Fracture 1977, Volume 3, ICF4, Waterloo, Canada, June 19 - 24, 1977

A PATH INDEPENDENT INTEGRAL FOR SYMMETRIC STRESS-DIFFUSION FIELDS SURROUNDING LINE CRACKS

E. C. Aifantis* and W. W. Gerberich**

INTRODUCTION

A careful thermomechanical analysis of crack propagation in continuous media led Cherepanov [1] to propose an integral form as a general fracture criterion. For slow crack growth, and in the absence of heat flux and body forces, this integral form is identified as the J-integral. This was independently discovered and popularized by Rice [2] as an outgrowth of the work of Eshelby [3], for calculating the path-independent fracture toughness of cracked metal sheets subjected to an elasto-plastic stress field. Knowles and Sternberg [4] discovered two more path independent integrals by applying Noether's theorem to the theory of linear elastostatics. Aifantis [5] extended the above results to generate conservation laws for linear isotropic stress fields in the presence of body forces derived from harmonic potentials.

The results of the present investigation may be considered as a partial answer to the question: Do path independent integrals exist when, in addition to the stress field, a diffusion field is present?

At first glance, it appears natural to attack the problem by modelling microscopically the change of the energetics of the solid-matrix, due to the motion of the diffusing species. But the uncertainty involved in specifying the details of the elastic interaction energy, suggests a continuum mechanics treatment. Thus, diffusion effects are taken into account by postulating the existence of an internal diffusion force which is properly introduced into the equation of motion to describe the exchange of momentum between the solute and solvent atoms. A steady state diffusion is considered and a simplified model for the diffusion force is adopted. The kinematics of the diffusing species are restricted by the principles of mass and momentum balance and their mechanical response is modelled with a constitutive law [6] for an elastic fluid. The mechanical response of the solid is determined within the theory of linear isotropic elastostatics.

We consider symmetric configurations for both the stress and solute density fields surrounding the line crack. The conservation law that we discover is independent of symmetry considerations but the path-independence is particularly sensitive to symmetry arguments.

Thus, for the cases under consideration, we derive a path-independent integral which includes terms due to diffusion. If we assume that the diffusion effects are negligible at infinity, then the first component of this integral, evaluated at infinity, is reduced to the familiar J-integral [2].

* Assistant Professor of Theoretical and Applied Mechanics, University of Illinois at Urbana-Champaign, Illinois, U. S. A.

** Professor of Materials Science, University of Minnesota, Minneapolis, Minnesota, U. S. A.

DIFFUSION MODELLING

We imagine a central crack in an infinite linear elastic isotropic medium, subjected to a symmetric tensile stress field at infinity, as shown in Figure 1. The crack tip serves as a source of solute atoms (e.g., dissociated hydrogen ions) which diffuse in the elastic medium symmetrically with respect to y-axis. The cloud of diffusing species is modelled to behave as a perfect fluid obeying the mechanical principles of mass and momentum balance. These principles, in local form, are expressed by the differential equations [6],

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho \underset{\sim}{v}) = 0 \tag{1}$$

and

$$\mathrm{div}\underset{\sim}{S} + \rho\underset{\sim}{\psi} = \dot{v} \tag{2}$$

respectively.

In the field equation (1) and (2), ρ, $\underset{\sim}{v}$, $\dot{\underset{\sim}{v}}$ and $\underset{\sim}{S}$ stand for the density, the velocity, the acceleration and the stress tensor of the gas and $\underset{\sim}{\psi}$ represents the diffusive force vector.

The following simple constitutive model is adopted for the stress tensor $\underset{\sim}{T}$ and diffusion force $\underset{\sim}{\psi}$:

$$\underset{\sim}{S} = -A\rho\underset{\sim}{1} \quad ; \quad \underset{\sim}{\psi} = B\underset{\sim}{v} \tag{3}$$

where A and B are constants, and $\underset{\sim}{1}$ represents the unit tensor. The above model is a special case of a general constitutive structure proposed in [6].

Next, we insert (3) into (2) and neglect the acceleration $\dot{\underset{\sim}{v}}$ in order to conform with classical diffusion theories [8]. The result is

$$\rho\underset{\sim}{v} = -\frac{A}{B}\nabla\rho . \tag{4}$$

Introducing (4) into (1) we obtain

$$\frac{\partial\rho}{\partial t} = \frac{A}{B}\nabla^2\rho , \tag{5}$$

which is the classical diffusion equation, if A/B is identified with the diffusivity D. We are interested in steady-state situations and in these cases (5) is reduced to

$$\nabla^2\rho = 0 . \tag{6}$$

STATIC EQUILIBRIUM OF THE SOLID

The infinite medium is modelled to behave as a linear elastic isotropic solid, obeying the familiar constitutive law

$$\underset{\sim}{T} = \lambda\, \mathrm{tr}\underset{\sim}{\varepsilon}\underset{\sim}{1} + 2\mu\underset{\sim}{\varepsilon} \tag{7}$$

where $\underset{\sim}{T}$ and $\underset{\sim}{\varepsilon}$ are the stress and strain tensors of the solid correspondingly; λ and μ are the Lamé constants; and $\mathrm{tr}\underset{\sim}{\varepsilon}$ represents the trace of $\underset{\sim}{\varepsilon}$. The strain tensor $\underset{\sim}{\varepsilon}$ is defined in the usual way by

$$\underset{\sim}{\varepsilon} = \frac{1}{2}\left[\nabla\underset{\sim}{u} + (\nabla\underset{\sim}{u})^T\right] \tag{8}$$

where $\underset{\sim}{u}$ is the displacement vector of the solid, the symbol "T" denotes transposition and ∇ is the gradient operator either for vector or scalar fields. The strain energy, W, of the solid is also given by the familiar relationship

$$W = \frac{1}{2}\mathrm{tr}(\underset{\sim}{T}\underset{\sim}{\varepsilon}) . \tag{9}$$

The solid is considered to be in static equilibrium, in the presence of the diffusive force field, $-\rho\underset{\sim}{\psi}$, which acts as a body force. Thus the equilibrium equations, in vector form, are given by

$$\mathrm{div}\,\underset{\sim}{T} - \rho\underset{\sim}{\psi} = 0 . \tag{10}$$

Using $(3)_2$ and (4) we can write (10) in the form

$$\mathrm{div}\,\underset{\sim}{T} + A\nabla\rho = 0 . \tag{11}$$

From now on we will consider the constant A to be given.

Substitution of (7) into (11) yields

$$\mathrm{div}\underset{\sim}{\varepsilon} = -\frac{A}{2\mu}\nabla\rho - \frac{\lambda}{2\mu}\nabla\,\mathrm{tr}\underset{\sim}{\varepsilon} . \tag{12}$$

Operating with the divergence in (12) and using a direct consequence of compatibility [8], we obtain

$$(\lambda + 2\mu)\nabla^2\mathrm{tr}\underset{\sim}{\varepsilon} = -A\nabla^2\rho . \tag{13}$$

We are interested in steady-state diffusion processes. Then, in light of (6), equation (13) results to

$$\nabla^2\mathrm{tr}\underset{\sim}{\varepsilon} = 0 . \tag{14}$$

Another relationship that will be useful in the subsequent analysis is the solution of (7) with respect to strain tensor $\underset{\sim}{\varepsilon}$. This solution is well known [9] and may be written as

$$\underset{\sim}{\varepsilon} = \frac{1+\nu}{E}\underset{\sim}{T} - \frac{\nu}{E}\sigma\underset{\sim}{1} \qquad (15)$$

where ν and E are the Poisson's ratio and the modulus of elasticity and σ is the trace of the stress tensor of the solid, i.e.,

$$\sigma = \mathrm{tr}\,\underset{\sim}{T}\,. \qquad (16)$$

A CONSERVATION LAW

In this section, we establish a conservation law holding for any sub-region, free from singularities, of the infinite isotropic linear elastic medium under consideration. Towards this aim we prove the following theorem.

Theorem: If an isotropic linear elastic domain supports a diffusive force field, of the form $\underset{\sim}{f} = A\nabla\rho$, exerted by diffusing species of concentration ρ, then the following conservation law holds:

$$\underset{\sim}{J}^* = \oint_C \left\{\left[W + \frac{1}{2\mu}\left(\rho^{*2} + 2\tau^*\rho^* + \frac{1}{2\nu}\tau^{*2}\right)\right]\underset{\sim}{n} + \left[2\rho^*\underset{\sim}{\varepsilon} + \tau^*\nabla\underset{\sim}{u} - \underset{\sim}{u}\otimes\nabla(\rho^*+\tau^*) - (\nabla\underset{\sim}{u})^T\underset{\sim}{T}\right]\underset{\sim}{n}\right\}d\ell = 0 \qquad (17)$$

for every surface C that is the boundary of a finite regular closed sub-region of the elastic domain, provided $\underset{\sim}{n}$ is the unit outward normal vector of C. In the case of zero diffusion, i.e., $\nabla\rho = 0$, it may be shown that

$$\underset{\sim}{J}^*_{\nabla\rho=0} = \oint_C \left[W\underset{\sim}{n} - (\nabla\underset{\sim}{u})^T\underset{\sim}{T}\underset{\sim}{n}\right]d\ell = 0 \qquad (18)$$

which is the familiar conservation law discussed in [4]. The scalar fields ρ^* and τ^* in (17) are defined by

$$\rho^* = A\rho \quad ; \quad \tau^* = \lambda\,\mathrm{tr}\underset{\sim}{\varepsilon}\,. \qquad (19)$$

Proof: For convenience we use the familiar indicial notation to define a vector $\underset{\sim}{J}$ by its components as

$$J_i = \int_{\partial D}\left(Wn_i - t_j u_{j,i}\right)d\ell\,, \qquad (20)$$

where $\underset{\sim}{n}$ is the outward normal of the surface ∂D that is the boundary of the finite regular closed subregion D; and $\underset{\sim}{t}$ is the traction vector defined by

$$t_i = T_{ij}n_j\,. \qquad (21)$$

The divergence theorem [10], the chain rule and the definitions (8), (9) and (21) allow for successive transformations of (20), as follows:

$$J_i = \int_D\left\{W_{,i} - \left[(\nabla\underset{\sim}{u})^T\underset{\sim}{T}\right]_{ij,j}\right\}dv = \int_D\left\{\frac{\partial W}{\partial\varepsilon_{mj}}\varepsilon_{mj,i} - \left(u_{m,i}T_{mj}\right)_{,j}\right\}dv$$

$$= \int_D\left\{T_{mj}\varepsilon_{mj,i} - u_{m,ij}T_{mj} - u_{m,i}T_{mj,j}\right\}dv = -\int_D u_{m,i}T_{mj,j}\,dv \qquad (22)$$

where the identity

$$T_{mj}\varepsilon_{mj,i} = \frac{1}{2}T_{mj}u_{m,ji} + \frac{1}{2}T_{mj}u_{j,mi} = \frac{1}{2}T_{mj}u_{m,ji} + \frac{1}{2}T_{jm}u_{j,mi} = T_{mj}u_{m,ji} \qquad (23)$$

was used.

Upon substitution of (11) into (22) we obtain

$$J_i = A\int_D u_{m,i}\rho_{,m}\,. \qquad (24)$$

With the aid of definition (8) and a trivial algebraic manipulation the last equation is written, in direct notation, as

$$\underset{\sim}{J} = 2A\int_D \underset{\sim}{\varepsilon}\,\nabla\rho\,dv - A\int_D(\nabla\underset{\sim}{u})\nabla\rho\,dv\,. \qquad (25)$$

It is convenient to introduce the definitions

$$\underset{\sim}{J}_1 = \int_D(\nabla\underset{\sim}{u})\nabla\rho\,dv \quad ; \quad J_2 = \int_D\underset{\sim}{\varepsilon}\nabla\rho\,dv\,. \qquad (26)$$

Then using the easily shown identity,

$$\mathrm{div}(\underset{\sim}{u}\otimes\nabla\rho) = (\nabla\underset{\sim}{u})\nabla\rho + \underset{\sim}{u}\,\mathrm{div}(\nabla\rho)\,, \qquad (27)$$

when the symbol "$\otimes$" denotes the dyadic, we obtain

$$J_1 = \int_D \mathrm{div}(u\otimes\nabla\rho)dv - \int_D\underset{\sim}{u}\,\nabla^2\rho\,dv\,, \qquad (28)$$

which with the aid of (6) and the divergence theorem [10], gives

$$J_1 = \int_{\partial D}(\underset{\sim}{u}\otimes\nabla\rho)\underset{\sim}{n}\,d\ell\,. \qquad (29)$$

The divergence theorem and the identity

$$\operatorname{div}(\rho\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}\nabla\rho + \rho \operatorname{div} \boldsymbol{\varepsilon} \tag{30}$$

serve to write $(26)_2$ in the following form

$$J_2 = \int_{\partial D} \rho\boldsymbol{\varepsilon}\mathbf{n}\, d\ell - \int_D \rho \operatorname{div} \boldsymbol{\varepsilon}\, dv \ . \tag{31}$$

Our efforts will be directed next in transforming the second term of the right hand side of (31) to a surface integral. Thus, the equilibrium equations (12) are used to write

$$\int_D \rho \operatorname{div} \boldsymbol{\varepsilon}\, dv = -\frac{A}{4\mu}\int_D \nabla\rho^2 dv - \frac{\lambda}{2\mu}\int_D \nabla(\rho \operatorname{tr} \boldsymbol{\varepsilon}) dv + \frac{\lambda}{2\mu}\int_D \operatorname{tr}\boldsymbol{\varepsilon}\, \nabla\rho\, dv \ , \tag{32}$$

which after the use of the divergence theorem is reduced to

$$\int_D \rho \operatorname{div} \boldsymbol{\varepsilon}\, dv = -\frac{A}{4\mu}\int_{\partial D} \rho^2 \mathbf{n}\, d\ell - \frac{\lambda}{2\mu}\int_{\partial D} \rho \operatorname{tr}\boldsymbol{\varepsilon}\, \mathbf{n}\, d\ell + \frac{\lambda}{2\mu}\int_D \operatorname{tr}\boldsymbol{\varepsilon}\, \nabla\rho\, dv \ . \tag{33}$$

Defining a new integral

$$\mathbf{J}_3 = \int_D \operatorname{tr}\boldsymbol{\varepsilon}\, \nabla\rho\, dv \ , \tag{34}$$

and expressing the equilibrium equations in terms of the displacement [9] we obtain

$$\mathbf{J}_3 = -\frac{\mu}{A}\int_D \operatorname{tr}\boldsymbol{\varepsilon}\, \nabla^2 \mathbf{u} - \frac{\lambda+\mu}{A}\int_D (\operatorname{tr}\boldsymbol{\varepsilon})\nabla \operatorname{tr}\boldsymbol{\varepsilon}\, dv \ . \tag{35}$$

Next, we use the identity

$$\int_D \operatorname{tr}\boldsymbol{\varepsilon}\, \nabla^2 \mathbf{u} = \int_{\partial D} [\operatorname{tr}\boldsymbol{\varepsilon}(\nabla\mathbf{u})\mathbf{n} - (\mathbf{u}\otimes \nabla \operatorname{tr}\boldsymbol{\varepsilon})\mathbf{n}]d\ell \ , \tag{36}$$

a proof of which is given in [5]. Thus, with the aid of the divergence theorem again to transform the second term of the left hand side of (36) to a surface integral, equation (35) may be written as

$$\mathbf{J}_3 = -\frac{\mu}{A}\int_{\partial D} [\operatorname{tr}\boldsymbol{\varepsilon}(\nabla\mathbf{u})\mathbf{n} - (\mathbf{u}\otimes \nabla \operatorname{tr}\boldsymbol{\varepsilon})\mathbf{n}]d\ell - \frac{\lambda+\mu}{2A}\int_{\partial D} (\operatorname{tr}\boldsymbol{\varepsilon})^2 d\ell \ . \tag{37}$$

Combination of (33), (34) and (37) yields

$$\int_D \rho \operatorname{div}\boldsymbol{\varepsilon}\, dv = -\frac{1}{2\mu}\int_{\partial D}\left\{\left[\frac{A}{2}\rho^2 + \lambda\rho \operatorname{tr}\boldsymbol{\varepsilon} + \frac{\lambda(\lambda+\mu)}{2A}(\operatorname{tr}\boldsymbol{\varepsilon})^2\right]\mathbf{n} + \frac{\lambda\mu}{A}\left[\operatorname{tr}\boldsymbol{\varepsilon}\nabla\mathbf{u} - \mathbf{u}\otimes\nabla \operatorname{tr}\boldsymbol{\varepsilon}\right]\mathbf{n}\right\} \tag{38}$$

and this way the following expression for J_2 is derived

$$\mathbf{J}_2 = \frac{1}{2\mu}\int_{\partial D}\left\{\left[\frac{\lambda\mu}{A}\rho\boldsymbol{\varepsilon} + \operatorname{tr}\boldsymbol{\varepsilon}\nabla\mathbf{u} - \mathbf{u}\otimes\nabla \operatorname{tr}\boldsymbol{\varepsilon}\right]\mathbf{n} + \left[\frac{A}{2}\rho^2 + \lambda\rho \operatorname{tr}\boldsymbol{\varepsilon} + \frac{\lambda(\lambda+\mu)}{2A}(\operatorname{tr}\boldsymbol{\varepsilon})^2\right]\mathbf{n}\right\} . \tag{39}$$

Defining new scalar fields ρ^* and τ^* as in (19) and combining (25), (26), (29) and (39) we finally obtain

$$\mathbf{J} = \int_{\partial D}\left\{[2\rho^*\boldsymbol{\varepsilon} + \tau^*\nabla\mathbf{u} - \mathbf{u}\otimes\nabla(\rho^*+\tau^*)]\mathbf{n} + \frac{1}{2\mu}\left(\rho^{*2} + 2\rho^*\tau^* + \frac{1}{2\nu}\tau^{*2}\right)\mathbf{n}\right\}d\ell \ . \tag{40}$$

Also equation (20) may be written in direct notation as

$$\mathbf{J} = \int_{\partial D}\left[W\mathbf{n} - (\nabla\mathbf{u})^T \mathbf{T}\mathbf{n}\right]d\ell \ . \tag{41}$$

The results (40) and (41) establish the validity of the conservation law (17), if the boundary ∂D is identifying with the closed surface C.

It is easily seen from (41) that the first component of the vector $\mathbf{J}$ is the well-known J-integral. Equation (40) indicates that the value of J integral along a closed path is not zero when diffusion is considered. The appropriate form which replaces the J-integral, in the cases under consideration, is provided by the first component of the vector $\mathbf{J}^*$ in equation (17).

It is natural to expect that the vector $\mathbf{J}^*$ is identified to the vector $\mathbf{J}$ when diffusion effects are neglected. In this case the density of the diffusing species is uniform, i.e.,

$$\nabla\rho^* = 0 \quad ; \quad \rho^* = \rho^*_o \ . \tag{42}$$

Under these conditions, equation (17) combined with (41) gives

$$\mathbf{J}^*_{\nabla\rho=0} = \mathbf{J} + \int_{\partial D}\left(2\rho^*_o\boldsymbol{\varepsilon} + \frac{1}{2\mu}\rho^{*\,2}_o + \frac{1}{\mu}\rho^*_o\tau^*\right)\mathbf{n}\,d\ell + \int_{\partial D}(\tau^*\nabla\mathbf{u} - \mathbf{u}\otimes\nabla\tau^*)\mathbf{n}\,d\ell + \frac{1}{4\mu\nu}\int_{\partial D}\tau^{*2}\mathbf{n}\,d\ell \ . \tag{43}$$

First observe that use of the divergence theorem and $(19)_2$ yields

$$\int_{\partial D}\left(2\rho^*_o\boldsymbol{\varepsilon} + \frac{1}{2\mu}\rho^{*\,2}_o + \frac{1}{\mu}\rho^*_o\tau^*\right)\mathbf{n}\,d\ell = \int_D\left(2\rho^*_o \operatorname{div}\boldsymbol{\varepsilon} + \rho^*_o\frac{\lambda}{\mu}\nabla \operatorname{tr}\boldsymbol{\varepsilon}\right)dv = 0. \tag{44}$$

This is true because the integrand in $(44)_2$ has a zero value, as it is easily seen by combining equations (12) and $(42)_1$. Next, we use the definition $(19)_2$, the identity (36) and the divergence theorem, to write

$$\int_{\partial D}\left(\tau^*\nabla \mathbf{u}-\mathbf{u}\nabla\tau^*\right)\mathbf{n}\,d\ell + \frac{1}{4\mu\nu}\int_{\partial D}\tau^{*2}\mathbf{n}\,d\ell = \lambda\int_D\left[(tr\boldsymbol{\varepsilon})\nabla^2\mathbf{u} + \frac{\lambda+\mu}{\mu}(tr\boldsymbol{\varepsilon})\nabla tr\boldsymbol{\varepsilon}\right]dv \ . \qquad (45)$$

But the equilibrium equations (12) are expressed in terms of the displacement vector in the form

$$\nabla^2 u + \frac{\lambda+\mu}{\mu}\nabla tr\varepsilon = -\frac{A}{\mu}\nabla\rho \ , \qquad (46)$$

which when is combined with $(42)_1$ and (45) results into

$$\int_{\partial D}(\tau^*\nabla \mathbf{u}-\mathbf{u}\nabla\tau^*)\mathbf{n}\,d\ell + \frac{1}{4\mu\nu}\int_{\partial D}\tau^{*2}\mathbf{n}\,d\ell = 0 \ . \qquad (47)$$

Then combination of (41), (43), (44) and (47) yields

$$\mathbf{J}^*_{\nabla\rho=0} = \oint_C\left[W\mathbf{n}-(\nabla\mathbf{u})^T\mathbf{T}\mathbf{n}\right]d\ell = 0 \qquad (48)$$

and this way the result (18) is established.

A PATH INDEPENDENT INTEGRAL

In the case without diffusion, it is shown [2] that the first component of vector $\mathbf{J}$ has the same value for any closed curve surrounding the singularity.

In the present investigation diffusion effects are introduced and we derive results analogous to those contained in [2]. We confine attention to a two dimensional symmetric configuration. Thus, we consider an infinite plate loaded symmetrically under plane strain conditions (Mode I) and containing a central crack acting as source of diffusing species symmetrically distributed with respect to x-axis. The scheme under examination is shown in Figure 1.

The objective is to show that the $\mathbf{J}^*$ integral, defined in (17), has the same value for all paths surrounding the line crack. Towards this aim we consider the closed curve, $C'+\Gamma^++C+\Gamma^-$, as indicated in Figure 1. Then the conservation law (17) insures path independence if we show that

$$\int_{\Gamma^++\Gamma^-}\left\{[2\rho^*\boldsymbol{\varepsilon}+\tau^*\nabla\mathbf{u}-\mathbf{u}\otimes\nabla(\rho^*+\tau^*)]\mathbf{n} + \frac{1}{2\mu}\left[\rho^{*2}+2\rho^*\tau^* + \frac{1}{2\nu}\tau^{*2}\right]\mathbf{n}\right\}d\ell = 0 \ . \qquad (49)$$

Employing symmetry arguments, as well as traction free boundary condition, we deduce the following relations holding on the crack surface.

$$T_{xx} = T_{yy} = T_{xy} = 0 \ ; \quad u_x = 0 \ ; \quad u_y\big|_{y=0^+} = -u_y\big|_{y=0^-}$$
$$\rho^*\big|_{y=0^+} = \rho^*\big|_{y=0^-} \ ; \quad \frac{\partial\rho^*}{\partial y}\Big|_{y=0^+} = -\frac{\partial\rho^*}{\partial y}\Big|_{y=0^-} \ ; \quad \frac{\partial\tau^*}{\partial y}\Big|_{y=0^+} = -\frac{\partial\tau^*}{\partial y}\Big|_{y=0^-} \ . \qquad (50)$$

Then τ^* vanishes on the crack surface and condition (49) is equivalent to

$$\int_{\Gamma^++\Gamma^-}\left\{[2\rho^*\boldsymbol{\varepsilon}-\mathbf{u}\otimes\nabla(\rho^*+\tau^*)]\mathbf{n} + \frac{1}{2\mu}\rho^{*2}\mathbf{n}\right\}d\ell = 0 \ . \qquad (51)$$

With the aid of (15) we obtain

$$\int_{\Gamma^++\Gamma^-}2\rho^*\boldsymbol{\varepsilon}\mathbf{n}\,d\ell = \int_{\Gamma^++\Gamma^-}2\rho^*\left[\frac{1+\nu}{E}\mathbf{t} - \frac{\nu}{E}\sigma\mathbf{n}\right]d\ell = 0 \qquad (52)$$

since $\mathbf{t}$ and σ vanish on the crack surface. If $\hat{\imath}_y$ is the base vector in the y-direction then

$$\int_{\Gamma^++\Gamma^-}\frac{1}{2\mu}\rho^{*2}\mathbf{n}\,d\ell = \hat{\imath}_y\frac{1}{2\mu}\left\{\int_{\Gamma^+}\rho^{*2}dx + \int_{\Gamma^-}\rho^{*2}dx\right\} = 0 \qquad (53)$$

since the integrand ρ^{*2} has the same value as it is integrated over the same interval in opposite directions. Also,

$$\int_{\Gamma^++\Gamma^-}[\mathbf{u}\otimes\nabla(\rho^*+\tau^*)]\mathbf{n}\,d\ell = \int_{\Gamma^++\Gamma^-}\mathbf{u}[\mathbf{n}\cdot\nabla(\rho^*+\tau^*)]d\ell =$$
$$= \hat{\imath}_x\int_{\Gamma^++\Gamma^-}u_x\left[\frac{\partial\rho^*}{\partial y}+\frac{\partial\tau^*}{\partial y}\right]dx + \hat{\imath}_y\int_{\Gamma^++\Gamma^-}u_y\left[\frac{\partial\rho^*}{\partial y}+\frac{\partial\tau^*}{\partial y}\right]dx \ . \qquad (54)$$

But $u_x = 0$ on the crack surface and the integrand $u_y\left[\frac{\partial\rho^*}{\partial y}+\frac{\partial\tau^*}{\partial y}\right]$ has the same value, because of the last two relations of (50), as it is integrated over Γ^+ and Γ^-. Thus

$$\int_{\Gamma^++\Gamma^-}[\mathbf{u}\otimes\nabla(\rho^*+\tau^*)]\mathbf{n}\,d\ell = 0 \ . \qquad (55)$$

The results (51), (52) and (55) establish the validity of condition (49), and therefore the path independence of $\mathbf{J}^*$ integral.

It has been shown that when J_x^* is integrated over paths away from the crack tip where the diffusion effects are neglected, then it has the same value as the familiar J-integral. This furnishes approximate knowledge of the value of J_x^* for these configurations in which the J-integral has been already evaluated. Then approximate estimates for the stress-diffusion field in the neighbourhood of the singularity may be attempted.

REFERENCES

1. CHEREPANOV, G. P., PPM, 31, No. 3, 1967, 476.
2. RICE, J. R., J. Appl. Mech., 35, 1968, 379.
3. ESHELBY, J. D., "Solid State Physics", Vol. 3, Academic Press, New York, 1956, 79.
4. KNOWLES, J. K. and STERNBERG, E., Arch. Rational Mech. Anal., 44, No. 3, 1972, 187.
5. AIFANTIS, E. C., J. Appl. Mech., (to be submitted).

6. AIFANTIS, E. C., Mech. Res. Communications, 3, No. 4, 1976, 245.
7. CHEREPANOV, G. P., Eng. Fract. Mechanics, 5, 1973, 1041.
8. AIFANTIS, E. C., Ph. D. Thesis, University of Minnesota, 1975.
9. SOKOLNIKOFF, I. S., "Mathematical Theory of Elasticity", McGraw Hill, New York, 1956, 73.
10. GURTIN, M. E., "The Linear Theory of Elasticity", Handbuch der Physik, VI a/2, Springer-Verlag, Berlin, Heidelberg, New York, 1972, 16.

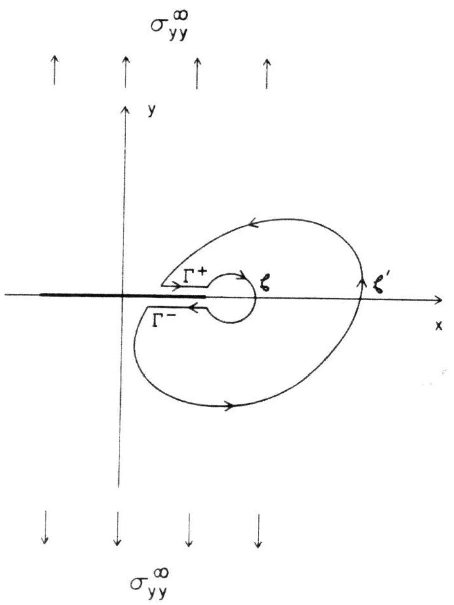

Figure 1