# On the Validness of Critical Stress - Intensity Factors

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Summary

The ratio p between the largest principle stress $\sigma_1$ and the yield strength Y for a yielding situation can be used as a (admittedly rather incomplete) characterization of the stress configuration.
For the plane-stress and plane-strain state in their for yielding most unfavourable configuration, like at the tip of sharp notches, $p = \frac{2}{3}\sqrt{3}$ ($\approx$ 1.15) and $p = (2 + \pi)\frac{\sqrt{3}}{3}$ ($\approx$ 2.97) resp.
During quasi-static crack growth - and also for a larger crack length than in an otherwise comparable situation - a decrease of a possible existing plane-strain condition in favour of the plane-stress condition occurs for materials like metals, showing a plastic zone ahead of the cracktip, when loaded.
When dealing with an unstable fracture, while the plastic flow at the cracktip remains contained at initiation, a critical stress-intensity factor $K_{pc}$ can be indicated, which will depend on the state of stress, in general being somewhere in between plane strain and plane stress and to be referred to by p.

The ratio between the plastic zone size $2r_Y$ ahead of the cracktip in a loaded plate and the plate thickness B, determines p, in the sense that when $\frac{2r_Y}{B}$ is very small, $p \approx 3$ and when $\frac{2r_Y}{B}$ exceeds a certain value (of the order of, but less than 1), then $p \approx 1$.

From estimates of the plastic-zone size at instability, e.g. by means of a suitable displacement recording and the quasi-static crack extension from the fracture appearance, in addition to the reading of the instability load, the state of stress can be quantitatively found, viz. by indicating p.
Thus obtained $K_{pc}$-values remain obviously dependent on $\sigma$ (and slightly dependent on p). If validness is interpreted as allowing a prediction to be made from a critical crack length under a certain load to the critical crack length under a different load, rather than referring to a material property not dependent on thickness (like $K_{Ic}$), these $K_{pc}$-values can be considered valid.

## ON THE VALIDNESS OF CRITICAL STRESS-INTENSITY FACTORS

When trying to determine the fracture toughness of practical construction materials like b.c.c. or f.c.c. steel plates at the service temperature or somewhat below, the $K_{Ic}$-values often prove to be non-valid, which means that the relevant ASTM-condition, viz. $2.5(\frac{K_{Ic}}{Y})^2$ < crack length and thickness for the investigated specimen is not satisfied. ($Y$ = yield strength; $K_{Ic}$ = critical stress-intensity factor for the plane-strain situation.)
Trivially this is due to the plasticity of the material, which postpones failure till well outside the elastic response region. It can also be stated that in those cases the test temperature was too high, or/and the thickness too small, or anyway that the constraint of plastic flow was not sufficiently realized to achieve the plane-strain condition.
Also the conditions for a valid $K_{IQ}$-determination referring to 2% stable crack growth with failure occurring outside the elastic response region is often not satisfied.
In fact the non-validness means that it turns out impossible to make predictions of failure load for an arbitrary crack length in a plate at a certain temperature, when this failure load was determined for a certain crack length at that temperature and for a certain thickness of the relevant material, while yet the plastic region near the cracktip remained contained in the considered cases.
In this respect it might be noted, that for each plate thickness of a material showing plastic flow apparently a temperature exists above which $K_{Ic}$ even does not exist. For when the plastic zone size $2r_Y$ becomes of the order of the thickness B, the plane-stress configuration becomes predominating. Estimating this plastic-zone size for the plane-stress configuration in the usual way as: $2r_Y = \frac{K_c^2}{\pi Y^2}$ (according to the Irwin or Dugdale approach) apparently, as $K_{Ic} < K_c$, an upper limit of $K_{Ic}$-values for a certain thickness is indicated by $Y\sqrt{\pi B\alpha}$, when for $\frac{2r_Y}{B}.a \simeq 1$, the plane-stress situation is realized.

The temperature, where the $Y\sqrt{\pi B\alpha}$ vs. T - and $K_{Ic}$ vs. T - curves intersect is the thickness dependent plane-strain / plane-stress transition temperature. The validness of critical stress-intensity factor determinations referring to plane stress appear to rely on the condition that failure occurs in the elastic response region.
In general it can be concluded that predictions appear possible from a valid





critical stress-intensity factor figure in a certain plane-stress situation towards another plane-stress situation or from a valid $K_{Ic}(K_{IQ})$ figure in the plane-strain situation towards another plane-strain situation.
For situations in between, the prediction possibilities appear poorly analysed in literature.

If, admittedly in an incomplete way, one characterizes a stress configuration by the parameter $p = \frac{\sigma_1}{Y}$ - with $\sigma_1$ largest principal stress, one has in the for yielding most unfavourable situation like near sharp crack tips in the plane-stress situation, $p = \frac{2}{3}\sqrt{3} \simeq 1.15 \simeq 1$ and in the plane-strain situation: $p = (2 + \pi)\frac{1}{3}\sqrt{3} = 2.97 \simeq 3.$
(Note: Obviously for uniaxial loading, $p = 1$; for pure torsion, $p = \frac{1}{3}\sqrt{3}$.)
If the yield stress is stronger temperature-dependent than the critical cleavage (or twinning) stress $\sigma_f$, like for b.c.c. metals where its temperature coefficient is negative, the p-values define transition temperatures as follows:

$p = \frac{1}{3}\sqrt{3}$ physical transition temperature: intersection of $\sigma_f(T)$ with $k(T) = \frac{1}{3}\sqrt{3}\ Y(T)$.

$p = 1$ embrittlement temperature: intersection of $\sigma_f(T)$ with $Y(T)$ = uniaxial yield strength.

$p = \frac{2}{3}\sqrt{3}$ plane stress notch embrittlement temperature: intersection of $\sigma_f(T)$ with $\frac{2}{3}\sqrt{3}\ Y(T)$ = yield stress in for yielding most unfavourable plane-stress situation.

$p = (2+\pi)\frac{1}{3}\sqrt{3}$ plane strain notch embrittlement temperature: intersection of $\sigma_f(T)$ with $(2+\pi)\frac{1}{3}\sqrt{3}\ Y(T)$ = yield stress in for yielding most unfavourable plane-strain situation ("highest" or absolute transition temperature).

In general one can try to describe the elastic stress field outside the plastic region for contained plastic flow at the tip of a line crack by an elastic line crack (if this plastic region is at least shaped like in the Dugdale approach, otherwise one will have to accomodate the border of the elastic crack to the elastic-plastic boundary line).
For a cracked plate uniaxially loaded transversally to the line crack and assuming the plate dimensions considerably larger than the crack length, one will then have to comply with (e.g. analogous to Well's consideration [1]):

$$(r_{Y_1} + r_{Y_2})pY = \int_0^{r_{Y_2}} \sigma_y dr = \int_0^{r_{Y_2}} \frac{K_p}{\sqrt{2\pi r}} dr = \frac{pK_p}{\pi}\sqrt{2r_{Y_2}} \quad \ldots\ldots\ldots\ldots (1)$$

using e.g. Sneddon's approximation of Westergaard's solution for the elastic case or the more realistic Dugdale approach. This refers to:

$2\ell_o^*$ = crack length (possibly enlarged by stable crack growth with respect to the original crack length $2\ell_o$).

$2\ell_o^* + 2r_{Y_1}$ = elastic crack length = $2\ell$.

$r_{Y_1} + r_{Y_2}$ = plastic zone size at cracktip.

$K_p$ = stress intensity factor, referring to elastic crack = $\sigma\sqrt{(\ell_o^* + r_{Y_1})}.F$, when in the plastic region $p = \frac{\sigma_1}{Y}$, with $\sigma_1$ = yield stress.

$\sigma$ = gross stress

$r$ = distance to tip of elastic crack

$W$ = specimen width

$F$ = $F(\frac{\ell}{W})$ = geometrical correction factor for relevant specimens.

$$r_{Y_2} = \frac{K_p^2}{2\pi p^2 Y^2}\,;\quad \left(r_{Y_1} + \frac{K_p^2}{2\pi p^2 Y^2}\right)pY = \frac{K^2}{\pi p Y}\,;\quad r_{Y_1} = \frac{K^2}{2\pi p^2 Y^2} = r_{Y_2} = r_Y \quad \ldots(2)$$

From the last equations one easily derives, with $y = Y/\sigma$, :

$$K_p^2 = \sigma^2 \ell F^2 = 2\pi p^2 Y^2 (\ell - \ell_o^*)\,;\quad \frac{\ell - \ell_o^*}{\ell_o^*} = \frac{r_Y}{\ell_o^*} = \frac{F^2}{2\pi p^2 Y^2 - F^2} \quad \ldots\ldots\ldots\ldots (3)$$

$$\frac{\sigma}{pY} = \frac{1}{F}\sqrt{\frac{2\pi(\ell - \ell_o^*)}{\ell}}\,;\quad K_p = Y\sqrt{\ell_o^*}\sqrt{\frac{2\pi(\ell - \ell_o^*)}{\ell_o^*}} = Y\sqrt{\ell_o^*}\sqrt{\frac{2\pi p^2 F^2}{2\pi p^2 y^2 - F^2}} \quad \ldots\ldots (4)$$

In fig. 3 the function $\frac{1}{F}\sqrt{\frac{2\pi(\ell - \ell_o^*)}{\ell_o}}$ from equation (4) is plotted as a function of $\frac{\ell}{W}$ for some current fracture mechanics test specimens.

It illustrates why the estimate of the critical stress intensity factor by iterating for $r_Y$ in plane-strain or plane-stress, sometimes shows a minimum difference between $\frac{K^2}{2\pi Y^2}$ and $r_Y$, which remains > 0.





If, besides the load, the effective crack length can be estimated, e.g. by a crack-opening displacement recording, either at the tip or at a convenient place at the crack edges (cf. [2]) one can apparently estimate $K_p$. For from a certain $\ell$ and $\sigma$ follows p (in fig. 3); the same holds at unstability. This evaluation of p, or $r_Y$, from a crack-opening displacement recording, can proceed, according to Westergaard, as:

plane strain c.o.d. $\delta = 2\eta = 4(1-\nu^2)\,\frac{\sigma}{E}\sqrt{\ell^2 \ell_o^{*2}}$

plane stress c.o.d. $\delta = 2\eta = 4\,\frac{\sigma}{E}\sqrt{\ell^2 - \ell_o^{*2}}$

with $\eta$ = crack-edge displacement at real cracktip (in the assumed load direction transversal to crack).

For both cases and intermediate ones: c.o.d. $\delta \simeq \frac{4\sigma}{E}\sqrt{\ell^2 - \ell_o^{*2}}$ and thus

$$\ell = \ell_o^* \left(1 + \frac{E^2\delta^2}{16\sigma^2 \ell_o^{*2}}\right)^{\frac{1}{2}},\qquad p^2 = \frac{\ell F^2}{2\pi y^2 (\ell - \ell_o^*)}$$

(For an infinite plate with $F = \sqrt{\pi}$, one has $p^2 \simeq \frac{1}{2y^2}\left(1 + \frac{32\sigma^2 \ell_o^{*2}}{E^2\delta^2}\right)$, if $\frac{\delta}{\ell_o} \ll \frac{4\sigma}{E}$.)

From the alternative Dugdale approach, which appears somewhat more realistic, $\ell$ can be deduced, according to:

c.o.d. $\delta = \frac{8\ell_o^* Y}{\pi E} \ln \sec \frac{\pi\sigma}{2Y}$ and $\cos \frac{\pi\sigma}{2Y} = \frac{\ell_o^*}{\ell}$

$$\frac{r_Y}{\ell_o^*} = \exp\left(\frac{E\pi\delta}{8\ell_o^* Y}\right) - 1\,,\qquad p^2 = \frac{F^2 \exp \frac{E\pi\delta}{8\ell_o^* Y}}{2\pi Y^2 \left\{\exp\left(\frac{E\pi\delta}{8\ell_o^* Y}\right) - 1\right\}}$$

It is concluded that for $\sigma \leq \frac{2}{3}Y$ and $\frac{\ell}{W} < 0.3$ the stress-intensity factor does not depend on p. For the critical $K_{pc}$, one has then $Y\sqrt{\ell_o^*}\sqrt{\frac{2\pi p^2 F^2}{2\,p^2 y^2 - F^2}}$ fairly independent on p, and allowing predictions in the sense as stipulated above.

This was revealed by calculation for a (infinite) plane with $F = \sqrt{\pi}$, cf. fig. 1 and 2, for a central cracked plate with $F = \left(\frac{\mathrm{tg}\,\frac{\pi\ell}{W}}{\frac{\ell}{W}}\right)^{\frac{1}{2}}$ and for a single-edge notched plate with F as indicated in [3].

References:

[1] Wells, A.A.- British Welding Journ. 10(1963)-11, pp. 563-570.
[2] Gross, B.; Roberts E. Jr. & Srawley, J.E.- NASA-TND-4232.
[3] Benthem, P. and Koiter, W.T.- to be published (in Methods of analysis and solutions of crack prolems, edited by G.C. Sih at Wolters-Noordhoff Publ.)

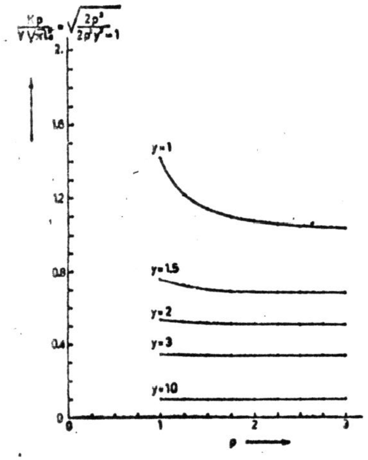


fig. 1

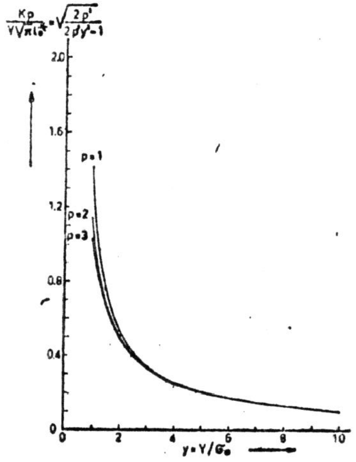


fig. 2

$$\frac{\sigma}{pY} = \frac{\sqrt{2\pi(1-\ell_o^{*}/\ell)}}{F(\ell/W)}; \quad \frac{\ell}{W} = \frac{\ell_o^{*}+r_Y}{W}; \quad K = \sigma\sqrt{\ell}\, F(\frac{\ell}{W})$$

$$F(\frac{\ell}{W}) = A_o + A_1(\frac{\ell}{W}) + A_2(\frac{\ell}{W})^2 + A_3(\frac{\ell}{W})^3 + A_4(\frac{\ell}{W})^4 .$$

| TYPE OF SPECIMEN | $A_0$ | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|
| CENTER-CRACKED (tension) ——— | + 1.77 | + 0.227 | − 0.51 | + 2.70 | 0.00 |
| DOUBLE-EDGE-CRACKED (tension) - - - - - | + 1.93 | + 0.36 | − 2.12 | + 3.42 | 0.00 |
| SINGLE-EDGE-CRACKED (tension) —·—·— | + 1.99 | − 0.41 | + 18.70 | − 38.48 | + 53.85 |
| SINGLE-EDGE-CRACKED (pure bending) ——— | + 1.93 | − 2.47 | + 12.97 | − 23.17 | + 24.80 |
| COMPACT CRACKLINE LOADED — — — | + 7.40 | − 46.37 | + 163.82 | − 254.25 | + 159.72 |

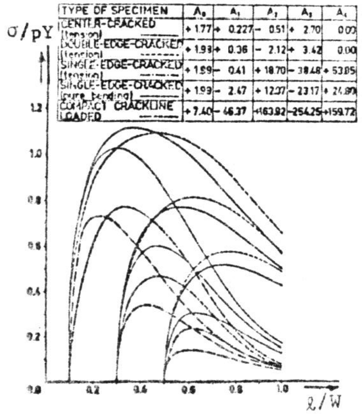


fig. 3