Stresses in Nonhomogeneous Materials with a Crack Terminating at and Crossing the Interface

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In studying the fracture of composite materials one may approach the problem from two different view points. In the first approach the primary interest is in estimating the "bulk strength" of the given material under a known system of external loads and environmental conditions. In this type of studies it is usually assumed that the existing material imperfections such as voids, inclusions and cracks are randomly distributed and the medium is statistically homogeneous. Thus, the very nature of the problem requires that some kind of a statistical strength theory be used as a quide in the investigations.

In the second approach to studying the fracture of composites one is basically interested in the initiation of fracture from the "localized" imperfections which are known (or assumed) to exist in the material. In this type of studies it is generally assumed that the composite medium consists of perfectly bonded homogeneous elastic materials and the localized imperfection may be idealized as a plane crack or as a flat elastic inclusion. Since the structural strength of the composite medium is dependent to a considerable extent on the size, the shape, and the orientation of these flaws, in designing with the composite materials it is necessary to have a fairly good estimate of the stress disturbances which may be caused by the flaws of varying sizes and orientations. Considering the nature of most of the currently accepted fracture criteria, it is particularly useful to have a means of evaluating the stress intensity factors and the crack opening displacement.

Previous studies on the subject deal almost entirely with the problem of a crack either imbedded in one of the homogeneous phases or located along a bimaterial interface. The characteristic features of both of these crack geometries are that the power of the stress singularity at the crack tips is -0.5 and as the crack propagates, the stress state in the neighborhood of the crack front remains selfsimilar, the only change being in the stress amplitude. The measure of this stress amplitude is the stress intensity factor(s) which is a continous function of the crack size. On the other hand when the crack front terminates at a bimaterial interface in the composite medium, the power of the stress singularity is no longer -0.5 and the angular distribution of the stresses differs considerably from that of a crack tip imbedded in a homogeneous medium. After reaching the interface further propagation of the crack may be in the form of (a) a cleavage crack into the adjacent medium, (b) a debonding crack along the interface, or (c) a "reflected" crack back into the first medium. Also, in a composite with a crack intersecting the bimaterial interface, the point of intersection is a location of very high stress concentration at which a debonding crack may initiate. It is thus clear that in none of these crack geometries will the stress state around the propagating crack front remain self-similar. Consequently in analyzing the fracture of the composite medium the conventional criteria based on the concepts of the energy balance or the fracture toughnes will not be applicable.

In this paper we consider the plane strain (or the generalized plane stress) and the antiplane shear problems for a crack running into, terminating at, and crossing the interface of two bonded elastic half planes. Even though the technique used to derive the intergral equations is quite general, only the case of a crack perpendicular to the interface (for which the kernels can be expressed in closed form) is studied. First the problem of two cracks, one on each side of the interface, occupying ($a_1 < r < b_1$, $\theta = \pi$) and ($a_2 < r < b_2$, $\theta = 0$) is considered. The elastic constants of the two materials are assumed to be μ_1, κ_1 for ($0 \leq r < \infty$, $\pi/2 < \theta < 3\pi/2$) and μ_2, κ_2 for ($0 \leq r < \infty$,

$-\pi/2 < \theta < \pi/2$). By using the Mellin transforms and assuming a loading symmetry with respect to the plane ($\theta = 0$, $\theta = \pi$), the problem is formulated in terms of the following unknown functions:

$$f_1(r) = \frac{\partial}{\partial r}\left[u_1(r, \pi + 0) - u_1(r, \pi - 0)\right], \quad (a_1 < r < b_1),$$

$$f_2(r) = \frac{\partial}{\partial r}\left[u_2(r, +0) - u_2(r, -0)\right], \quad (a_2 < r < b_2), \tag{1}$$

where u_k, ($k = 1,2$) is the θ-component of the displacement vector in the in-plane problems and the z-component in the anti plane shear problem. After somewhat lengthy manipulations, the integral equations of the problem are obtained as follows:

$$\frac{1}{\pi}\int_{a_1}^{b_1} \frac{f_1(s)}{s-r}\,ds + \frac{1}{\pi}\int_{a_1}^{b_1} k_{11}(r,s) f_1(s)\,ds + \frac{1}{\pi}\int_{a_2}^{b_2} k_{12}(r,s) f_2(s)\,ds = A_1 p_1(r), \quad (a_1 < r < b_1),$$

$$\frac{1}{\pi}\int_{a_2}^{b_2} \frac{f_2(s)}{s-r}\,ds + \frac{1}{\pi}\int_{a_1}^{b_1} k_{21}(r,s) f_1(s)\,ds + \int_{a_2}^{b_2} k_{22}(r,s) f_2(s)\,ds = A_2\, p_2(r), \quad (a_2 < r < b_2) \tag{2}$$

where in the perturbation problem considered p_1 and p_2 are the known crack surface tractions given by

$$p_1(r) = \tau_{1\theta\theta}(r,\pi), \quad p_2(r) = \tau_{2\theta\theta}(r,0),$$

$$A_k = (1+\kappa_k)/2\,\mu_k, \quad (k = 1,2), \tag{3}$$

for the in-plane problems, and

$$p_1(r) = \tau_{1\theta z}(r,\pi), \quad p_2(r) = \tau_{2\theta z}(r,0),$$

$$A_k = 2/\mu_k, \quad (k = 1,2), \tag{4}$$

for the anti-plane shear problem. The kernels k_{ij} are

$$k_{11}(r,s) = \sum_{n=1}^{3} \frac{c_{1n}\, r^{n-1}}{(s+r)^n}, \quad k_{22}(r,s) = \sum_{n=1}^{3} \frac{c_{2n}\, r^{n-1}}{(s+r)^n},$$

$$k_{12}(r,s) = \sum_{n=1}^{2} \frac{d_{1n}\, r^{n-1}}{(s+r)^n}, \quad k_{21}(r,s) = \sum_{n=1}^{2} \frac{d_{2n}\, r^{n-1}}{(s+r)^n}, \qquad (5)$$

where the constants c_{in} and d_{in} are functions of the modulus ratio μ_2/μ_1 and (in the case of in-plane problems) the elastic constants $\varkappa_1$ and $\varkappa_2$. Structurally, the only difference between the integral equations for the in-plane problems and that for the anti-plane problem is that in the latter $c_{1n} = 0 = c_{2n}$, $(n = 2,3)$ and $d_{12} = 0 = d_{22}$.

Expressing the solution of the system of singular integral equations in terms of bounded functions g_1, g_2 as

$$f_k(s) = \frac{g_k(s)}{(s-a_k)^{\alpha_k}\,(b_k-s)^{\beta_k}}, \quad (0 < \mathrm{Re}(\alpha_k, \beta_k) < 1,\ k=1,2), \qquad (6)$$

and applying the function-theoretic method directly to the integral equations, we obtain the following systems of characteristic equations for various in-plane crack geometries:

(a) Imbedded cracks, $a_1 > 0$, $a_2 > 0$:

$$\cot \pi \alpha_k = 0, \quad \cot \pi \beta_k = 0, \quad (k = 1,2)\ ; \qquad (7)$$

(b) One crack tip at the interface, $a_1 = 0$, $a_2 > 0$:

$$\cos \pi \alpha_1 + c_{11} + \alpha_1 c_{12} + \frac{\alpha_1(\alpha_1+1)}{2} c_{13} = 0,$$

$$\cot \pi \alpha_2 = 0, \quad \cot \pi \beta_1 = 0, \quad \cot \pi \beta_2 = 0\ ; \qquad (8)$$

(c) Crack crossing the interface, $a_1 = 0 = a_2$:

$$\cot \pi \beta_k = 0, \quad (k = 1,2), \quad \alpha_1 = \alpha_2 = \alpha,$$

$$(\cos \pi \alpha + c_{11} + \alpha\, c_{12} + \frac{\alpha(1+\alpha)}{2} c_{13})(\cos \pi \alpha + c_{21} +$$

$$\alpha\, c_{22} + \frac{\alpha(\alpha+1)}{2} c_{23}) - (d_{11} + \alpha d_{12})(d_{21} + \alpha d_{22}) = 0 \qquad (9)$$

Similar but somewhat simpler results are obtained for the anti-plane shear problem. From (7-9) it is seen that whenever the crack tip is imbedded into a homogeneous medium, one obtains the expected -0.5 power singularity. However, for the crack terminating at or crossing the interface at the singular point $r = 0$ the power of singularity is different than -0.5. Analytically, this difference comes from the contribution of the kernels k_{ij}, $(i,j = 1,2)$ in (2) which also become unbounded (as r^{-1}) as the related variables r and s go to zero simultaneously.

For the composite medium loaded outside the perturbation zone of the cracks, i.e., for constant p_1, p_2 satisfying

$$\frac{p_1}{p_2} = \frac{(1+\varkappa_2)\mu_1}{(1+\varkappa_1)\mu_2}, \quad \text{(in-plane problems)},$$

$$\frac{p_1}{p_2} = \frac{\mu_1}{\mu_2}, \quad \text{(anti-plane problems)}, \qquad (10)$$

the system of integral equations is solved by using a numerical method developed recently. The details of the analysis and the complete results for various material combinations, crack geometries and relative dimensions will be published separately in a series of papers. The published numerical results will include the crack surface displacements, the density functions, and the stress intensity factors which, for the irregular cases, are defined by

$$k(a_1) = \lim_{r\to 0} \sqrt{2}\, r^{\alpha_1}\, \tau_{2\theta\theta}(r,0) \qquad (11)$$

for $a_1 = 0$, $a_2 > 0$, and

$$k_\theta = \lim_{r\to 0} r^{\alpha}\, \tau_{1\theta\theta}(r, \pi/2), \quad k_r = \lim_{r\to 0} r^{\alpha}\tau_{1r\theta}(r,\pi/2) \qquad (12)$$

for $a_1 = a_2 = 0$.