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A DESCRIPTION OF THE DEPENDENCE OF FRACTURE RESISTANCE ON CRACKLENGTH

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INTRODUCTION

By evaluation of the absorbed plastic energy at the cracktip up to the moment of instability an expression for the fracture resistance in critical cracklength, critical gross stress and average reciprocal modulus over the plastic zone proves possible. Taking into account some conditions relevant for instability (like e.g. the Orowan conditions) and using the relevant expression for the energy release rate the fracture resistance curve appears from a small number of instability tests.

It is assumed according to Kraaft [1] that the absorbed energy pro cm crack extension at stable crack growth does not depend on initial crack-length, but only on the increase of cracklength a. By observing the stable crack extension at any time one can find dU/da = elastic energy rate and for an infinite plate of unit thickness, uniaxially loaded transversal to a crack a up to a gross stress σ_0 for the elastic case with E = Young's modulus:

$$R = \frac{dW}{da} = \frac{dU}{da} = \frac{2\pi\sigma_0{}^2 a}{E} \tag{1.1a}$$

For a finite plate $R = \frac{1}{2}\sigma_0{}^2 w^2 (dC/da)$ with w = width and C = compliance, and for the infinite plate $C = 1/EwB + 2\pi a^2/Ew^2B$ with B = thickness, yielding (1.1a).

When the increasing $2\pi\sigma_0{}^2 a/E$ for increasing a and σ_0 can be no longer matched by an increase of R = dW/da of the material, instability occurs and (cf. Figure 1, Orowan [2]):

$$R_c = \left\{\frac{dW}{da}\right\}_c = \frac{2\pi\sigma_{oc}{}^2 a_c}{E}, \text{ while } \left\{\frac{d^2W}{da^2}\right\}_c = \frac{2\pi\sigma^2{}_{oc}}{E} \tag{1.1b}$$

The R-curve can thus be obtained by a series of tests revealing instability at some cracklengths. If a reliable R vs. a description proves possible, the number of required instability tests might be greatly reduced.

CRITICAL ENERGY EVALUATION NEAR CRACKTIP AT INSTABILITY

For the absorbed energy W (cf. Figure 2) is written:

$$\frac{W}{2} = \int_{-\pi}^{\pi} d\Theta \int_{0}^{R(\Theta)} r(\Theta)dr(\Theta) \int_{0{\cdot}2}^{\varepsilon^*_c\{r(\Theta)\}} Y[\varepsilon^*\{r(\Theta)\}]\, d\varepsilon^*\{r(\Theta)\} \tag{2.1}$$

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with:

$r(\Theta)$ = radius from notchtip to considered point in plastic zone

Θ = angle between notch direction and radius

$R(\Theta)$ = elastic-plastic boundary at instability

$\varepsilon\{r(\Theta)\}$ = effective deformation in $r(\Theta)$

c = index referring to instability

$Y[\varepsilon^*\{r(\Theta)\}]$ = uniaxial yield stress for effective deformation

Take: $Y[\varepsilon^*\{r(\Theta)\}] = \sigma_0\, f(s)$, with

$\sigma_o = \zeta[\varepsilon^*\{r(\Theta)\}]\sigma_0$ = gross stress, when in $r(\theta)$ the effective deformation is $\varepsilon^*\{r(\theta)\}$

$$0 < \zeta < 1$$

$\zeta \to 1$ for $\varepsilon^* \to \varepsilon_c^*$ and $\sigma_0 \to \sigma_{0c}$

$$f(s) = \Sigma_{k,j} \sqrt{1/2 \left[s_k\ [\varepsilon^*\{r(\Theta)\}] - s_j[\varepsilon^*\{r(\Theta)\}]\right]^2}$$

$k, j = 1, 2, 3$; $k \neq j$

$$s_k[\varepsilon^*\{r(\Theta)\}]\ \zeta\ [\varepsilon^*\{r(\Theta)\}]\ \sigma_{0c} = \sigma_k\ [\varepsilon^*\{r(\Theta)\}]$$

k = 1, 2, 3

σ_k = principal stress

If: $E^* = E^*[\varepsilon^*\{r(\Theta)\}]$ = local modulus of the Y vs. ε^* curve

$$\frac{W}{2} = \sigma_{0c}{}^2 \int_{-\pi}^{\pi} d\Theta \int_0^{R(\Theta)} r(\Theta)dr(\Theta) \int_0^{\varepsilon_c^*\{r(\Theta)\}} \zeta f(s) d\left\{\frac{\zeta f(s)}{E^*}\right\} \qquad (2.2a)$$

$$\frac{W}{2} = \sigma_{0c}{}^2 \int_{-\pi}^{\pi} d\Theta \int_0^{R(\Theta)} r(\Theta)dr(\Theta) \int_0^{\varepsilon_c^*\{r(\Theta)\}} \left[\zeta^2\left\{\frac{f}{E^*} - \frac{f^2}{E^{*2}}\frac{dE^*}{df}\right\}\frac{df}{d\varepsilon^*} + \frac{\zeta f^2}{E^*}\frac{d\zeta}{d\varepsilon^*}\right]d\varepsilon^* \qquad (2.2b)$$

Taking:

$$1/2\beta\left[\varepsilon_c^*\ \{r(\Theta)\}\right] \equiv \int_0^{\varepsilon_c^*\{r(\Theta)\}} \left[\zeta^2\left\{\frac{f}{E^*} - \frac{f^2}{E^{*2}}\frac{dE^*}{df}\right\}\frac{df}{d\varepsilon^*} + \frac{\zeta f^2}{E}\frac{d\zeta}{d\varepsilon}\right]d\varepsilon^* \qquad (2.3)$$

$$\frac{W}{2} = \sigma_{0c}{}^2 \int_{-\pi}^{\pi} d\Theta \int_0^{R(\Theta)} r(\Theta)dr(\Theta)\ 1/2\beta\left[\varepsilon_c^*\{r(\Theta)\}\right] \qquad (2.4a)$$

$$\frac{W}{2} = 1/2\bar{\beta}_c\ \sigma_{0c}{}^2 \int_{-\pi}^{\pi} \frac{R(\Theta)^2}{2}\, d\Theta = 1/2\bar{\beta}\ \sigma_{0c}{}^2 \int_{-\pi}^{\pi} \frac{\rho(\Theta)^2}{2}\, d\Theta \qquad (2.4b)$$

THE EVALUATION OF R FROM W_c IN THE ELASTIC CASE

In the case that the plastic zone is sufficiently small an easy calculation of the plastic zone boundary can proceed from the goniometric Sneddon approximations of Westergaard's description of the stresses near the cracktip in the elastic case.

Then one has, using reduced stresses $s = \sigma/\sigma_o$:

$$s_x = \sqrt{\frac{a}{2r}} \cos\frac{\Theta}{2}\left\{1 - \sin\frac{\Theta}{2}\sin\frac{3\Theta}{2}\right\}$$

$$s_y = \sqrt{\frac{a}{2r}} \cos\frac{\Theta}{2}\left\{1 + \sin\frac{\Theta}{2}\sin\frac{3\Theta}{2}\right\}$$

$$t_{xy} = \sqrt{\frac{a}{2r}} \sin\frac{\Theta}{2}\cos\frac{\Theta}{2}\cos\frac{3\Theta}{2}$$

with as principal reduced stresses:

$$s_{1,2} = \sqrt{\frac{a}{2r}} \cos\frac{\Theta}{2}\left\{1 + \sin\frac{\Theta}{2}\right\};$$

$s_3 = 0$ for plane stress (3.1)

$= 2\nu\sqrt{\frac{a}{2r}}\cos\frac{\Theta}{2}$ for plane strain

Combining this with the Von Mises-Hencky yield criterion:

$$2Y_0{}^2 = (\sigma_1 - \sigma_2)^2 + (\sigma_1 - \sigma_3)^2 + (\sigma_2 - \sigma_3)^2 \qquad (3.2)$$

one arrives at:

$$R(\Theta) = \frac{K^2}{2rY^2}\cos^2\frac{\Theta}{2}\left\{1 + 3\sin^2\frac{\Theta}{2} - p\right\} \qquad (3.3)$$

$p = 0$ for plane stress; $p = 4\nu\ (1 - \nu)$ for plane strain with K referring to the mathematical cracklength a_e.

$$a_e = a + \alpha\frac{K^2}{2\pi Y^2} \qquad (3.4)$$

$\alpha = 1$ for plane stress; $\alpha = 1/3$ for plane strain.

At plane stress:

$$\rho(\Theta) \equiv \frac{R(\Theta)}{a} = \frac{\sigma_{0c}{}^2}{2Y_0{}^2}\left\{1 + \frac{\sigma_{0c}{}^2}{2Y_0{}^2}\right\}\cos^2\frac{\Theta}{2}\left\{1 + 3\sin^2\frac{\Theta}{2}\right\} \qquad (3.5)$$

and

$$W = \bar{\beta}_c\ \sigma_{0c}{}^2\ a^2\frac{\sigma_{0c}{}^4}{4Y_0{}^4}\left\{1 + \frac{\sigma_{0c}{}^2}{2Y_0{}^2}\right\}^2 \int_{-\pi}^{\pi} \cos^4\frac{\Theta}{2}\left\{1 + \sin^2\frac{\Theta}{2}\right\}^2 d\Theta \qquad (3.6)$$

$$= \bar{\beta}_c\ \sigma_{0c}{}^2\ a^2\left\{\frac{f_0{}^4}{4} + \frac{f_0{}^6}{6} + \frac{f_0{}^8}{16}\right\}\frac{123\pi}{64} = CY_0{}^2a^2\bar{\beta}_cF_0 \qquad (3.7)$$

with: $f_0 = \frac{\sigma_{0c}}{Y_0}$; $C = \frac{123\pi}{64} = 6.038$; $F_0 = \frac{f_0^6}{4} + \frac{f_0^8}{6} + \frac{f_0^{10}}{16}$

cF_0a^2/f_0^2 obviously represents the surface (volume) of the plastic zone (cf. Table 1).

$\overline{\beta}_c$ = "plastic zone averaged" reciprocal modulus.

$$\overline{\beta}_c Y^2 f_0^2 = \overline{\beta}_c \sigma_{0c}^2 \qquad (3.8)$$

represents the average energy density.

At instability

$$R_a = \left\{\frac{dW}{da}\right\}_c = \frac{dU}{da} \text{ and } \left\{\frac{d^2U}{da^2}\right\}_c = \left\{\frac{d^2W}{da^2}\right\}_c, \text{ (Orowan [2])} \qquad (3.9)$$

It is moreover assumed that for the critical stress σ_{0c} at instability:

$$\frac{d\sigma_{0c}}{da} = 0, \text{ implying } \frac{df_0}{da} = 0 \text{ and } \frac{dF_0}{da} = 0 \qquad (3.10)$$

Then:

$$R_a = \left\{\frac{dW}{da}\right\}_c = CaF_0Y_0^2\left\{2\overline{\beta}_c + a\frac{d\overline{\beta}}{da}\right\} \qquad (3.11)$$

For an infinite plate this equals:

$$\frac{2\pi}{E} Y_0^2 f_0^2 a \qquad (3.12)$$

while

$$2CF_0Y_0^2\left\{\overline{\beta}_c + 2a\frac{d\overline{\beta}_c}{da} + a^2\frac{d^2\overline{\beta}_c}{da^2}\right\} = \frac{2\pi}{E} Y_0^2 f_0^2 \qquad (3.13)$$

Then:

$$2\overline{\beta}_c + a\frac{d\overline{\beta}_c}{da} = 2\left\{\overline{\beta}_c + 2a\frac{d\overline{\beta}_c}{da} + a^2\frac{d^2\overline{\beta}_c}{da^2}\right\} \qquad (3.14)$$

or

$$\frac{d^2\overline{\beta}_c}{da^2}\Big/\frac{d\overline{\beta}_c}{da} = -\frac{3}{2a} \rightarrow \frac{d\overline{\beta}_c}{da} = Aa^{-3/2};\ \frac{d^2\overline{\beta}_c}{da^2} = -\frac{3}{2}Aa^{-5/2} \qquad (3.15)$$

$$\overline{\beta}_c = -2Aa^{-1/2} + \beta_0 \qquad (3.16)$$

(A and β_0 constants)

It might be noted that for:

$$\frac{a_0}{\Delta a_0} = \frac{R_a}{\left\{\frac{dR}{da}\right\}\Delta a_0} - 1 \qquad (3.17)$$

$$\left.\begin{aligned} R_a &= cF_0a(2\beta_0 - 3Aa^{-1/2}) \\ \frac{dR_a}{da} &= 2cF_0(\beta_0 - 3/2\ Aa^{-1/2}) \end{aligned}\right\} \rightarrow R_a\Big/\frac{dR_a}{da} = a \qquad (3.18)$$

For a cracked infinite plate in the elastic case:

$$f_0^2 = \frac{\sigma_0^2}{Y_0^2} = \frac{ER}{2\pi aY_0^2}\left\{\text{as } \frac{dU}{da} = \frac{2\pi\sigma_0^2 a}{E}\right\} \qquad (3.19)$$

$$F_0 = \frac{(f_0^2)^3}{4} + \frac{(f_0^2)^4}{4} + \frac{(f_0^2)^5}{16} = 1/4\left\{\frac{ER}{2\pi aY_0^2}\right\}^3 + 1/4\left\{\frac{ER}{2\pi aY_0^2}\right\}^4 + \frac{1}{16}\left\{\frac{ER}{2\pi aY_0^2}\right\}^5 \qquad (3.20)$$

Thus:

$$R = CaF_0Y_0^2\left\{2\overline{\beta}_c + a\frac{d\overline{\beta}_c}{da}\right\} = C\ F_0Y_0^2a\ (2\beta_0 - 3Aa^{-1/2}) \qquad (3.21)$$

$$R = 1/4CY_0^2a(2\beta_0 - 3Aa^{-1/2})\left[\left\{\frac{ER}{2\pi aY_0^2}\right\}^3 + \left\{\frac{ER}{2\pi aY_0^2}\right\}^4 + 1/4\left\{\frac{ER}{2\pi aY_0^2}\right\}^5\right] \qquad (3.22)$$

or:

$$\frac{EC}{8\pi} = (2\beta_0 - 3Aa^{-1/2})\left[\left\{\frac{ER}{2\pi aY_0^2}\right\}^2 + \left\{\frac{ER}{2\pi aY_0^2}\right\}^3 + 1/4\left\{\frac{ER}{2\pi aY_0^2}\right\}^4\right] = 1 \qquad (3.23)$$

Incorporating the geometrical correction factor for the finite plate, one has:

$$\frac{dU}{da} = \frac{2\phi^2\sigma^2 a}{E} + \frac{2\sigma^2 a}{E}\phi\frac{d\phi}{da} \text{ with } \phi = \frac{K}{\sigma\sqrt{a}} \qquad (3.24)$$

$$f_0^2 = \frac{\sigma_0^2}{Y_0^2} = \frac{ER}{\left\{2\phi^2 a + 2a^2\phi\frac{d\phi}{da}\right\}Y_0^2} = \frac{ER}{w\left\{2\phi^2\lambda + 2\lambda^2\phi\frac{d\phi}{d\lambda}\right\}} \equiv \mathscr{R} \qquad (3.25)$$

$$f_0 = \frac{(f_0{}^2)^3}{4} + \frac{(f_0{}^2)^4}{4} + \frac{(f_0{}^2)^5}{16} = 1/4\,\mathscr{R}\left\{1+\mathscr{R}+\frac{\mathscr{R}^2}{4}\right\} \qquad (3.26)$$

$$R = 1/4\; C Y_0{}^2 a (2\beta_0 - 3Aa^{-1/2})\,\mathscr{R}^3 \left\{1+\mathscr{R}+\frac{\mathscr{R}^2}{4}\right\} \qquad (3.27)$$

With that a description of R in a, and in adaptable material parameters β_0, A, Y, E (and width w) appears possible.

For further references cf. [3]. Related ideas were developed in [4], [5], [6].

REFERENCES

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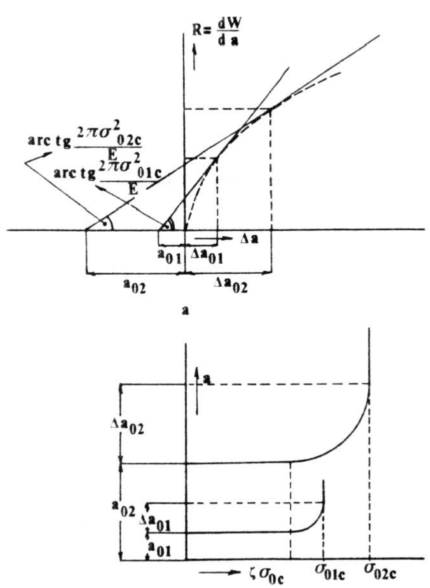
Table 1

$f_0 = \frac{\sigma_0}{Y_0}$	$\frac{F_0}{f_0} = \frac{f_0{}^4}{4} + \frac{f_0{}^6}{6} + \frac{f_0{}^8}{16}$	$CF_0/f_0{}^2 = \frac{123\pi}{64}\frac{F_0}{f_0}$
0	0	0
0.1	0.0000252^5	0.000152^5
0.2	0.000416	0.00251
0.3	0.00221	0.0133^5
0.4	0.00746^5	0.0450^5
0.5	0.0198	0.1194
0.6	0.0451	0.2724
0.7	0.0930	0.5618
0.8	0.178	1.0777
0.9	0.324	1.955
1.0	0.5625	3.396



b

Figure 1 a) Assumed increase of $R = \frac{dW}{da}$ with Δa matched by $\frac{dU}{da}$ up to instability.

b) Assumed gross stress and cracklength behaviour before and after instability

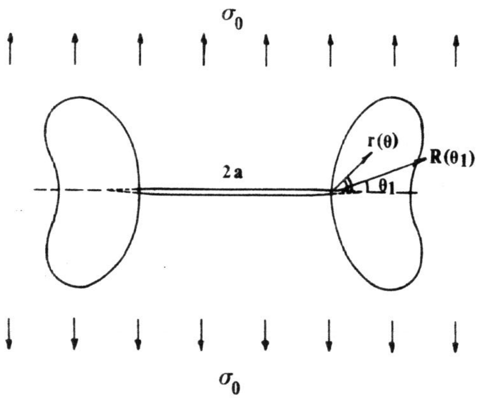


Figure 2   Plastic zone at crack tip in loaded plate, absorbing energy