Fracture 1977, Volume 3, ICF4, Waterloo, Canada, June 19 - 24, 1977

A SIMPLE ANALYTICAL MODEL FOR THE THICKNESS DEPENDENT TRANSITION FROM PLANE STRESS TO PLANE STRAIN IN BODIES WITH A CRACK

G. Prantl*

INTRODUCTION

From experimental work an influence of the dimensions of specimens containing cracks on their fracture behaviour is apparent. While there is a large amount of data on size effects, and there are empirically derived criteria for the dimensions of laboratory test specimens used to obtain results applicable to full scale structures, the quantitative understanding of the problem and of the effect of the various parameters is not yet perfect. A possible consequence of increasing the dimensions from small specimens to large structures is a transition from a plane stress to a plane strain state of deformation in the critical region of the material and a connected transition in fracture mode.

The purpose of the present paper is to construct a simple analytical model, using continuum mechanics principles, which is able to describe the influence of specimen thickness on the stress state in the neighbourhood of a crack, assuming linear elastic material. For the more important practical case of elastic plastic material behaviour, where yield zones develop, some limited conclusions can be drawn on the basis of an analogy between the elastic and the elastic plastic distribution of the in-plane stresses in front of the crack.

Although it is possible to calculate the stresses and strains in a body of finite dimensions using three dimensional finite element techniques, the present work is justified, because it allows the effect of the various parameters of the problem (e.g., thickness, external load, Poisson's constant, crack length) to be estimated in a direct closed form.

DEVELOPMENT OF AN ELASTIC MODEL

The work described here is restricted to the following class of problems: A disc, made of linear elastic material, contains a crack penetrating through the thickness. The dimensions of the disc in its main plane are large compared to the length of the crack, 2c. The thickness 2t can take any value between zero and infinity. The body is loaded by uniformly distributed tensile stresses, σ_∞, perpendicular to the plane of the crack. Figure 1 explains the coordinates used, as well as the notation of the stresses.

In the vicinity of the crack tip, there is a concentration of the stresses σ_r and σ_ϕ, which is known from the solution of the respective plane problem (1), if t/c is either approaching zero or infinity. In the first case $\sigma_z \equiv 0$, in the second case $\sigma_z = \ (\sigma_r+\sigma_\phi)$. In real bodies σ_z is within

* Swiss Federal Institute for Reactor Research, CH-5303 Würenlingen, Switzerland

MS229

these limits.

It is assumed for the sake of argument, that the disc is divided into two zones, as shown in Figure 2. The inner zone I contains the material under relatively high stresses, while the ringshaped outer zone II includes the rest of the body within the region of the stress concentration. In each of the two zones the sum of the stresses $\sigma_r+\sigma_\phi$ is averaged.

If the zones were free to deform laterally, each would suffer a contraction, caused by the average stresses in it. This individual deformation of each zone results in a misfit at their common boundary. Before they are reassembled, the difference in lateral contraction must be cancelled. This can be done by the application of shear stresses $\tau_{I,II}$ with the proper magnitude and distribution to the common boundary of the zones. Still considering the separated zones, it is noticed that these shear stresses, which are necessary to maintain the continuity of the whole body, induce tensile stresses σ_z in zone I and compressive stresses in zone II. It is assumed in the model, that the stresses σ_z are uniformly distributed over the cross section of the respective zone.

The calculated stresses σ_{Iz} and σ_{IIz} are mean values over the cross section of the respective zone, and therefore depend on the zone size chosen. For the mathematical formulation of this idea, the following conditions are used:

- Equilibrium of forces in direction z on zone I alone.
- Equilibrium of forces in direction z on zone I and zone II together.
- A compatibility relation at the common boundary of the two zones, formulated with a series expansion with respect to r of the displacement in direction z.
- Hooke's law.
- Boundary conditions for σ_z at the median plane and at the surfaces of the disc.

In order to simplify the treatment of the problem, the cross sections of the zones are assumed to be circular. The radius of inner zone I is 'a', the radius of outer zone II is 'b', (see Figure 2).

Using the conditions listed leads to a differential equation for the average stress σ_{Iz} in zone I:

$$\sigma_{Iz}'' - \lambda^2\sigma_{Iz} = -\lambda^2 B(z) . \qquad (1)$$

The double prime denotes the second derivative of σ_{Iz} with respect to z. The parameter λ contains the dimensions of the zones and Poisson's constant ν:

$$\lambda = \sqrt{\frac{3}{2(1+\nu)}}\,\frac{1}{\sqrt{a(b-a)}} \qquad (2)$$

B(z) is a function, which consists of the difference of the stresses σ_r and σ_ϕ between the zones, and a term that depends on the radial deformation at the common boundary of the zones. While the first part can be determined from the distribution of the stresses σ_r and σ_ϕ, the second part remains unknown.

The boundary conditions are:

$$\sigma_{Iz}'(0) = 0 , \quad \sigma_{Iz}(t) = 0 , \quad \sigma_{Iz}'(t) = 0 . \qquad (3)$$

The prime denotes the first derivative of σ_{Iz} with respect to z. The solution of equation (1), using the first two boundary conditions (3), is:

$$\sigma_{Iz} = \lambda\left\{\left[-\int_z^t B(z)\sinh\lambda z\,dz + \text{tangh}\,\lambda t\int_0^t B(z)\cosh\lambda z\,dz\right]\cosh\lambda z - \sinh\lambda z\int_0^z B(z)\cosh\lambda z\,dz\right\} . \qquad (4)$$

Equation (4) does not satisfy the third boundary condition. The reason is that equation (1), which results from the simplified one dimensional analysis, is of the second order.

The complete solution, accounting for the radial stresses at the borders of the zones too, would be a differential equation of the fourth order.

In order to introduce the retroaction of σ_z on the stresses σ_r and σ_ϕ, a specific function is proposed for B(z), which makes it possible, to satisfy this boundary condition too:

$$B(z) = \sigma_{Iz_{max}}\left(1 - \frac{2\cosh\lambda z}{\cosh\lambda t + \frac{\lambda t}{\sinh\lambda t}}\right) . \qquad (5)$$

Here, $\sigma_{Iz_{max}}$ is the greatest possible value of σ_{Iz} for the problem under consideration, namely:

$$\sigma_{Iz_{max}} = \nu(\Delta\sigma_r + \Delta\sigma_\phi)\cdot\left(1 - \frac{a^2}{b^2}\right) . \qquad (6)$$

The differences of the stresses between the two zones, $\Delta\sigma_r$ and $\Delta\sigma_\phi$, can be determined from the respective stress distribution. The form of equation (5) is indicated by still unpublished photoelastic measurements, made by the author. Further, it is based on the assumption, that the z-distribution of the in-plane stresses and deformations is also governed by the parameter λ.

Inserting equation (5) into equation (4) results in the final equation for σ_{Iz}, the average stress in zone I:

$$\frac{\sigma_{Iz}}{\sigma_{Iz_{max}}} = 1 - \frac{(\lambda t\cosh\lambda t+\sinh\lambda t)\cosh\lambda z-\lambda z\sinh\lambda t\sinh\lambda z}{\sinh\lambda t\cosh\lambda t + \lambda t} . \qquad (7)$$

From equation (7) the average stress in zone II, σ_{IIz}, and the shear stress $\tau_{I,II}$ at the common boundary can easily be deduced, if necessary.

If the thickness of the disc is large compared to the dimension b, it is also large compared to the half crack length c, because b and c are of the same order of magnitude. In this special case equation (7) reduces to:

$$\frac{\sigma_{Iz}}{\sigma_{Iz_{max}}} \simeq 1 - (1+\lambda\xi)\; e^{-\lambda\xi} \;. \qquad (8)$$

In this equation the distance ξ from the surface is used instead of z, where $\xi = t-z$.

APPLICATION OF THE ELASTIC MODEL

Before the stresses can be calculated, the input data must be derived from the distribution of the in-plane stresses. The parameter λ, equation (2) and the stress $\sigma_{Iz_{max}}$, equation (6) are determined using the two dimensional solution of the crack problem (1). The results are presented in Figure 3 as a function of the relative zone size a/b.

The mean stresses within the two zones and the parameter λ are strongly influenced by a/b. The dimension 'b', according to the model, defines that part of the body which is able to constrain the material in zone I. It is specific to the particular problem treated and it can be estimated from the distribution of the in-plane stresses. It is reasonable to assume 'b' proportional to the fading distance of the perturbation of the stress field. In this work, b = 0.5 c is used. This is an arbitrary choice, suited to the application of Sneddon's elastic solution. The dimension 'a', which fixes the size of zone I, is varied in the numerical analysis. If 'a' approaches zero, then σ_{Iz} becomes infinitely large and the ratio $\sigma_{Iz}/\nu(\sigma_r+\sigma_\phi)_I$ approaches unity. With this data, equation (8) is evaluated. It yields the distribution of σ_{Iz} over the thickness coordinate ξ, according to Figure 4.

Figure 5 illustrates the influence of the choice of 'a', with 'b' kept constant, as determined by the distribution of σ_r and σ_ϕ. The condition of equilibrium of the forces in direction z, formulated on zone I and zone II together, requires that the integral of σ_z over any cross section of the two zones must vanish. Therefore tension in zone I causes compressive stresses in zone II. Finally, Figure 6, derived from equation (7), gives an example of the rise of the out-of-plane stress σ_{Iz} in the mid plane with increasing relative thickness t/c.

TREATMENT OF PROBLEMS WITH PLASTIC ZONES

When real materials are considered, plastic zones are found ahead of the crack tip, even at very small loads. According to the crack model of Dugdale [2] for instance, perfect plastic behaviour of the material confines the stress within the yield zone to the uniaxial yield stress 's'. As this crack model gives no information on the magnitude of the stresses σ_r and σ_ϕ within the plastic zone except at the y-axis ahead of the crack tip, evaluation of the input data, needed to calculate σ_{Iz} analogous to the elastic case, is not possible.

But, setting the radius 'a' of zone I equal to the length of the yield zone according to the Dugdale model enables some conclusions to be drawn

concerning the effect of the external load on σ_{Iz}. To our present knowledge, the length of the plastic zone varies through the thickness of the body. The same should reasonably be assumed for the dimension 'a', but the model can deal only with a constant 'a'. As long as the radius 'a' is smaller than or equal to the radius of the plastic zone, it has a very small influence on the stress difference $(\Delta\sigma_r+\Delta\sigma_\phi)$, because the stress is constant within the plastic zone. Therefore the effect of the external load on a given body, which determines the plastic zone size, can be investigated separately from the choice of the radius 'a'.

Figure 7 compares λ and $(\Delta\sigma_r+\Delta\sigma_\phi)$ with the respective values of the elastic case. The term $\lambda t/t/c\,\sqrt{3/2(1+\nu)}$ is the ratio between the effective thickness and the geometrical thickness of the body. The smaller this term, the smaller is σ_{Iz}. The term $(s-1,2\sigma_\infty)/\sigma_\infty$ is the analogue of the elastic stress difference $(\Delta\sigma_r+\Delta\sigma_\phi)\sigma_\infty$. The smaller its value, the smaller is the maximum stress, $\sigma_{Iz_{max}}$, that can be achieved with a sufficiently large thickness. Using this data, Figure 8 finally demonstrates the fall off of σ_{Iz} with rising external load, when the dimensions of the body remain constant.

It should be pointed out again, that this kind of treatment of elastic-plastic problems cannot be expected to give absolute values of the stress σ_{Iz}. It only indicates the effect of external load on the stress σ_z in a specimen of given dimensions.

ACKNOWLEDGEMENTS

The permission of the Swiss Federal Institute for Reactor Research (EIR) and the Swiss Federal Institute of Technology Zürich (ETHZ) to publish this work, which is part of a dissertation submitted to ETHZ, is acknowledged.

REFERENCES

1. SNEDDON, I. N., Proc. Roy. Soc., A 187, 1946, 229.
2. DUGDALE, D. S., J. Mech. Phys. Solids, 8, 1960, 100.

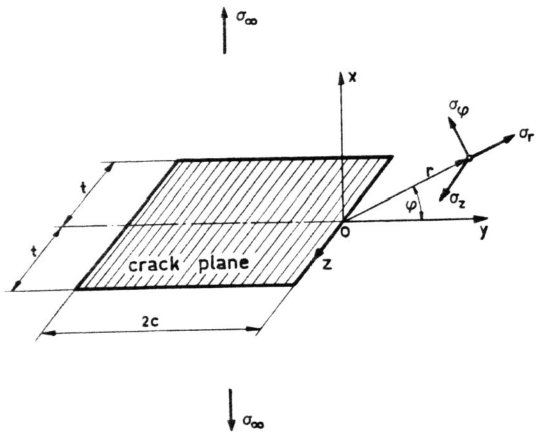

Figure 1 Dimensions, Coordinates and Notation of Stresses

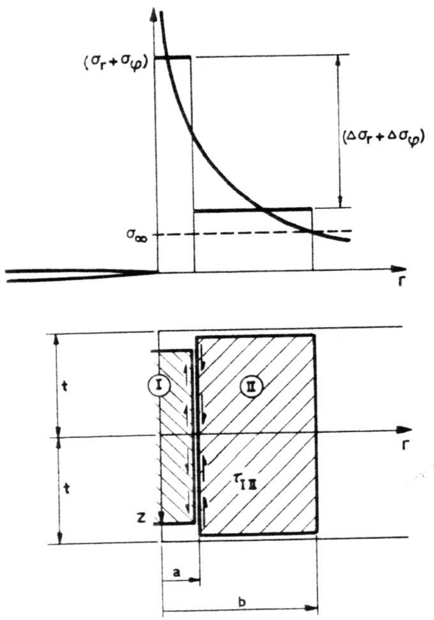

Figure 2 Illustration of the Proposed Model. Half Cross Section of the Separated Zones

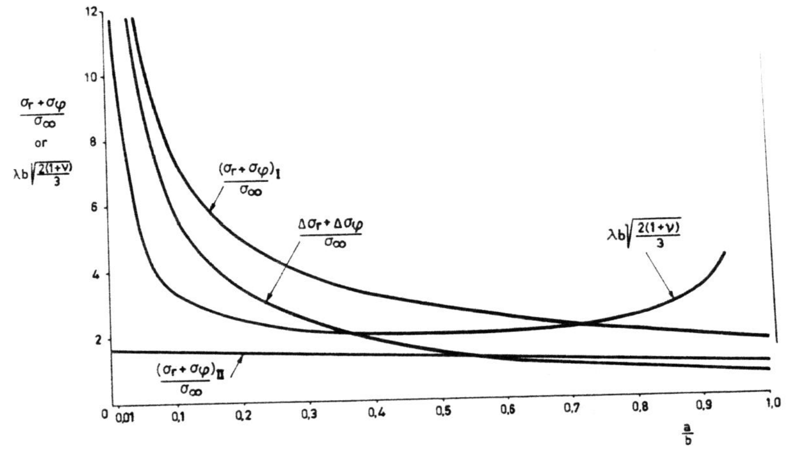

Figure 3 Average In-Plane Stresses Within the Zones and Parameter λ for a Crack in a Large Disc

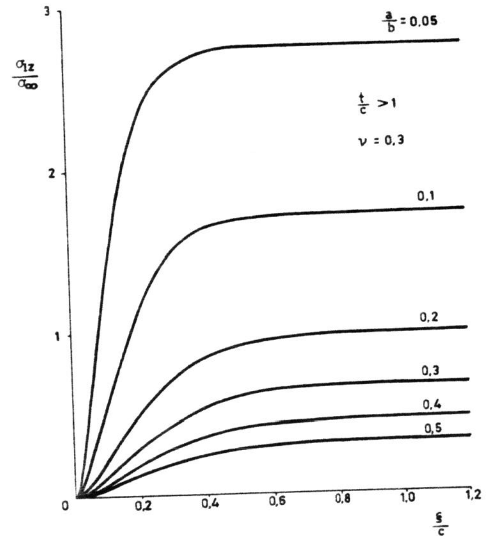

Figure 4 Distribution of σ_{Iz} through the Thickness for Various Values of the Relative Zone Size a/b

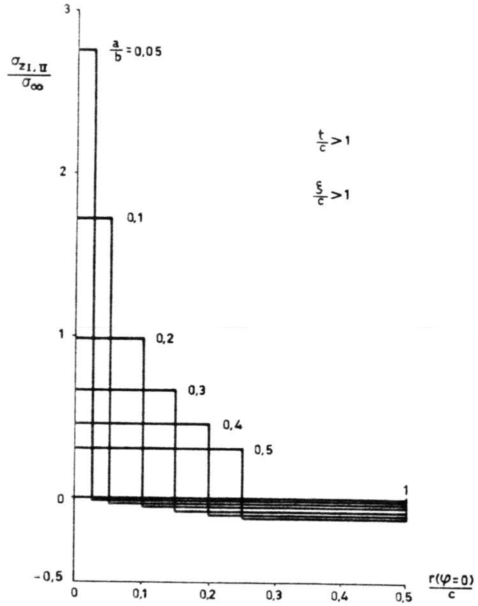

Figure 5 Average σ_z in Zone I and Zone II for Various Values of a/b

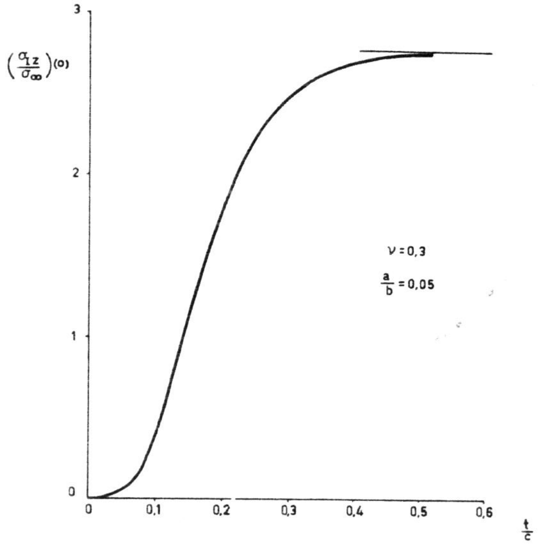

Figure 6 σ_{Iz} in the Mid-Plane as a Function of Relative Thickness

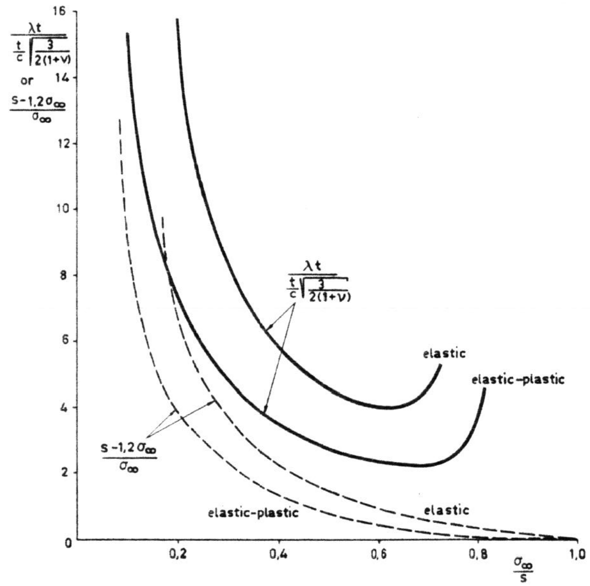

Figure 7 $(\Delta\sigma_r+\Delta\sigma_\phi)$ and λ as a Function of Load for Elastic and Elastic-Perfectly Plastic Material

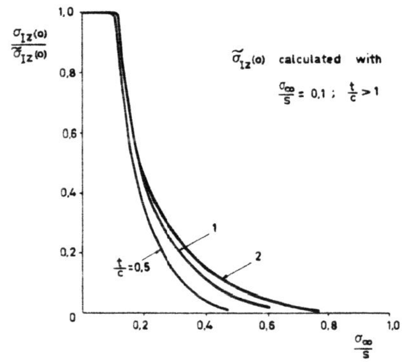

Figure 8 σ_{Iz} at the Median Plane as a Function of Load for Various Relative Thicknesses