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TENSILE INSTABILITIES IN STRAIN-RATE DEPENDENT MATERIALS

J. W. Hutchinson and H. Obrecht*

ABSTRACT

Neck development in bars of strain-rate dependent materials under tensile load is analyzed in several ways. For a circular bar of a power-law creeping material, an exact linearized solution is produced for the velocities and stresses due to a small nonuniformity in the radius which varies sinusoidally in the axial direction. This solution is used to assess the accuracy of a long wavelength approximation commonly used to analyze neck development. The long wavelength approximation is then adopted to study the growth of nonuniformities in bars of a general class of strain-rate dependent materials. This analysis, together with two illustrative examples, elucidates Hart's stability criterion for such materials. Attention is drawn to the connection between the criterion and experimental observation in tension testing and the closely parallel situation in creep buckling of compressed columns.

INTRODUCTION

The development of necks in bars of strain-rate dependent materials under tensile load is analyzed from several points of view. To start, power-law creeping materials are considered. An exact linearized solution for the stresses and strain-rates is given for a solid round bar with a slight nonuniformity in its current radius which varies sinusoidally in the axial direction. The rate of growth of the nonuniformity is examined as a function of the ratio of its wavelength to the bar radius. Long wavelength imperfections are found to grow faster, at least in the early stages of growth when the linearized analysis is valid. This growth rate can be calculated accurately by a widely used long wavelength approximation in which the rate of contraction of any cross-sectional area is determined by taking the stress to be locally uniaxial and uniform over the section. The long wavelength approximation is used to study the growth of geometric nonuniformities in a general class of strain-rate dependent materials. This analysis elucidates Hart's [1] stability criterion for such materials. Two illustrative examples are chosen to show that the criterion is not universally useful in that it may greatly underestimate the critical strain or time for noticeable necking to set in.

Hart's criterion is closely related to an analogous stability criterion proposed by Rabotnov and Shesterikov [2] for the creep buckling of columns under compressive load. Attention is also drawn to the connection between the stability criterion and experimental observation in tension testing of rate-dependent materials and the parallel situation noted in column buckling almost twenty years ago.

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* Division of Engineering and Applied Physics, Harvard University, Cambridge, Massachusetts 02138, U. S. A.

GROWTH-RATE OF SMALL GEOMETRIC NONUNIFORMITIES IN A CREEPING BAR

Consider an infinitely long, axisymmetric solid bar whose current radius varies along its axial direction z according to

$$R = R_o[1+\xi \cos(2\pi z/\lambda)] \tag{1}$$

as depicted in Figure 1. Elastic strain-rates are neglected and creep strain-rates are assumed to be given by the well known power-law relation

$$\dot\varepsilon_{ij} = \frac{3}{2}\alpha\sigma_e^{n-1}s_{ij} \quad , \quad \sigma_e = \left(\frac{3}{2}s_{ij}\,s_{ij}\right)^{1/2} \tag{2}$$

where s_{ij} is the stress deviator and σ_e is the effective stress. In simple tension (2) reduces to $\dot\varepsilon = \alpha\sigma^n$.

The bar carries a total axial load P. The perfect bar serves as a convenient reference. With $\xi = 0$ and $R = R_o$, the stress and strain-rate in the perfect bar are

$$\sigma^o_{zz} \equiv \sigma_o = P/\pi R_o^2 \quad , \quad \dot\varepsilon^o_{zz} \equiv \dot\varepsilon_o = \alpha\sigma_o^n \tag{3}$$

where throughout the paper a subscript or superscript o will denote quantities associated with the perfect bar under the same load P. This reference solution will also be used in later sections. Denote the difference between values for the imperfect bar and the perfect bar at the same instant of time as

$$\Delta\dot\varepsilon_{ij} = \dot\varepsilon_{ij} - \dot\varepsilon^o_{ij} \quad , \quad \Delta\sigma_{ij} = \sigma_{ij} - \sigma^o_{ij} \ . \tag{4}$$

The boundary value problem for these quantities is formulated in the Appendix. There it is shown that for small ξ the problem may be linearized and solved in closed form.

Of primary interest here is the rate of growth of the nonuniformity in the cross-sectional area $\Delta\dot A = \dot A - \dot A_o$. A revealing form for expressing this result is (see Appendix)

$$\Delta\dot A = (n-1)\dot\varepsilon_o\,\Delta A\, f(n,q) \tag{5}$$

where

$$\Delta A = 2\pi R_o^2\xi\cos(2\pi z/\lambda) \tag{6}$$

is the discrepancy in cross-sectional area at z in the linearized problem and

$$q = 2\pi R_o/\lambda \ . \tag{7}$$

The expression for f is given in the Appendix. Numerical values for f as a function of q for fixed n are plotted in Figure 2. An expansion of f in small q gives

$$f(n,q) = 1 - \frac{1}{8}\left(\frac{n}{n-1}\right)q^2 - \frac{1}{876}\left(\frac{n^2+3n-9}{n-1}\right)q^4 + \ldots \tag{8}$$

As seen from (8) and Figure 2, $f \to 1$ as $q \to 0$. In this limit (5) reduces to

$$\Delta\dot A = (n-1)\dot\varepsilon_o\,\Delta A \ . \tag{9}$$

Equation (9) is precisely the result obtained by assuming that at any cross-section with a small nonuniformity in area ΔA the stress is uniaxial and uniform across the section, as will be shown later. Calculations based on these assumptions, which are quite common in the material science literature, will be referred to here as long wavelength approximations. Nonuniformities of long wavelength experience the fastest growth-rate in a power-law creeping material and (9) is reasonably accurate if $q < 1$. From (5) and Figure 2, nonuniformities with sufficiently short wavelengths (i.e., q sufficiently large) decay in the sense that $\Delta\dot A > 0$ if $\Delta A < 0$ and $\Delta\dot A < 0$ if $\Delta A > 0$, or more briefly that $\Delta\dot A/\Delta A < 0$. For n = 3, for example, it can be seen from Figure 2 that $\Delta\dot A/\Delta A < 0$ for wavelengths shorter than about 2 1/2 times R_o. A more meaningful measure of decay or growth is the rate of change of the relative size of the nonuniformity, $a(t) = \Delta A(t)/A_o(t)$,

$$\dot a = \Delta\dot A/A_o - \Delta A\dot A_o/A_o^2 = \Delta\dot A/A_o + \dot\varepsilon_o\Delta A/A_o = (\Delta A/A_o)[\Delta\dot A/\Delta A + \dot\varepsilon_o] \tag{10}$$

Thus a negative value of $\Delta\dot A/\Delta A + \dot\varepsilon_o$ or, from (5), a negative value of $(n-1)f+1$ implies a decay of the nonuniformity relative to the evolving area of the perfect bar. Such relative decay never occurs since it can be shown that the limit of f for large q is $-1/(n-1)$ and f approaches this limit from above. In summary, the amplitude of a sinusoidal nonuniformity may decrease in absolute magnitude if q is sufficiently large, but the relative amplitude does not decrease.

A discussion of the behaviour in the vicinity of n = 1 is included in the Appendix since the $(n-1)f$ in (5) has a nonzero limit as $n \to 1$.

NONLINEAR GROWTH OF LONG WAVELENGTH NONUNIFORMITIES IN A CREEPING BAR

The analysis just carried out involves linearization about the solution for the perfect bar. We now restrict attention to nonuniformities with wavelengths which are sufficiently long such that the long wavelength approximation is valid over a substantial portion of the lifetime of the bar. As previously indicated, the stress is assumed to be uniaxial and uniform across each cross-section according to

$$\sigma = P/A \tag{11}$$

where A is the area of the cross-section in question. In this approximation simple formulas can be obtained for the fully nonlinear behaviour, including the time to rupture. As the rupture time is approached necking may become highly localized and then the long wavelength approximation may no longer be accurate. However it is reasonable to expect that the rupture time obtained by this approximation should provide a lower bound to the actual rupture time since the growth-rates associated with shorter wavelengths are slower.

As before, let $A_o(t)$ denote the area of the perfect bar as a function of time, and let A(t) denote the area of the section of the imperfect bar with the smallest cross-sectional area. With

$$\Delta A(t) = A(t) - A_o(t) \tag{12}$$

let η be the measure of the initial discrepancy in area between the section of the imperfect bar in question and the perfect bar defined according to

$$\eta = -\Delta A(0)/A_o(0) \ . \tag{13}$$

Positive η thus corresponds to an initially imperfect bar with its smaller cross-section satisfying $A(0) < A_o(0)$.

Let ℓ denote the current length of a material line element aligned with the tensile axis and $\dot{\ell}$ denote its rate of change. Assuming the deformation is incompressible, the strain-rate (natural-rate) is

$$\dot{\varepsilon} = \dot{\ell}/\ell = -\dot{A}/A \ . \tag{14}$$

It is related to the stress by

$$\dot{\varepsilon} = \alpha\sigma^n \tag{15}$$

For constant P, the equation for A(t) can be obtained by eliminating σ and ε from (11), (14) and (15) and integrating with the result

$$\alpha n[P/A(0)]^n t = 1-[A(t)/A(0)]^n \ . \tag{16}$$

First, the critical time of the perfect bar t_o^c (i.e., the time at which $A \to 0$) is found from (16) to be

$$1/t_o^c = \alpha n[P/A_o(0)]^n = n\dot{\varepsilon}_o(0) \tag{17}$$

as noted originally in [3]. Then with $\tau = t/t_o^c$, (16) may be written for the perfect bar as

$$A_o(t)/A_o(0) = (1-\tau)^{1/n} \ . \tag{18}$$

Next, introducing (12) and (13) and retaining the definition of τ based on the critical time for the perfect bar, one can rewrite (16) in the form most instructive for the imperfect bar as

$$\Delta A(t)/A_o(0) = -(1-\tau)^{1/n} + [(1-\eta)^n - \tau]^{1/n} \ . \tag{19}$$

The rupture time for the imperfect bar is

$$t^c = t_o^c(1-\eta)^n \ . \tag{20}$$

Solid line curves in Figure 3 are calculated from (19). They approach vertical asymptotes given by (20). Also included in Figure 3 as dashed line curves are predictions based on a linearization of (19) in small η, i.e.,

$$\Delta A(t)/A_o(0) = -\eta/(1-\tau)^{\frac{n-1}{n}} \ . \tag{21}$$

(Alternatively, (21) can be obtained by integrating (9).) For small non-uniformities, e.g., $\eta \leq 0.01$, the linearized solution is accurate over most of the lifetime of the bar. Nevertheless, it is incapable of giving any indication of the dependence of the rupture time on η.

A fully nonlinear long wavelength analysis, largely based on numerical computation, has been presented by Burke and Nix [4] for this same class of materials. They give details of the spatial development of the neck from some specific initial nonuniformities in finite length specimens. Further discussion of power-law creeping materials will be given later in the paper following discussion of a more general class of strain-rate dependent materials.

ANALYSIS OF GROWTH OF LONG WAVELENGTH NONUNIFORMITIES IN BARS OF A MORE GENERAL CLASS OF STRAIN-RATE DEPENDENT MATERIALS

Following Hart [1] we consider a constitutive relation in simple tension of the form

$$\sigma = F(\varepsilon, \dot{\varepsilon}) \tag{22}$$

where σ and $\dot{\varepsilon}$ are defined as before. The bar is subject to an axial load P(t) which will be assumed to satisfy $P(0) = 0$. As in [1], the long wavelength assumption is invoked so that (11) and (14) apply. As before A(t) denotes the smallest cross-sectional area of the imperfect bar and $A_o(t)$ that of the perfect bar. Both bars are subject to the same load history P(t). The initial nonuniformity measure η is again defined by (13) with $\Delta A(t)$ by (12). Denote the solution for the stress and strain associated with the perfect bar by $\sigma_o(t)$ and $\varepsilon_o(t)$ and let $\sigma(t)$ and $\varepsilon(t)$ be associated with the smallest cross-section of the imperfect bar. Let

$$\Delta\sigma(t) = \sigma(t) - \sigma_o(t) \quad , \quad \Delta\varepsilon(t) = \varepsilon(t) - \varepsilon_o(t) \ . \tag{23}$$

We proceed to obtain the linearized equation for the growth of ΔA by systematically linearizing with respect to η and the Δ-quantities. Note that by virtue of the definition of the Δ-quantities the linearization is with respect to the evolving configuration of the perfect bar. From (11)

$$\Delta\sigma A_o + \sigma_o \Delta A = 0 \tag{24}$$

and from (14)

$$\Delta\dot{\varepsilon} = -\Delta\dot{A}/A_o + \dot{A}_o\Delta A/A_o^2 = -(\Delta A/A_o)^{\bullet} \ . \tag{25}$$

Integration of (25) subject to the initial condition (13) gives

$$\Delta\varepsilon = -\eta - \Delta A/A_o \ . \tag{26}$$

From (22),

$$\Delta\sigma = \left(\frac{\partial F}{\partial \varepsilon}\right)_o \Delta\varepsilon + \left(\frac{\partial F}{\partial \dot{\varepsilon}}\right)_o \Delta\dot{\varepsilon} \tag{27}$$

where it is to be understood that the partial derivatives in (27) are functions of time through their evaluation at $\varepsilon_o(t)$ and $\dot{\varepsilon}_o(t)$. Elimination of $\Delta\sigma$, $\Delta\varepsilon$ and $\Delta\dot{\varepsilon}$ from (24), (25), (26) and (27) gives the linearized differential equation for $\Delta A(t)$. It is

$$\Delta\dot{A}+h(t)\Delta A = (\gamma/m)\dot{A}_o\eta \tag{28}$$

where

$$\gamma(t) = \left(\frac{\partial F}{\partial \varepsilon}\right)_o \sigma_o^{-1}\ , \quad m(t) = \left(\frac{\partial F}{\partial \dot{\varepsilon}}\right)_o \frac{\dot{\varepsilon}_o}{\sigma_o} \tag{29}$$

and

$$h(t) = \frac{\dot{\varepsilon}_o}{m}\,[-1+\gamma+m] \tag{30}$$

Hart's original investigation proceeds along lines similar to the above except that he does not explicitly bring in the effect of the initial nonuniformity which enters as the nonhomogeneous term in (28). (Note that only if $\gamma = 0$ will this term drop out). Consequently Hart arrives at just the homogeneous part of (28) and proposes that linear stability hinges on the sign of $\Delta\dot{A}/\Delta A$. This, in turn, leads to his condition for stability, $(-1+\gamma+m) > 0$. Apparently his suggestion that $\Delta\dot{A}/\Delta A$ is negative if $(-1+\gamma+m) > 0$ has led some to conclude that nonuniformities should tend to decrease with time under these conditions. In general, this is not correct. Indeed, it can readily be shown from (28) that the sign of $\Delta\dot{A}/\Delta A$ is not tied to the sign of $-1+\gamma+m$.

The load history enters into (28) through $h(t)$ and the nonhomogeneous term which both depend on the deformation history of the perfect bar. The solution to (28) subject to the initial condition (13) is

$$\Delta A = -\eta e^{-H(t)}\left[A_o(0) - \int_o^t [\gamma(\tau)/m(\tau)]\dot{A}_o(\tau)e^{H(\tau)}d\tau\right]$$

where

$$H(t) = \int_o^t h(\tau)d\tau\ .$$

Depending on $P(t)$ and on the nature of (22), h may change signs during the course of the time history, as will be illustrated by example in the next section. It is clear from the above solution that exponential growth or decay depends not only on the instantaneous sign of h but on the integrated history of h as well. More important to our discussion, however, is the magnitude of h which relates to the characteristic time associated with exponential growth or decay. If m is not very small, for example $m = O(1)$, then, from (30), $h = O(\dot{\varepsilon}_o)$. That is, the characteristic time is of the same order as the time scale associated with development of axial strains of order unity in the perfect bar. A change in sign of $-1+\gamma+m$ from positive to negative is not likely to signal any noticeable increase in the growth of a small nonuniformity until an additional axial strain of order unity takes place. In other words, when m is not very small the early stage of the necking process is extremely long-lived, and the sign of $-1+\gamma+m$ cannot be expected to serve as a useful indicator.

On the other hand, the sign of h may be a sharp indicator if the magnitude of h is large so that the characteristic time is short compared to the relevant time scale associated with deformation of the perfect bar. This can be the situation for materials which are nearly insensitive to the strain-rate (i.e., $0 \le m \ll 1$) and which are subject to the history of a standard tension test at some nominal strain-rate $\dot{\varepsilon}_o$. Then $h = O(\dot{\varepsilon}_o/m)$. For $m \ll 1$ the sign of h depends essentially on the sign of $-1+\gamma$. Furthermore, from (29),

$$-1 + \gamma = 0 \quad \Longrightarrow \quad \sigma_o = d\sigma_o/d\varepsilon_o \tag{31}$$

which is the well-known necking criterion of Considère. Condition (31) is a reasonably sharp criterion for predicting the onset of necking in bars of materials which are strain-rate insensitive or nearly so. It is also known that the long wavelength approximation used in deriving (31) is adequate as long as the wavelength of the nonuniformity is not less than several radii of the specimen [5].

Doubt can also be cast on the significance of basing a criterion on $(-1+\gamma+m)$ by noting that the criterion depends heavily on the choice of ΔA as the measure of the nonuniformity. Consider, as before, the measure, $a(t) = \Delta A(t)/A_o(t)$, of the evolving relative size of the nonuniformity, which is at least as acceptable as the absolute size ΔA. Equation (28) is readily converted using (10) to

$$\dot{a} + \tilde{h}(t)a = -(\gamma/m)\dot{\varepsilon}_o\eta \tag{32}$$

where

$$\tilde{h}(t) = \frac{\dot{\varepsilon}_o}{m}\,[-1+\gamma] \tag{33}$$

thus suggesting the sign of $-1+\gamma$ as a possible criterion. Note from (25) that $\Delta\dot{\varepsilon} = -\dot{a}$, so that a criterion based on growth or decay of a is identical to one based on localization of axial strain. This alternative criterion, tied to the sign of $-1+\gamma$, has been derived by Jonas, et al [6]. Only when m is very small will any such similar criterion be essentially independent of the choice of the measure of the nonuniformity, and then the criterion is essentially that of Considère (31). Of course, the shortcomings for instances when m is not very small apply equally well to any such criterion, as pointed out by Jonas et al.

A NUMERICAL EXAMPLE

To illustrate some of the above remarks we take the simple functional relation between σ, ε and $\dot{\varepsilon}$ which is frequently used to represent nonsteady creep

$$\dot{\varepsilon} = \alpha\sigma^n/\varepsilon^p \tag{34}$$

or

$$\sigma = F(\varepsilon,\dot{\varepsilon}) = \alpha^{-1/n}\varepsilon^{p/n}\dot{\varepsilon}^{1/n}\ . \tag{35}$$

The parameters in (29) and (30) are

$$m = 1/n \quad , \quad \gamma(t) = p/[n\varepsilon_o(t)] \tag{36}$$

$$h(t) = \{-1+1/n+p/[n\varepsilon_o(t)]\}\,\dot{\varepsilon}_o/m \ . \tag{37}$$

The response curves of Figure 4 are for constant applied load P. They were calculated numerically using a straightforward incremental method with due allowance for the initial time step. The nondimensional time τ is defined by

$$\tau = t[n\alpha\sigma_o^n(0)] \ . \tag{38}$$

With a dot denoting differentiation with respect to τ, the response of the perfect bar is governed by

$$\frac{\dot{A}_o}{A_o(0)} = -\frac{1}{n\varepsilon_o^p}\left(\frac{A_o}{A_o(0)}\right)^{1-n} \quad , \quad \dot{\varepsilon}_o = \frac{1}{n\varepsilon_o^p}\left(\frac{A_o}{A_o(0)}\right)^{-n} \ . \tag{39}$$

In the long wavelength approximation a similar set of equations governs the time history of the smallest cross-section of the imperfect bar. The initial nonuniformity is again defined by (13). Curves for the imperfect bar, $\eta > 0$, in Figure 4 are all plotted against the nondimensional time τ defined in (38).

From (37) it can be seen that h will be positive when ε_o is sufficiently small. The solid dots in Figure 4 indicate the time at which h becomes negative. It is apparent in these examples at least that no special significance can be ascribed to the time at which h = 0. The Considère criterion, $-1+\gamma = 0$, is met prior to the time indicated by the dot.

An experimental investigation of necking in a strain-rate sensitive metal system was conducted by Sagat and Taplin [7]. In one case in which experimentally measured values of m and γ implied that $(-1+\gamma+m)$ became negative for axial strains greater than about 0.02, the first detectable nonuniformities were not observed until strains about 0.7 were attained. Hart [1] noted a similar delay in necking beyond his stability limit for the special class of materials where $\gamma = 0$ and m is not very small. The pure power-law creep relation (15) is such a material. In this case, $h = \dot{\varepsilon}_o(1-n) < 0$ with $\Delta\dot{A}/\Delta A > 0$ (see (28)) for all long wavelength nonuniformities. Nevertheless, the time for any incipient neck to develop is on the order of the rupture time of the perfect bar, as has been noted in connection with Figure 3. Burke and Nix [4] have also emphasized this feature. In addition, they have performed calculations to show that a small nonuniformity has little influence on the overall load-elongation history over most of the lifetime of the specimen.

RELATION TO WORK IN CREEP BUCKLING

The history of creep buckling of axially compressed columns is parallel in some respects to events in the study of tensile instabilities in strain-rate sensitive materials. In 1957 Rabotnov and Shesterikov [2] proposed a criterion for creep buckling of columns based on an analysis of the

exponential growth or decay of small disturbances obtained by perturbing from the evolving solution for the perfect column. These authors considered dynamic disturbances and therefore included inertia terms in their stability equation, which turned out to be a third order ordinary differential equation. However, their stability criterion is determined by the quasi-static terms in their equation (i.e., the inertia-independent terms) and the quasi-static equation is first order, analogous to (28). (Rabotnov and Shesterikov do not introduce small initial imperfections in their analysis. If they had, their resulting quasi-static equation would be unaltered except for a nonhomogeneous term analogous to that in (28)). The Rabotnov-Shesterikov criterion attracted considerable attention. Yet comparison of their criterion with test data on creep buckling times revealed that the experimentally measured critical time was usually greatly in excess of the prediction of the criterion, similar to the situation for tensile testing of strain-rate sensitive materials. An extensive review of creep buckling of columns has been given by Hoff [8].

APPENDIX: ANALYSIS OF CREEPING BAR WITH AN AXISYMMETRIC IMPERFECTION

The boundary value problem for the stresses and velocity fields in a bar of material characterized by the power-law creep relation (2) is a nonlinear, viscous flow problem which depends on the current geometry and is otherwise independent of prior history. The problem considered (Figure 1) is an infinite axisymmetric bar with a sinusoidal variation of the radius according to (1).

In a cylindrical coordinate system (r,θ,z) with z directed along the axis of the bar, the field equations for axisymmetric, incompressible flow are

$$\left.\begin{aligned} &\dot{\varepsilon}_r = v_{r,r} \ , \quad \dot{\varepsilon}_\theta = r^{-1}v_r \ , \quad \dot{\varepsilon}_z = v_{z,z} \\ &\dot{\varepsilon}_{rz} = (v_{r,z}+v_{z,r})/2 \ , \quad \dot{\varepsilon}_r+\dot{\varepsilon}_\theta+\dot{\varepsilon}_z = 0 \end{aligned}\right\} \tag{40}$$

$$r^{-1}(r\sigma_r)_{,r}+\sigma_{rz,z}-r^{-1}\sigma_\theta = 0 \tag{41}$$

$$\sigma_{z,z}+r^{-1}(r\sigma_{rz})_{,r} = 0 \tag{42}$$

where v_r and v_z are the velocity components in the r and z directions, and the stress and strain-rate components are denoted with a standard notation. These equations are supplemented by the constitutive law (2). Traction-free conditions on the lateral surface, r = R, can be written as

$$\sigma_r\cos^2\psi+\sigma_z\sin^2\psi-2\sigma_{rz}\sin\psi\cos\psi = 0 \tag{43}$$

$$(\sigma_r-\sigma_z)\sin\psi\cos\psi+\sigma_{rz}(\cos^2\psi-\sin^2\psi) = 0 \tag{44}$$

where $\psi(z)$ is the angle made by the tangent vector to the surface with a vector in the z-direction as depicted in Figure 1. The bar is assumed to support a total axial load P so that for any z

$$P = 2\pi \int_0^R \sigma_z \, r \, dr \, . \tag{45}$$

The amplitude ξ of the nonuniformity is assumed to be small which permits the governing equations to be linearized in ξ and the Δ-quantities. Equations (40), (41) and (42) are already linear so the Δ-quantities simply replace their counterparts in these equations. Equation (2) becomes

$$\left.\begin{aligned}
\Delta\dot{\varepsilon}_z &= k^{-1} n \Delta s_z \\
\Delta\dot{\varepsilon}_r &= k^{-1}[\Delta s_r - (n-1)\Delta s_z/2] \\
\Delta\dot{\varepsilon}_\theta &= k^{-1}[\Delta s_\theta - (n-1)\Delta s_z/2] \\
\Delta\dot{\varepsilon}_{rz} &= k^{-1}\Delta s_{rz}
\end{aligned}\right\} \tag{46}$$

where $k = 2/(3\alpha\sigma_o^{n-1})$. These equations may be inverted to give

$$\left.\begin{aligned}
\Delta\sigma_z &= k\Delta\dot{\varepsilon}_z/n - \Delta p \\
\Delta\sigma_r &= k[\Delta\dot{\varepsilon}_r + (n-1)\Delta\dot{\varepsilon}_z/(2n)] - \Delta p \\
\Delta\sigma_\theta &= k[\Delta\dot{\varepsilon}_\theta + (n-1)\Delta\dot{\varepsilon}_z/(2n)] - \Delta p \\
\Delta\sigma_{rz} &= k\Delta\dot{\varepsilon}_{rz}
\end{aligned}\right\} \tag{47}$$

where $\Delta p = -(\Delta\sigma_z + \Delta\sigma_r + \Delta\sigma_\theta)/3$.

A systematic linearization of the boundary conditions (43) and (44) gives

$$\Delta\sigma_r = 0 \quad \text{on} \quad r = R_o \tag{48}$$

$$\Delta\sigma_{rz} = -\xi q \sigma_o \sin(2\pi z/\lambda) \quad \text{on} \quad r = R_o \tag{49}$$

where q is defined in (7). In the linearized problem the conditions may be applied at $r = R_o$ as indicated. Condition (49) is consistent with applied load P. To see this write (45) as

$$\begin{aligned}
P &= 2\pi \int_0^{R_o} (\sigma_o + \Delta\sigma_z) r \, dr + 2\pi \int_{R_o}^{R} (\sigma_o + \Delta\sigma_z) r \, dr \\
&= \pi R_o^2 \sigma_o + 2\pi \int_0^{R_o} \Delta\sigma_z r \, dr + 2\pi R_o^2 \xi \sigma_o \cos(2\pi z/\lambda) + O(\xi^2, \xi\Delta\sigma_z)
\end{aligned} \tag{50}$$

Thus the linearized version of (50) implies

$$\int_0^{R_o} \Delta\sigma_{z,z} r \, dr = (2\pi R_o^2 \xi \sigma_o/\lambda)\sin(2\pi z/\lambda) \, . \tag{51}$$

But by integrating the equilibrium equation (42) from 0 to R_o, one finds (51) reduces to (49).

Incompressibility permits the introduction of a velocity potential Φ such that

$$\Delta v_r = -\Phi_{,z} \quad , \quad \Delta v_z = r^{-1}(r\Phi)_{,r} \, . \tag{52}$$

The governing field equations can be reduced to a single partial differential equation in Φ along lines similar to those carried out in [5] and [9]. We find

$$\tilde{L}^2(\Phi) - \left(\frac{n-3}{n}\right)\tilde{L}(\Phi_{,zz}) + \Phi_{,zzzz} = 0 \tag{53}$$

where

$$\tilde{L}(\Phi) = (r^{-1}(r\Phi)_{,r})_{,r} \tag{54}$$

and where $\tilde{L}^2$ indicates two applications of $\tilde{L}$. The boundary conditions (48) and (49) become

$$r^{-1}[r\tilde{L}(\Phi)]_{,r} + \left(\frac{3}{n}\right)\Phi_{,zzr} + \left(\frac{3-2n}{n}\right) r^{-1}\Phi_{,zz} = 0 \, , \quad r = R_o \tag{55}$$

$$\tilde{L}(\Phi) - \Phi_{,zz} = -3\xi q \dot{\varepsilon}_o \sin(2\pi z/\lambda) \quad , \quad r = R_o \, . \tag{56}$$

Equation (53), with the boundary conditions, admits a separated solution

$$\Phi = \phi(r)\sin(2\pi z/\lambda) \, . \tag{57}$$

The equation for ϕ can be written as

$$(L + q^2\rho^2)(L + q^2\bar{\rho}^2) = 0 \tag{58}$$

where the operator L is defined in terms of the nondimensional coordinate $\zeta = r/R_o$ by

$$L(\phi) = (\zeta^{-1}(\zeta\phi)_{,\zeta})_{,\zeta} \, . \tag{59}$$

With $i = \sqrt{-1}$, ρ is defined as the first quadrant root of

$$\rho^2 = \frac{n-3}{2n} + i\sqrt{1 - \left(\frac{n-3}{2n}\right)^2} \tag{60}$$

and $(\bar{\ })$ denotes complex conjugation.

The general solution to (58) for a real, bounded ϕ can be written in terms of an unknown complex constant c as

$$\phi = [-\xi\dot{\varepsilon}_o\lambda R_o/(2\pi)]R_e\{cJ_1(q\rho\zeta)\} \tag{61}$$

where R_e denotes the real part and the terms in the square brackets have been introduced for later convenience. Here, J_n is the Bessel function of order n of the first kind with complex argument. Substitution of (61) into the boundary conditions (55) and (56) and use of identities for Bessel functions leads to the following pair of equations for determining c:

$$R_e\{c[J_1(q\rho) - q\rho(\rho^2+3/n)J_0(q\rho)/2]\} = 0 \tag{62}$$

$$R_e\{c(1-\rho^2)J_1(q\rho)\} = 3 \ . \tag{63}$$

To determine $\Delta\dot{A}$, and thus $f(n,q)$ defined in (5), first note from (52) that

$$\Delta v_r(R_o) = \xi\dot{\varepsilon}_o R_o R_e\{cJ_1(q\rho)\}\cos(2\pi z/\lambda) \ . \tag{64}$$

Next, from

$$\dot{A}_o + \Delta\dot{A} = \dot{A} = 2\pi R[v_r^o(R) + \Delta v_r(R)] = 2\pi R[-R\dot{\varepsilon}_o/2 + \Delta v_r(R)]$$

one can obtain the linearized expression

$$\Delta\dot{A} = -\Delta A\dot{\varepsilon}_o + 2\pi R_o\Delta v_r(R_o) \ . \tag{65}$$

Using (5), (6), (64) and (65) one finds

$$f(n,q) = \frac{1}{n-1}[R_e\{cJ_1(q\rho)\} - 1] \ . \tag{66}$$

Numerical results plotted in Figure 2, and those discussed below, were determined from (62), (63) and (66).

The definition of f in (5) is not convenient for displaying results for n in the vicinity of unity. Let $g = (n-1)f$ so that, instead of (5), one has

$$\Delta\dot{A} = \dot{\varepsilon}_o\Delta A\ g(n,q) \ . \tag{67}$$

Numerical results for g for n in the range $1 < n \leq 2$ are shown in Figure 5. Equations (62), (63) and (66) degenerate for $n = 1$ and this case must be treated specially, which we have not done. Nevertheless, the numerical results for $n = 1.001$ in Figure 5 are for all practical purposes identical to those one would obtain for $n = 1$. Furthermore, the expansion of g for small q obtained from (8) holds when $n = 1$; it is

$$g = -q^2/8 + 5q^4/876 + \dots \ .$$

As seen in Figure 5, $\Delta\dot{A}/\Delta A < 0$ for all finite q when $n = 1$. However, as already mentioned, the relative size of the nonuniformity $\Delta A/A_o$ always increases according to the linear theory.

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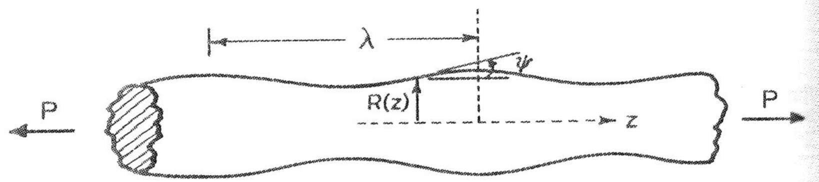


Figure 1

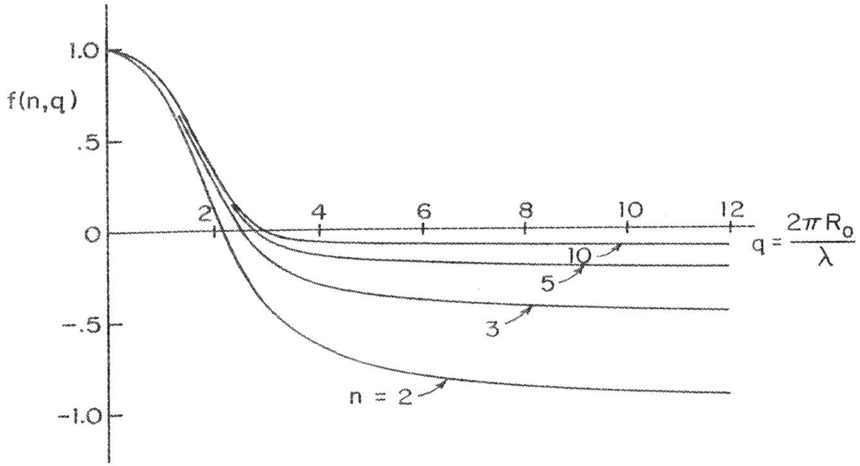


Figure 2

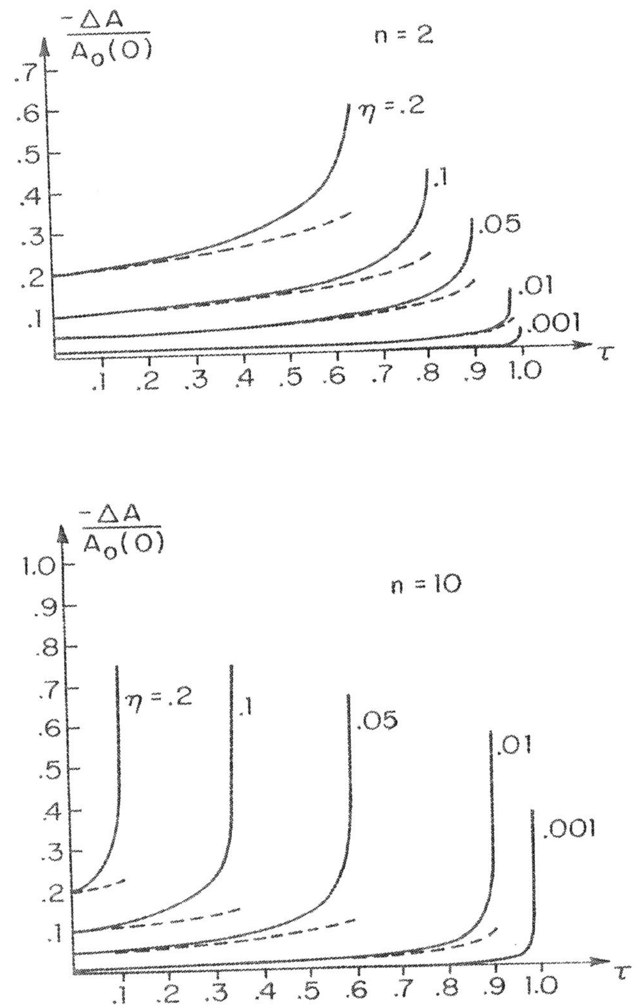
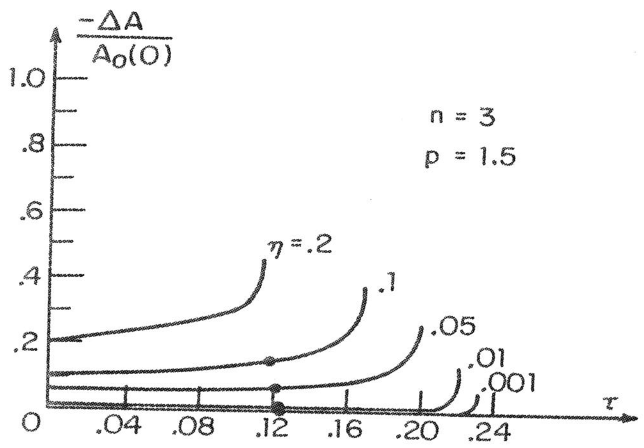
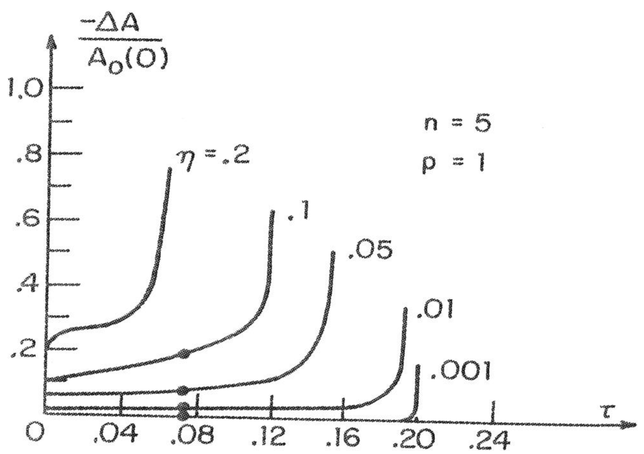


Figure 3



$\frac{-\Delta A}{A_o(0)}$
1.0
.8
.6
.4
.2
0
.04
.08
.12
.16
.20
.24
τ
n = 3
p = 1.5
η =.2
.1
.05
.01
.001



$\frac{-\Delta A}{A_o(0)}$
1.0
.8
.6
.4
.2
0
.04
.08
.12
.16
.20
.24
τ
n = 5
p = 1
η =.2
.1
.05
.01
.001

Figure 4