Analysis of Random Cracks and Elliptical Holes

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This paper presents a general analysis of interaction problems of elliptical holes which are arbitrarily distributed in an infinite body. All or some of the holes may be cracks, circular holes or elastic inclusions as special choice of the parameters. The analysis is based on Laurent series expansions of the complex potentials, where the expansion coefficients are determined from the boundary conditions with the aid of a perturbation technique.

1. Longitudinal Shear

The displacement and stress components are given in terms of the only complex potential $\varphi(z)$ as follows:

$$w = \frac{\tau d}{G} \mathrm{Re}\{\varphi(z)\}, \quad \tau_{xt} - i\tau_{yt} = \tau \varphi'(z) \tag{1}$$

where τ is a reference stress and z is a dimensionless complex variable defined by a multiple of a reference length d.

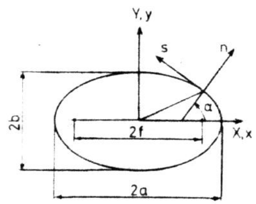

Fig. 1

Assume a free elliptical hole as shown by Fig. 1. From the single-valuedness of the displacement, $\varphi(z)$ can be expressed as the following Laurent series:

$$\varphi(z) = \sum_{n=0}^{\infty} \{(F_n^{\cdot} + iF_n')z^{-(n+1)} + (M_n^{\cdot} + iM_n')z^{n+1}\} \tag{2}$$

and the sress free conditions of the hole edge yield the relations among the coefficients of Eq. (2).

$$\left.\begin{aligned} F_{2n}^{\cdot} &= -\sum_{p=0}^{\infty} \lambda^{2n+2p+2} P_{2p}^{2n} M_{2p}^{\cdot}, \quad F_{2n+1}^{\cdot} = -\sum_{p=0}^{\infty} \lambda^{2n+2p+4} P_{2p+1}^{2n+1} M_{2p+1}^{\cdot} \\ F_{2n}' &= -\sum_{p=0}^{\infty} \lambda^{2n+2p+2} Q_{2p}^{2n} M_{2p}', \quad F_{2n+1}' = -\sum_{p=0}^{\infty} \lambda^{2n+2p+4} Q_{2p+1}^{2n+1} M_{2p+1}' \\ \lambda &= \frac{a}{d} \end{aligned}\right\} \tag{3}$$

where the coefficients P_{2p}^{2n} etc. in the righthand-side are known functions of a/b which are given in closed forms.

Now consider an infinite body containing random distribution of N elliptical holes and subjected to longitudinal shear stresses τ_{xt}^{∞} and τ_{yt}^{∞} at infinity. In Fig. 2 are shown the j-th and k-th holes, with the corresponding geometric parameters and coordinate systems (x, y) as well as (x_j, y_j), (x_k, y_k) associated with those typical holes.

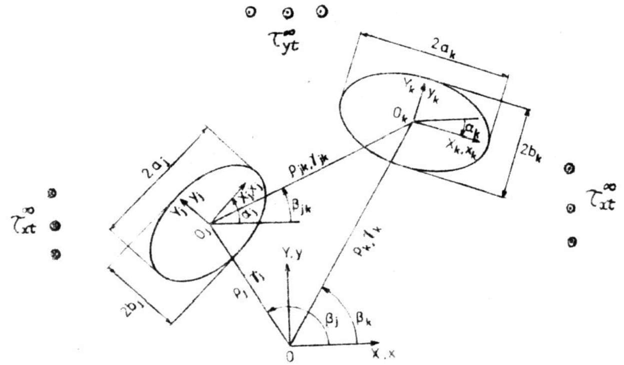

Fig. 2

The complex potential φ for the problem is first assumed

$$\left.\begin{aligned} &\varphi = \varphi_0(z) + \sum_{k=1}^{N} \varphi_k(z_k), \quad z = x + iy, \quad z_k = x_k + iy_k \\ &\varphi_0(z) = \frac{1}{\tau}(\tau_{xt}^{\infty} - i\tau_{yt}^{\infty})z, \quad \varphi_k(z_k) = \sum_{n=0}^{\infty}(\dot{F}_{n,k} + iF'_{n,k})\, z_k^{-(n+1)} \end{aligned}\right\} \quad (4)$$

In order to consider the boundary relations of the j-th hole, φ is reexpanded around its center O_j with the results

$$\varphi = \sum_{n=0}^{\infty}\left\{(\dot{F}_{n,j} + iF'_{n,j})\, z_j^{-(n+1)} + (\dot{M}_{n,j} + iM'_{n,j})\, z_j^{n+1}\right\} \quad (5)$$

$$\left.\begin{aligned} \dot{M}_{n,j} &= \frac{1}{\tau}(\tau_{xt}^{\infty}\cos\alpha_j + \tau_{yt}^{\infty}\sin\alpha_j)\Delta_n^0 + \sum_{p=0}^{\infty}\sum_{k\neq j}^{N}(e_{n,j}^{p,k}\dot{F}_{p,k} - f_{n,j}^{p,k}F'_{p,k}) \\ M'_{n,j} &= \frac{1}{\tau}(\tau_{xt}^{\infty}\sin\alpha_j - \tau_{yt}^{\infty}\cos\alpha_j)\Delta_n^0 + \sum_{p=0}^{\infty}\sum_{k\neq j}^{N}(f_{n,j}^{p,k}\dot{F}_{p,k} + e_{n,j}^{p,k}F'_{p,k}) \end{aligned}\right\} \quad (6)$$

where the coefficients $e_{n,j}^{p,k}$, $f_{n,j}^{p,k}$. are given in terms of the geometric parameters as shown in Fig. 2.

Since Eq. (5) is of the same form as (2), the stress free

relations (3) are rewritten as

$$\left.\begin{aligned} &\dot{F}_{2n,j} = -\sum_{p=0}^{\infty}\lambda_j^{2n+2p+2}(P_{2p}^{2n})_j\dot{M}_{2p,j}, \quad \dot{F}_{2n+1,j} = -\sum_{p=0}^{\infty}\lambda_j^{2n+2p+4}(P_{2p+1}^{2n+1})_j\dot{M}_{2p+1,j} \\ &F'_{2n,j} = -\sum_{p=0}^{\infty}\lambda_j^{2n+2p+2}(Q_{2p}^{2n})_jM'_{2p,j}, \quad F'_{2n+1,j} = -\sum_{p=0}^{\infty}\lambda_j^{2n+2p+4}(Q_{2p+1}^{2n+1})_jM'_{2p+1,j} \end{aligned}\right\} \quad (7)$$

where $\lambda_j = a_j/d$ and $(P_{2p}^{2n})_j$ etc. are known functions of a_j/b_j.

The next step of the analysis is to determine the unknown coefficients $\dot{F}_{n,j}$, $F'_{n,j}$, $\dot{M}_{n,j}$, $M'_{n,j}$ (n=0, 1, 2, $\cdots$; j=1, 2, $\cdots$, N) using a perturbation technique. Let $\lambda_j = \lambda\delta_j$ where $\lambda = \mathrm{Max}\,(a_j + a_k)/\rho_{jk}$ and δ_j are known constant representing the ratio of hole lengths. Then all the unknowns are assumed as power series of the only parameter λ, and their coefficients are determined successively from Eqs. (6) and (7) with $j = 1, 2, \cdots, N$.

Finally the stress intensity factors at crack tips and stresses along hole boundaries are given by double infinite series corresponding to the assumed Laurent series expansions and the perturbation procedure, but the series due to the Laurent expansions are summed up in closed forms. In this way all the necessary quantities are given by power series of λ whose coefficients are evaluated exactly.

It may be noted that the analysis is quite simplified in special cases of periodic cracks.

2. Plane problems and Plate Bending

In plane problems the Airy's stress function χ is assumed as the following sum:

$$\chi = \chi_0 + \sum_{k=1}^{N}\chi_k \quad (8)$$

χ_0 corresponds to stresses at infinity and χ_k are written as

$$\left.\begin{aligned} X_k &= \sigma d^2 \, Re\{\bar{z}_k \varphi_k(z_k) + \psi_k(z_k)\} \\ \varphi_k(z_k) &= \sum_{n=0}^{\infty} (F_{n,k} + i F'_{n,k}) \, z_k^{-(n+1)} \\ \psi_k(z_k) &= \sum_{n=0}^{\infty} (D_{n,k} + i D'_{n,k}) \, z_k^{-(n+1)} \end{aligned}\right\} \quad (9)$$

The analysis is rather complicated, but in principle it is similar to that in the previous case. The stress function is reexpanded around O_j and relations corresponding to Eqs. (6) and (7) in the previous case are obtained. Those relations with j=1 to N are then solved by means of a perturbation technique, and the necessary quantities are given in forms of power series of λ.

In the classical theory of plate bending, all the physical quantities are given by the two complex potentials defined by

$$D w = M_0 d^2 Re\{\bar{z}\varphi(z) + \psi(z)\}, \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad (10)$$

and the analysis is done in like manner as plane problems.

3. Some Numerical Results

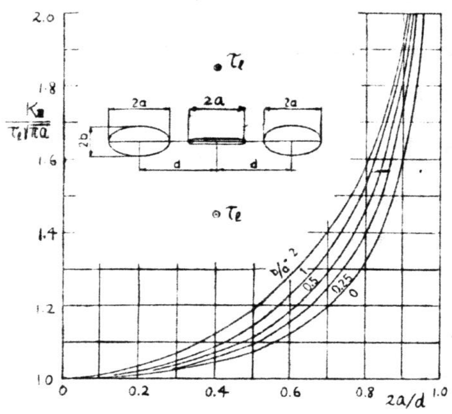

Fig. 3

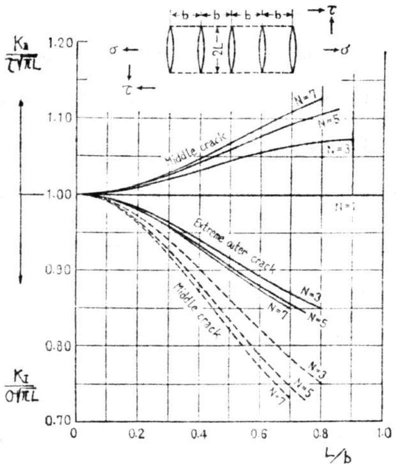

Fig. 4

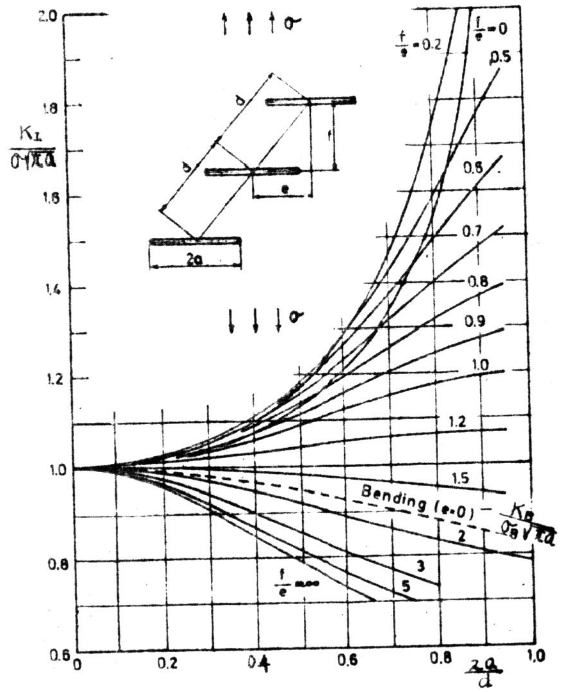

Fig. 5 Infinite parallel cracks

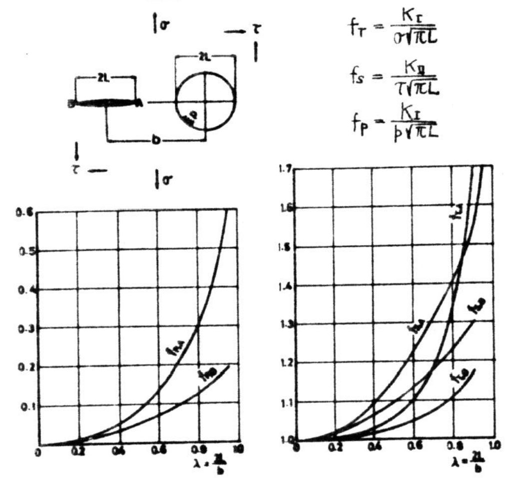

Fig. 6 Crack approaching circular hole