AN APPROXIMATE THREE-DIMENSIONAL STATE OF STRESS IN THE VICINITY OF A CRACK

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INTRODUCTION

The classical linear fracture mechanics solutions of the crack-tip stresses are based on two-dimensional formulation of the stress field under either plane stress or plane strain conditions. For contained plasticity the slip-line field is not enough for predicting the plastic zone size. Besides numerical treatment, several papers presented asymptotic analysis of the stress singularity at the crack tip [1, 2, 3]. Three-dimensional solutions are difficult and not available at the present time. The object of this paper is to study the effect of the plate thickness on the stress distribution in the plastic zone of a through crack with the assumption of an ideally plastic material.

MATHEMATICAL ANALYSIS

Consider the stress function defined by

$$F = \frac{1}{2} r^2 f(\theta) \ . \tag{1}$$

The stress components defined by equation (1) are [4]

$$\left.\begin{aligned} \sigma_r &= \frac{1}{r}\frac{\partial F}{\partial r} + \frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} = f + \frac{1}{2} f'' \ , \\ \sigma_\theta &= \frac{\partial^2 F}{\partial r^2} = f \ , \\ \tau_{r\theta} &= - \frac{\partial^2}{\partial r \partial \theta}\left(\frac{F}{r}\right) = - \frac{1}{2} f' \ . \end{aligned}\right\} \tag{2}$$

If one specifies that the octahedral shearing stress τ_o defined by [4]

$$9\tau_o^{\,2} = (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 \tag{3}$$

to be constant, one will get the classical solution of the plastic stress field at the crack tip. In the above expression σ_1, σ_2, σ_3 are principal.

To construct an approximate solution in three dimensions let us assume a set of two Maxwell function [5]

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$$X = Y = \psi H \quad \text{and} \quad Z = \phi F \tag{4}$$

where H and F are functions of x and y only, ψ and ϕ are functions of x, y and z except that these functions vary so slowly with x and y that their derivatives with respect to x and y can be ignored. The resulting stress components are given by the following:

$$\left.\begin{array}{ll} \sigma_x = \phi F_{yy} + \psi'' H\,, & \tau_{yz} = -\psi' H_y \\ \sigma_y = \phi F_{xx} + \psi'' H\,, & \tau_{zx} = -\psi' H_x \\ \sigma_z = \psi \nabla^2 H\,, & \tau_{xy} = -\phi F_{xy} \end{array}\right\} \tag{5}$$

where the subscripts represent partial differentiation and the primes denote differentiations with respect to z.

If we further introduce the assumption that the functions F and H are identical, then the above stress components will be reduced to four remaining components; namely,

$$\left.\begin{array}{ll} \sigma_x = \phi\,\bar\sigma_x\,, & \sigma_y = \phi\,\bar\sigma_y\,, \\ \sigma_z = \psi(\bar\sigma_x + \bar\sigma_y)\,, & \tau_{xy} = \phi\,\bar\tau_{xy}\,, \end{array}\right\} \tag{6}$$

where $\bar\sigma_x = F_{yy}$, $\bar\sigma_y = F_{xx}$, $\bar\sigma_{xy} = -F_{xy}$. (7)

It is important to note that the above set of stresses do not satisfy the equations of equilibrium exactly, but the quantities neglected will be small by assumption.

The task now is to suggest a function F that will yield a fan-shaped plastic region in the xy-plane. In this region the octahedral stress τ_o will be constant. This function F can be taken to be the same classical solution that solves the plane-strain case in the region $0 \le |\theta| \le \pi/4$ and a new solution outside the region. (See Figure).

Thus, for $0 \le |\theta| \le \frac{\pi}{4}$,

$$f(\theta) = \frac{p+q}{2} - \frac{p-q}{2}\cos 2\theta \tag{8}$$

and for $0 \le |\alpha| \le \frac{3\pi}{4}$, $(\alpha = \pi - \theta)$

$$f(\alpha) = \frac{p+q}{2} - \frac{3q}{4}\cos\frac{2\alpha}{3} + \frac{p-2p}{4}\cos 2\alpha \tag{9}$$

where p and q are the stresses at $\theta = 0$.

It can be verified that these solutions satisfy all the boundary conditions and at $\theta = \pi/4$ all the stress conditions are matched.

By using equation (6) we can calculate τ_o as

$$9\tau_o^2 = (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = \frac{1}{2}(\phi-2\psi)^2(\bar\sigma_x+\bar\sigma_y)^2 + \frac{3}{2}\phi^2 V \tag{10}$$

where V is defined by

$$V = (\bar\sigma_x - \bar\sigma_y)^2 + 4\,\bar\tau_{xy}^2\,. \tag{11}$$

For the case of plane strain $\phi = 2\psi$, we have

$$9\tau_o^2 = \frac{3}{2}\phi^2 V \qquad (\phi = 1)\,, \tag{12}$$

and for plane stress, $\psi = 0$, thus

$$9\tau_o^2 = 2\phi^2(\bar\sigma_x^2 - \bar\sigma_x\bar\sigma_y + \bar\sigma_y^2 + 3\bar\tau_{xy}^2)\,. \tag{13}$$

If we denote the value of ϕ at $z = 0$ by ϕ_m and that at $z = \pm c$ by ϕ_s and equate the two τ_o values, we obtain from equations (12) and (13) the ratio of the two ϕ-values as

$$\left(\frac{\phi_m}{\phi_s}\right)^2 = \frac{4}{3}\left[1 + \frac{p/q}{(1-p/q)^2}\right] \tag{14}$$

The ϕ-function is further restricted by

$$\int_{-c}^{c} \phi\, dz = 2c \tag{15}$$

The ψ-variation can also be estimated based on the condition τ_o being constant. The more specific form of ϕ and ψ can be suggested based on experimental results available in the literature, but will not be the subject of this paper.

CONCLUSION

An approximate three-dimensional solution of the state of stress in the vicinity of a crack has been suggested. Although these stresses do not satisfy the equilibrium equations exactly, the error is expected to be small for moderately thick plate. Variations of the stresses in the thickness are defined by two functions which can be estimated.

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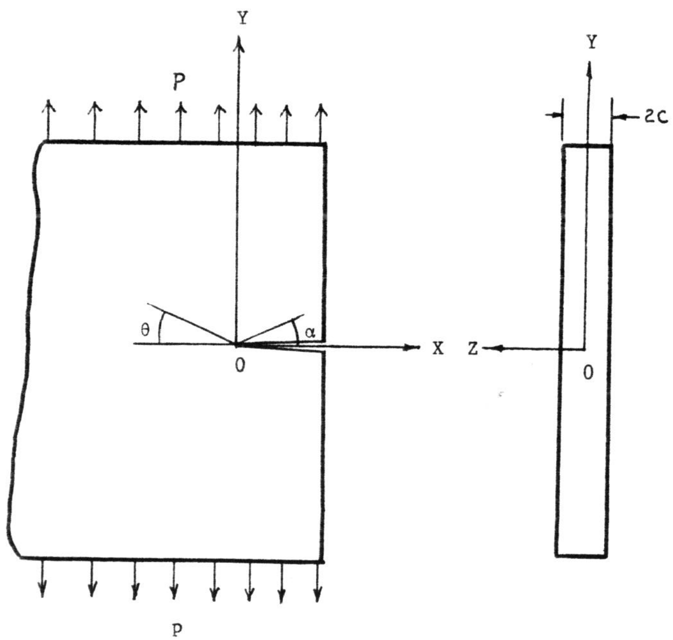

Figure 1 Plate with Finite Thickness with Crack Under Tension