# FATIGUE CRACK GROWTH MODEL PREDICTION WITH TWO COUPLED DIFFERENTIAL EQUATIONS

J. Pellas*, G. Baudin** and M. Robert***

## INTRODUCTION

In the case of periodic loading, the Paris-Forman law [1] is considered as an accurate phenomenological model for fatigue crack growth. This no longer holds if the loading amplitude is not constant or monotonously increasing with time. Under variable amplitude loading, many theories [2, 3] have been proposed all based on the Wheeler model [4] which shows the influence of the loading history by taking into account the yield zone ahead of the crack tip: nevertheless, comparison between calculation results and experimental data reveals that this model is not satisfactory. Then, it is necessary to introduce another parameter in the fatigue crack propagation law [5]: the crack propagation threshold. Finally the proposed fatigue crack propagation law consists of two coupled differential constitutive equations connecting crack propagation rate, stress intensity factor and loading history by means of crack propagation threshold. Calculation results are thus in good agreement with experimental data obtained from tests on bending specimen [6].

## WHEELER'S MODEL LIMITATIONS

The Wheeler crack propagation model [4] attempts to account for the reduction which occurs in the crack growth rate after an overload by considering two yield zones ahead of the crack tip: the current yield zone $\rho_{SS}$ and the real yield zone $\rho_{AS}$ due to the overload.

The model is based on Paris' law for crack growth, as shown in (1):

$$\frac{da}{dN} = C \cdot K^{\eta}_{max} \cdot \left(\frac{\rho_{SS}}{\rho_{AS}}\right)^{\beta} \qquad (1)$$

where:

C and $\eta$ are the empirical constants of Paris' law
$\beta$ is an exponent characteristic of each material.

This model has been tested, and calculation results have been compared with experimental data obtained from tests on bending specimen. This leads to the following conclusions:

* Research Engineer at O.N.E.R.A. (Office National d'Etudes et de Recherches Aérospatiales - 92320 Chatillon, France
** Research Engineer at O.N.E.R.A.
*** Senior Technician at O.N.E.R.A.

- Exponent $\beta$ is not intrinsic: it depends on test data, which is of course inconsistent with the existence of a constitutive law.
- This model takes into account only the loading history after the last overload; indeed, experimental results show that retardation is a function of a part of loading history before the overload.
- Furthermore, such a model does not incorporate the existence of crack arrest phenomenon.

In summary, it becomes apparent that an improvement is needed to the Wheeler model, especially to incorporate the effect of loading history. For that purpose, we introduce another parameter in the fatigue crack growth law: the crack propagation threshold.

## NOTION OF CRACK PROPAGATION THRESHOLD

Let us consider a variable amplitude loading: at any time, we can define the minimal load $P_{th}$ below which fatigue crack growth does not occur. This critical value is called crack propagation threshold and corresponds to a threshold stress intensity factor $K_{th}$ by means of the relation [7, 8]

$$K_{th} = P_{th} \cdot K_r(a)$$

where $K_r(a)$ is the stress intensity factor for a unit load, called reduced stress intensity factor.

### Threshold Effect

To illustrate crack propagation threshold effect, let us now consider the two following loadings (Figure 1):

- the case of a single overload $P_{MS}$,
- the case of a test with two load levels $P_{MS}$ and $P_{max}$ with $P_{MS} \geq P_{max}$.

In both cases the minimal load is equal to $P_{min}$ and $R_s$ is defined by

$$R_s = \frac{P_{MS}}{P_{max}} .$$

For a constant $P_{max}$, we increase $P_{MS}$ and we observe that:

- in the first case, crack arrest takes place for a critical value of $R_s$, $R_1(R)$ where R is defined by

$$R_R = \frac{P_{min}}{P_{max}} .$$

- in the second case, crack arrest occurs for a critical value of $R_s$, $R_2(R)$ which, for the same value of R, is different from $R_1(R)$.

Those experimental facts demonstrate:

1) the existence of crack propagation threshold, and
2) its dependence on loading history.





### Generalization

From the previous paragraph, we conclude that it is necessary to introduce two coupled differential equations connecting crack growth rate, stress intensity factor and crack propagation threshold.

Furthermore, we can assume that crack growth rate and crack propagation threshold rate are functions of:

- body's geometry and crack's length through the reduced stress intensity factor $K_r(a)$,
- external loads $P_{max}$ and $P_{min}$,
- loading history through the crack propagation threshold $K_{th}$ [9],
- strains and stresses at crack tip through the Wheeler retardation parameter.

Finally, the two coupled equations appear to be:

$$\frac{da}{dN} = h\left(K_{max}, K_{min}, K_{th}, \frac{\rho_{SS}}{\rho_{AS}}\right)$$

$$\frac{dK_{th}}{dN} = g\left(K_{max}, K_{min}, K_{th}, \frac{\rho_{SS}}{\rho_{AS}}\right) . \qquad (2)$$

## PROPOSED CRACK GROWTH MODEL

### Crack Growth Rate

The first differential equation is taken as follows:

$$\frac{da}{dN} = C_o\left(K_{max} - K_{th}\right)^n \qquad \text{if } K_{max} \geq K_{th}$$

$$\frac{da}{dN} = 0 \qquad \text{if } K_{max} \leq K_{th} \qquad (3)$$

which leads to the following remarks:

- crack arrest is well accounted for by the factor $(K_{max} - K_{th})$,
- in the case of periodic loading, it is possible to write [6]:

$$\frac{da}{dN} = C_o\left(K_{max} - K_{th}\right)^n = CK^n_{max} f^n(R) \qquad (4)$$

with

$$K_{th} = \frac{K_{max}}{R_2(R)}$$

and analytical expression for f(R) may be written as:

$$f(R) = \frac{R_2(0) \cdot (R_2(R)-1)}{R_2(R) \cdot (R_2(0)-1)} \tag{5}$$

with

$$C_o = C \cdot \left(\frac{R_2(0)}{R_2(0)-1}\right)^{\eta} . \tag{6}$$

Analytical expression for f(R) is in good agreement with experimental data, and thus we can assume that mean stress influence is well represented.

- Furthermore, it is possible to show that the first constitutive equation is independent of Wheeler's retardation parameter.

Crack Propagation Threshold

The second constitutive differential equation is more complicated. Nevertheless, several remarks must be done.

- Let us consider again the case of a single overload $P_{MS}$: before this overload, crack propagation threshold $K_{th}$ is equal to

$$\frac{K_{max}}{R_2(R)}$$

where $K_{max}$ is the maximum stress intensity factor corresponding to the maximum cyclic load $P_{max}$. R is the ratio

$$\frac{P_{min}}{P_{max}} .$$

When overload is applied, $K_{th}$ increases and following this single overload, crack propagation threshold decreases very slowly and becomes equal to

$$\frac{K_{max}}{R_2(R)}$$

when memory effect has disappeared (Figure 2).

Then, we must introduce the factor

$$\left(\frac{K_{max}}{R_2(R)} - K_{th}\right)$$

in the second constitutive equation in order to make crack fissuration threshold converge to

$$\frac{K_{max}}{R_2(R)} .$$

Furthermore, the exponent of this term will be taken equal to 1: consequently, in the case of a periodic loading, the variation of the crack propagation threshold $\Delta K_{th}$ for one cycle will be of the same order of





magnitude than the variation of the stress intensity factor $\Delta K_{max}$ for one cycle. Finally, the sign of the expression

$$\left(\frac{K_{max}}{R_2(R)} - K_{th}\right)$$

will provide the sign of the variation

$$\frac{dK_{th}}{dN} .$$

- In the case of a single overload, we notice that crack propagation threshold variation is more important when load increases than when load decreases. So, we introduce the Wheeler's retardation parameter in the second constitutive equation.

In summary, the second constitutive differential equation may be taken as follows:

$$\frac{dK_{th}}{dN} = \alpha(R) \cdot \left(\frac{K_{max}}{R_2(R)} - K_{th}\right) \cdot \left(\frac{\rho_{SS}}{\rho_{AS}}\right)^{\delta} \tag{7}$$

Note that $\delta$ is an exponent characteristic of material. $\alpha(R)$ is a function of the ratio

$$R = \frac{K_{min}}{K_{max}}$$

and depends on $R_1(R)$ and $R_2(R)$.

## APPLICATIONS

Theoretical results obtained through this model have been compared with experimental data recorded from tests on bending specimen. As a preliminary it has been shown that bending tests were well representative of crack growth under cyclic loading: Paris' law determined from tensile tests on cracked plates gave good results when applied to bending tests.

Calculation results are in good agreement with experimental ones for many cases: a single overload (Figures 3 and 4), two levels loading (Figure 5), etc.

All the results are reported on Figure 6.

## CONCLUSIONS

Calculation results are in good agreement with experimental data. Note that only the case of a constant minimal load has been considered. Work is being performed to extend the validity of this model to include minimal load as a parameter (compression effects in particular) and to the case of random stochastic loadings. Nevertheless, we consider that the principle of two coupled differential constitutive equations has already received sufficient experimental verifications and consequently the proposed model can be used with confidence for design and analysis.

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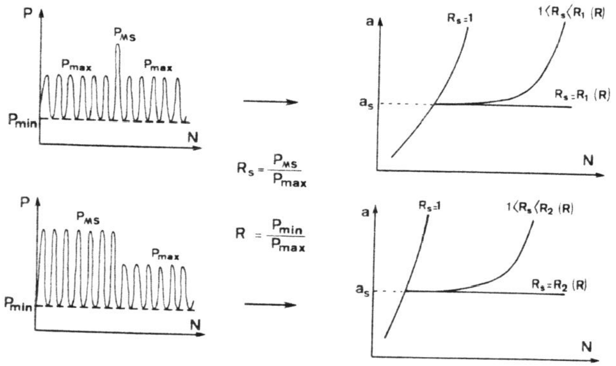


Figure 1 Threshold Effect





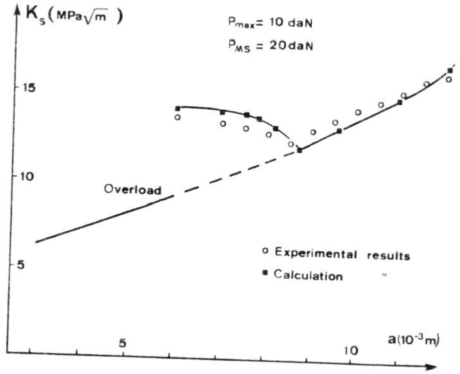


Figure 2 Crack Propagation Threshold Evolution After a Single Overload

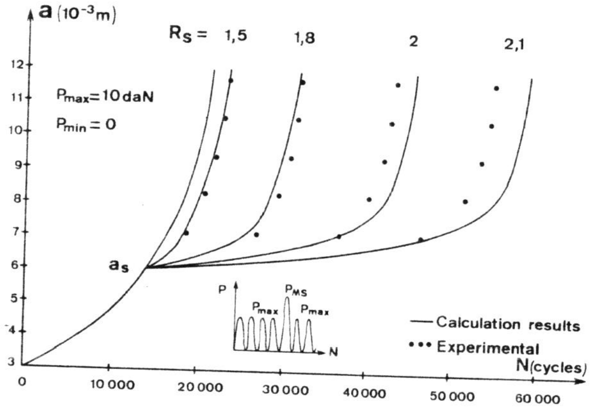


Figure 3 Tests with a Single Overload $\left(P_{min} = 0\right)$

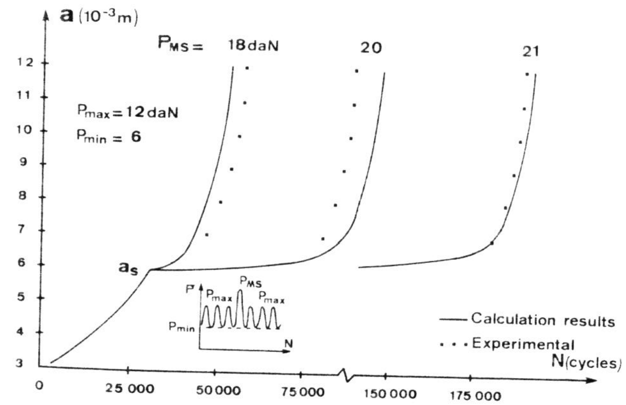


Figure 4 Tests with a Single Overload $\left(P_{min} \neq 0\right)$

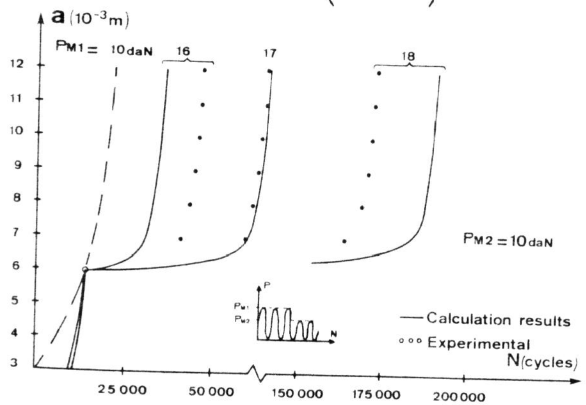


Figure 5 Tests with Two Levels Loading

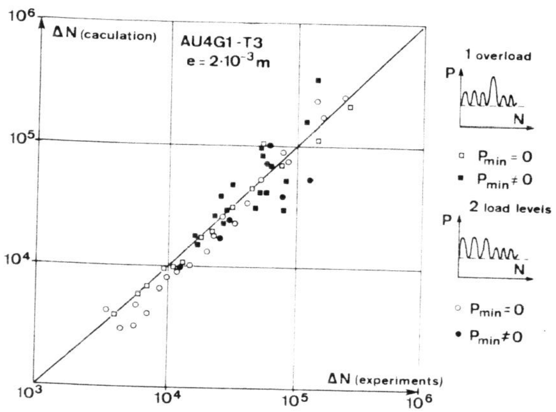


Figure 6 Comparison Between Calculations and Experiments