{2}{3}\ \varepsilon_{ij}'\varepsilon_{ij}'\right)^{1/2} \qquad (16)$$

Given $d\sigma_y$, dc* can be calculated numerically from equation (14), where σ is kept constant at $\sigma_i{}^c$:

$$d(\sigma_i{}^c) = 0 = \frac{dM}{dc}\bigg|_{c^*}\cdot\ \sigma_y\ dc^* + M(c^*)\ d\sigma_y + d\left(\frac{\Sigma_H}{1-f}\right) \qquad (17)$$

The last term above must be expanded in detail. Using the non-dimensional macroscopic stress $\underline{S}$, where

$$\underline{S} = \underline{\Sigma}/\sigma_y \qquad (18)$$

the chain rule gives

$$d\left(\frac{\Sigma_H}{1-f}\right) = \frac{S_H}{(1-f)}\cdot\ d\sigma_y + \frac{\sigma_y}{(1-f)}\cdot\ dS_H + \frac{\sigma_y\ S_H}{(1-f)^2}\cdot\ df \qquad (19)$$

Given enough information about a particular problem, dS_H can be put in terms of df. Then, equation (17) can be written as

$$-\frac{dM}{dc^*}\cdot\frac{\Gamma(c^*/\bar{c}+1)}{0.4412}\cdot df(\text{nucl}) + \frac{\sigma_y}{1-f}\left(\frac{dS_H}{df} + \frac{S_H}{1-f}\right)\Big(df(\text{nucl})+df(\text{growth})\Big)$$
$$+ M(c^*)\ d\sigma_y = 0 \qquad (20)$$

$d\sigma_y$ and df(growth) are both linear functions of the flow parameter. (See equations (1), (16), and (22).) Rearranging terms gives

$$df(\text{nucl})\cdot\left[\frac{dM}{dc^*}\frac{\Gamma(c^*/\bar{c}+1)}{0.4412} - \frac{\sigma_y}{1-f}\left(\frac{dS_H}{df} + \frac{S_H}{1-f}\right)\right] = \left(----\right)d\Lambda \qquad (21)$$

NUCLEATION INSTABILITY

Nucleation becomes unstable if the coefficient of df(nucl.) goes to zero. The burst of nucleation would cause instantaneous bulk softening, which under the proper circumstances [1,3] could lead to an unstable macroscopic flow field bifurcation (ductile fracture). Note that unstable nucleation of the type suggested in equation (21) could not occur given the flow dependent nucleation mechanism of equation (13). (A similar type of instability, corresponding to a macroscopic non-hardening state [1,3], could still occur.)

df(growth) can be expressed in terms of dE as follows:

$$df(\text{growth}) = (1-f)\ dE_{kk},\ = (1-f)\ \Phi,_{\Sigma_{kk}}\ d\Lambda \qquad (22)$$

In Figure 3, this is compared to df(nucl.) over a range of matrix hardening levels, for several values of $\sigma_i{}^c$. The instability suggested in equation (21) is quite evident. Φ as used in the calculation of these curves was derived in [3,4] for a long circular cylindrical void geometry. The form is similar to equation (11). Note that quantities on the vertical axis are normalized by f and $\bar{c}$.

The effect of nucleation on the slope of the yield function can be seen from Figure 4, for a specific case of equation (11). Note that here, $\sigma_i{}^c$ is normalized by σ_y, the current state of matrix hardening. As Σ_H increases, the value of $M(c^*)$ (and thus c*) needed to attain nucleation decreases, until $M(c^*) \to 1$ and nucleation takes place at all particles not yet debonded from the matrix. df(nucl.), as calculated in equation

(21), again becomes unstable. When this happens, b_2^f in equation (8) becomes infinite and the slope of the yield function becomes negative infinite, as suggested in the figure.

ACKNOWLEDGEMENTS

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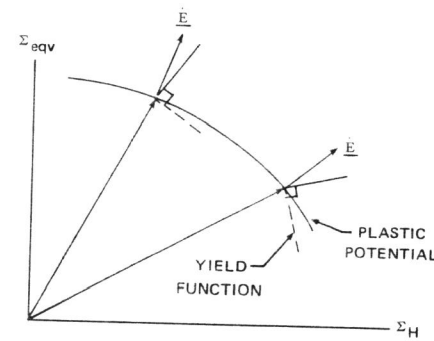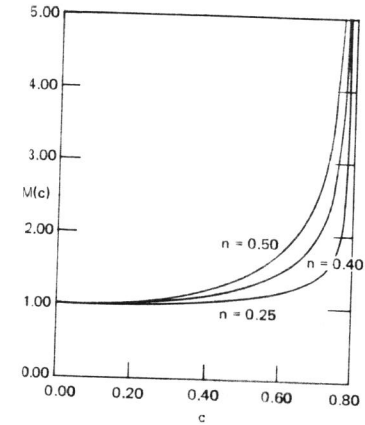7. EASTERLING, K. E., FISCHMEISTER, H. F. and NAVARA, E., "The Particle-to-Matrix Bond in Dispersion-Hardened Austenitic and Ferritic Iron Alloys", Powder Metallurgy 16, 1973, 128-145.
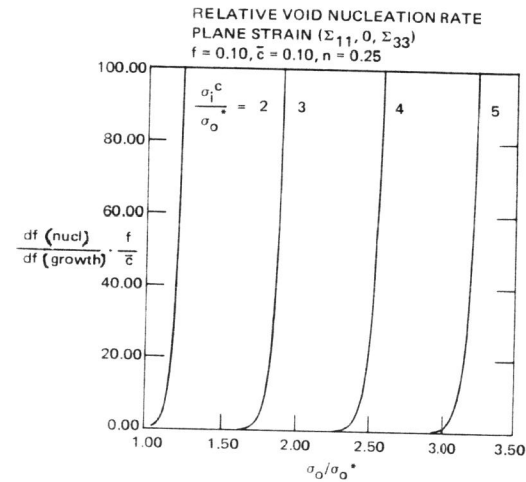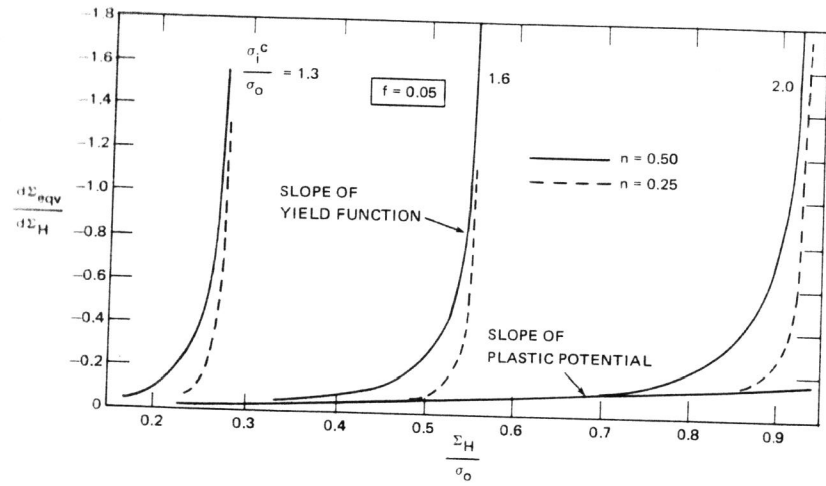
Figure 1

Figure 2

Figure 3

Figure 4