Stability of a Part-Through Crack in a Material Sensitive to Strain-Hardening

J. Dufresne

The part-through cracks are, as far as pressurized vessels are concerned, the most probable type of defects that can occur. Various studies concerning the stress distribution and the stability of this type of cracks have been undertaken. Rice has studied the case of a fully linear elastic material [1]), Reynolds [2]), Eiber and al. [3]) have done experimental studies on A106B steel tubes for L.W.R. Eiber [4]) proposes an empirical formula extending Hahn's flow stress criteria [5]) to part-through cracks. This formula has been found to be accurate for the specific material considered here (A106B).

In this publication an analysis of the phenomenon is proposed in the case of materials having strain-hardening characteristics. The method utilised is analogous to Dugdale's in his study of the plasticity zone at the crack tips [6]). In order to perform this analysis, the plate having a non-through crack is separated into two simple elements having the characteristics of a fully elastic material or of an elastic-plastic material sensitive or non-sensitive to work-hardening.
The boundary conditions between the two zones are satisfied on the basis of the conformal mapping method proposed by Muskhelishvili [7]). This study was first done for flat plates and has ulteriorly been extended to cylindrical vessels.

Analytical Model

A part-through crack in a flat plate submitted to a uniform stress field at infinity in a direction perpendicular to the crack axis is considered here. One can settle boundaries in this plate to the zones where the material is either plastically or elastically deformed. The whole structure can be described as follows (Fig. 1):

a) a plate having a thickness equal to the actual thickness and having a crack with a fictive length greater than the real one. This plate is submitted at infinity to the real loads and its deformations are fully elastic.

b) On the edges of this through-crack, perpendicular loads are applied that tend to close it; their intensities are equal in each point to the force developed by the ligament when fully plastic. If the material considered here is purely plastic (non work-hardening) and if the thickness of the ligament is constant, this closing force will be constant along the whole length of the crack and will correspond to the yield stress in the thickness and along the full length of the ligament. If on the contrary the material is strain-hardening sensitive the closing force will depend on the actual opening of the crack.

c) At the two extremities of the through-crack, additional closing forces are applied equivalent to the forces developed in the plastic zones existing at the crack tips of the real crack (Dugdale's zone). These loads correspond to the yield stress of the material in the zones considered. Strain-hardening has not been taken into account since, for continuity reasons the opening in the vicinity of the crack tip is small compared to that one at the centre, therefore the effect of work-hardening can be neglected in these zones.

Rupture Criteria

One admits that the rupture of the ligament of a part-through crack will initiate at the centre. One can then attempt to propose a rupture criteria for the ligament. These criteria implie two conditions, one is related to the deformation of the ligament in its middle (analogous to the COD concept), the other is related to the state of stress at the centre of the ligament:

The criteria can be expressed as follows:

Rupture of the ligament in its middle occurs when:

- the crack has opened an amount δ_c in its middle where δ_c is a material characteristic,
- the equivalent stress in the ligaments centre is equal to the ultimate stress of the material.

Analytical Solution

When the material is sensitive to strain-hardening a sinus law has been chosen to represent the variation of the closing forces along the ligament. To these forces correspond a stress applied along the crack and on the whole thickness of the plate equal to:

$$\sigma_1(x) = \frac{e-p}{2e}\left[(\sigma_u + \sigma_y) + (\sigma_u - \sigma_y)\cos\pi\frac{x}{\ell}\right] \qquad (1)$$

The three boundary conditions applied on the whole thickness of the elastic plate are as follows:

- Stress at infinity perpendicular to the crack: σ_h $\quad -\infty < x < +\infty$
- Variable stress along the crack $\sigma_1(x)$ (1) $\quad -\ell < x < +\ell$
- Stress at the crack tips: $\frac{p}{e}\sigma_y$ .... $\left\{\begin{array}{l}\ell < x < a\\ -a < x < -\ell\end{array}\right.$

Following a method utilised by Muskhelihvili one determines for every boundary conditions above mentioned two conjugate harmonic funtions ϕ and ψ that are related to the Airy function U so that:

$$U = \mathrm{Re}\,[\bar{z}\,\phi(z) + \psi(z)]$$

The functions compatible with the boundary conditions at infinity and with the crack-tip closing forces have been already determined [7,8]). The functions associated with a sinusoidal variation of the cloosing forces along the edge of the crack (see (1)) have been determined:

$$\varphi = \frac{\ell}{4}\left[\frac{\sigma_u+\sigma_y}{\zeta} + \frac{\sigma_u-\sigma_y}{2}\zeta\left(1-\frac{\zeta^2}{3}\right)\right]\left(1-\frac{p}{e}\right) \qquad (2)$$

$$\psi = \frac{\ell}{2}\left[(\sigma_u+\sigma_y)\frac{\zeta}{\zeta^2-1} + \frac{\sigma_u-\sigma_y}{2}\zeta\left(1+\frac{\zeta^2}{3}\right)\right]\left(1-\frac{p}{e}\right) \qquad (3)$$

One can determine by summing the three couples of conjugate functions above mentioned the functions ϕ and ψ that satisfy the 3 imposed boundary conditions.

Following the same reasoning as Dugdale's the coefficients of the term in $1/\zeta^2 - 1$ are annuled, in order to eliminate a discontinuity in the stress field at the crack tip; one finds the formula:

$$\frac{\cos\alpha}{2}\left(1-\frac{p}{e}\right)\left(\frac{\sigma_u}{\sigma_y} + 1 - 2\frac{\sigma_h}{\sigma_y}\right) + \frac{2\alpha}{\pi}\frac{p}{e} = \frac{\sigma_h}{\sigma_y}\frac{p}{e} \qquad (4)$$

with: $a \cos\alpha = 1$.

For low values of α:

$$\alpha = \frac{\pi}{2}\frac{e}{p}\left[\frac{\sigma_h}{\sigma_y} - \frac{1}{2}\left(1-\frac{p}{e}\right)\left(\frac{\sigma_u}{\sigma_y}+1\right)\right] \qquad (5)$$

Knowing the functions ϕ and ψ and α, one can determine the value of the crack opening δ in the centre of the crack:

$$\delta_c = \frac{8}{\pi}\frac{\ell\sigma_y}{E}\left[\frac{p}{e}\,\mathrm{Log.\,tg}\left(\frac{\alpha}{2}+\frac{\pi}{4}\right) - \frac{\pi}{6}\left(\frac{\sigma_u}{\sigma_y}-1\right)\left(1-\frac{p}{e}\right)\right] \qquad (6)$$

Considering that this aperture δ_c is a material characteristic, one can plot a family of curves interconnecting the dimensionless parameters: p/e, σ_u/σ_y, $\ell\sigma_y/\delta_c E$. These curves are represented on Fig. 2 for a hardening value of $\sigma_u/\sigma_y = 1,5$.

Application to part-through cracks in the shells of cylindrical vessels

The same reasoning as the one above presented is applicable to an internally pressurized cylindrical vessel. Nevertheless, in this case two supplementary considerations

must be made:

a) *Biaxiality*: if the Von Mises criterion concerning the equivalent stress in materials that are plastically deformed is applied, it is necessary to replace the values σ_u and σ_y in the equations (4), (5) and (6) by corrected values so that together with the axial hoop stress $\sigma_h/2$ an equivalent stress equal to σ_u and σ_y for monoaxial tensile tests is obtained. Thus:

$$\sigma^x = \sigma_m\left[\sqrt{1-\frac{3}{16}\left(\frac{\sigma_h}{\sigma_m}\right)^2} + \frac{1}{4}\frac{\sigma_h}{\sigma_m}\right]$$

σ_h = hoop stress
$\sigma_m = \sigma_u$ or σ_y monoaxial
$\sigma^x = \sigma_u$ or σ_y in the ligament

b) *Curvature*: It is not necessary to consider the usual corrections for through-cracks for the curvature of the shell; these corrections assume obviously that no bounds exist between the edges of the crack, this is not so for part-through cracks.

Experimental Verifications

Verifications of the various hypotheses have been undertaken on the basis of data collected by Reynolds for A106B steel [2]). These results give for every experiment the geometry of the crack, the σ_h stress at bursting and the stresses corresponding to the elastic limit obtained by pressurizing the tanks without cracks. The monoaxial stress at yield point can be derived by dividing by 1.15 these latter stresses. The ultimate mono-axial stress has been taken as $\sigma_u = 1{,}46\ \sigma_y$. On the basis of these data the critical value of δ_c has been calculated according to formula (6). The average value found is δ_c = 0,085 mm, the average of the square of the discrepancies is 0,05 mm. Considering this value of δ_c as a characteristic of the material the value of $(\sigma_h/\sigma_y)_{th}$ has been computed for each test. The value of $(\sigma_h/\sigma_y)_{th}$ being that one for which, according to the rupture criteria described, must cause the rupture of the ligament. These values are compared with the experimental values $(\sigma_h/\sigma_y)_{ex}$ in Fig. 3. About 9/10 of the calculated values are not more than 15% off the experimental values. One must note that the experimental results also show a large scattering, this probably is due to the fact that the test specimen have a radius at the borderline between crack and material and were not submitted to initial fatigue crack growth before the bursting tests. On the other hand, since value of σ_u was not given in the references, a value of $\sigma_u/\sigma_y = 1{,}46$ was taken in all cases.

References

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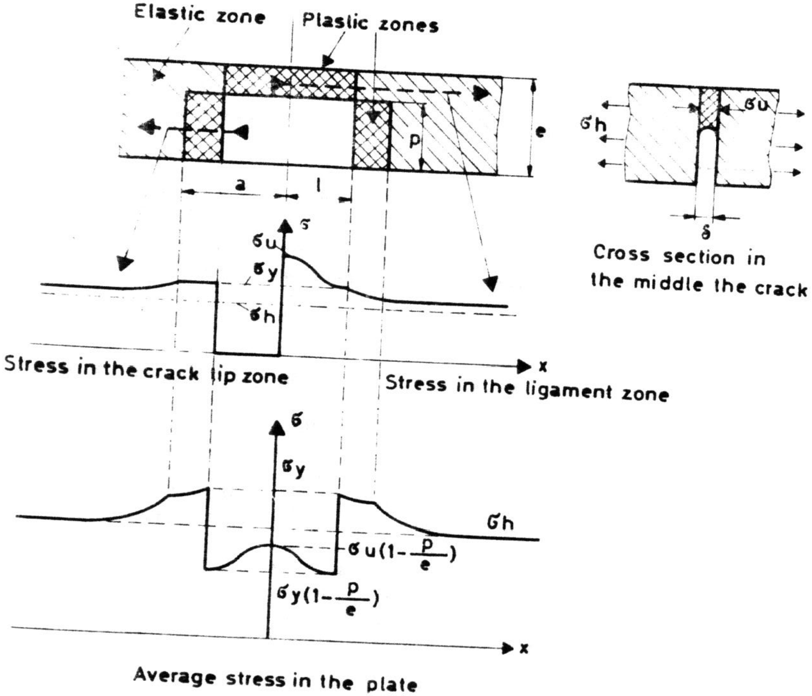

FIG. N°1

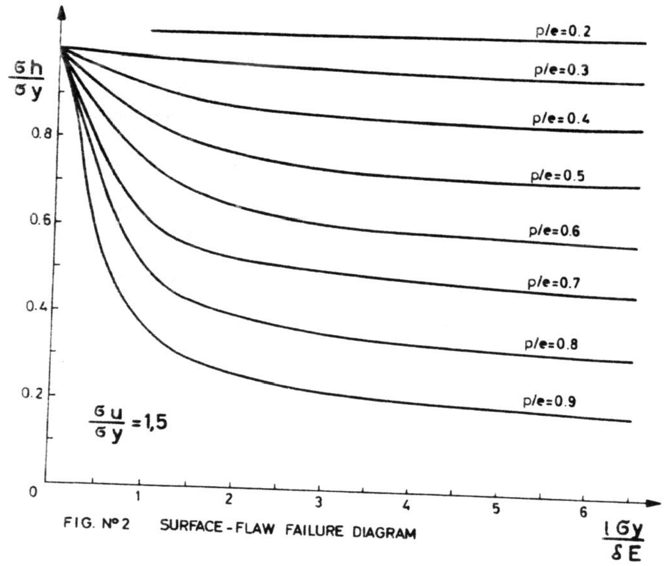

FIG. N°2 SURFACE-FLAW FAILURE DIAGRAM

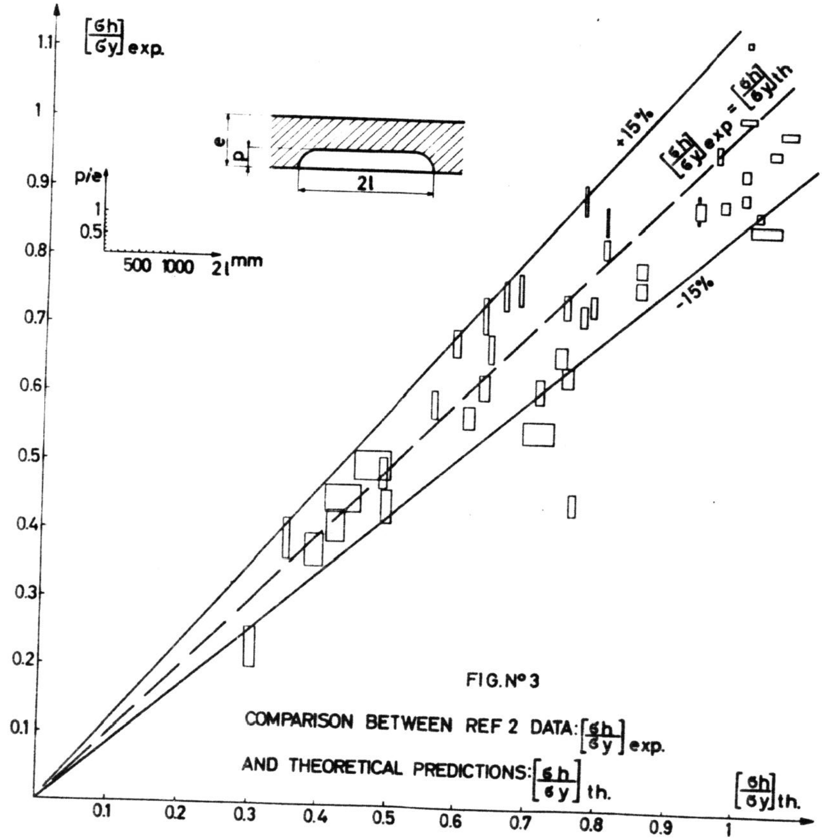

FIG.N°3

COMPARISON BETWEEN REF 2 DATA: $\left[\frac{\sigma h}{\sigma y}\right]_{exp.}$

AND THEORETICAL PREDICTIONS: $\left[\frac{\sigma h}{\sigma y}\right]_{th.}$