PAPER 2 (SESSION I)

The effect of the shape of plastic zone size and crack opening displacement

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Summary

For an elastic plastic material the stress distribution depends on the history of loading. The effect of such differences on the commonly used fracture criteria are considered for the case of anti plane strain for the following geometries: a crack under normal loading on the crack and at infinity and a wedge shaped notch under normal loading at infinity, by calculating the plastic zone size and the crack opening displacement on the alternative assumptions that plasticity is confined to the plane of the crack or that plasticity occurs in fans centred on the crack tips which corresponds to monotonic loading.

Introduction

For a given geometry and loading of a system of cracks in an elastic material the stress and displacement distribution and hence the condition for instability is uniquely determined. For an elastic plastic material the stress distribution depends on the history of loading. The present paper examines the effect on the commonly used criteria for failure of varying stress distribution for constant external loading. Attention is confined to the case of anti plane strain of a non work hardening material as the equations may more readily be solved than in the more practically important cases of plane stress and plane strain and of work hardening material. The stress distribution corresponding to the case of a uniform monotonic loading of a crack by stresses applied to infinity has been determined by Hult and McClintock [1], Koskinen [2] and Rice [3] for the cases of a crack and of a wedge and for an infinite row of equal and equally spaced collinear cracks or wedges. Kostrov and Nikitin [4] have done the same for a crack in which plasticity is confined to its plane. Arthur and Blackburn [5] extended this to two equal cracks or to an infinite row of equal equally spaced parallel or collinear cracks. Kostrov and Nikitin [6] have also considered a more complex case in which the slip in the plane is related to the cohesive stress. In all cases the crack surfaces are assumed to be stress free. However as the condition for failure involves the cohesive stresses on the crack tip, Arthur and Blackburn [7], it is important to extend the method of solution to include stresses applied on the surface of the crack.

Solutions are presented here on the alternative assumptions of monotonic loading and plasticity confined to the crack and the results are compared. A material containing a crack under uniform loading on the crack and at infinity and a wedge shaped notch loaded at infinity will be considered.

For anti plane strain of the elastic part of a material the stress may be expressed in terms of the transverse displacement w perpendicular to the x, y plane as $\tau_{xz} = \mu(\partial w/\partial x), \tau_{yz} = \mu(\partial w/\partial y)$, where μ is the modulus of rigidity. Thus the complex stress $\tau = \tau_{yz} + i\tau_{xz}$ is a function of $z = x + iy$. Two alternative assumptions are made for the plastic region: either plasticity is confined to a plane or the stress is a function of the strain. In the latter case, which corresponds to monotonic loading, the stress is constant on straight lines. If the displacement is continuous and the stress applied to the crack (or in the more general case to the wedge or step) is constant, these lines pass through the crack tip and the stress is of magnitude k (the yield stress in shear) and acts in a direction perpendicular to these lines. Thus if $z = c$ is the centre of one of these fans, on the elastic plastic boundary $\arg(z - c) + \arg \tau$ is zero, i.e. $\operatorname{Im} \tau(z - c) = 0$. When plasticity is confined to the real axis, on the elastic plastic boundary $\operatorname{Im} z = 0$.

The relationship between z and τ may thus readily be obtained for any given loading for which the boundary of τ and of z or of $(z - c)\tau$ may be conformally mapped onto the real upper half plane by a known conformal mapping. As an illustration we consider a crack extending to $\pm a$ under normal loading σ at infinity. By symmetry, only a quarter space need be considered. The boundary in the stress plane is a quarter circle which may be transformed on to the upper half plane by the transformation

$$t = \frac{(k^2+\sigma^2)^2\tau^2}{(k^4-\sigma^2\tau^2)(\tau^2-\sigma^2)}$$

(see Fig. 1). This particular transformation is chosen such that the points $0, -1$ and σ transform to 0, 1 and ∞. The point k at the tip of the plastic zone transforms to

$$T = \frac{(k^2+\sigma^2)^2}{(k^2-\sigma^2)^2}$$

Then as shown by Kostrov and Nikitin [4] the positive quarter plane in the z position space is related to the upper half t plane by

$$z = a\sqrt{t}$$

Loaded crack

We consider first a single crack of length $2a$ between $z = \pm a$ under a normal shear stress σ at infinity and a normal shear stress $-\lambda$ on the crack. **By symmetry, only the real quarter plane in the z space need be considered.**

The boundary of the stress plane is shown in Fig. 1. This is transformed into the upper half plane by the following mapping (Kober [8]):

$$t = \frac{(k^2+2\lambda\sigma+\sigma^2)^\alpha\{[\sqrt{(k^2-\lambda^2)}+i(\lambda+\tau)]^\alpha-[\sqrt{(k^2-\lambda^2)}-i(\lambda+\tau)]^\alpha\}^2}{(k^2+2\lambda\tau+\tau^2)^\alpha\{[\sqrt{(k^2-\lambda^2)}+i(\sigma+\lambda)]^{2\alpha}+[\sqrt{(k^2-\lambda^2)}-i(\sigma+\lambda)]^{2\alpha}\}}$$
$$\times (k^2+2\lambda\sigma+\sigma^2)\{[\sqrt{(k^2-\lambda^2)}+i(\lambda+\tau)]^{2\alpha}+[\sqrt{(k^2-\lambda^2)}-i(\lambda+\tau)]^{2\alpha}\} \qquad (2.1)$$

$$\tau = i\sqrt{(k^2-\lambda^2)}\,\frac{\{\sqrt{(t-\sec^2\gamma)}-i\sqrt{t}\tan\gamma\}^{1/\alpha}-\{\sqrt{(t-\sec^2\gamma)}+i\sqrt{t}\tan\gamma\}^{1/\alpha}}{\{\sqrt{(t-\sec^2\gamma)}-i\sqrt{t}\tan\gamma\}^{1/\alpha}+\{\sqrt{(t-\sec^2\gamma)}+i\sqrt{t}\tan\gamma\}^{1/\alpha}}-\lambda \qquad (2.2)$$

where $\alpha = \pi/(\frac{1}{2}\pi+\sin^{-1}\lambda/k)$ and the mapping is chosen to be such that σ, $-\lambda$ and $-\lambda-i\sqrt{(\lambda^2 - k^2)}$ map on to ∞, 0 and 1 respectively. T is used to denote the value of t at the end of the plastic zone where $\tau = k$. Then $T = \sec^2\gamma$

where
$$\gamma = \alpha\tan^{-1}\frac{\lambda+\sigma}{\sqrt{(k^2-\lambda^2)}} \qquad (2.3)$$

When plasticity is confined to the plane of the crack z and t are a quarter plane and a half plane respectively in which the origin and infinity coincide. Thus since $t = 1$ at the crack tip where $z = a$, they are related by

$$z = a\sqrt{t} \qquad (2.4)$$

The length of the plastic zone is

$$a(\sqrt{T} - 1) = a(\sec\gamma - 1) \qquad (2.5)$$

The crack opening displacement is $2w(a)$ i.e.

$$\operatorname{Im}\frac{2}{\mu}\int_T^1 \tau\frac{dz}{dt}\,dt = \frac{a\sqrt{(k^2-\lambda^2)}}{\mu}$$
$$\times\int_1^{\sec^2\gamma}\frac{\{\sqrt{t}\sin\gamma+\sqrt{(1-t\cos^2\gamma)}\}^{2/\alpha}}{2\cos\frac{\pi}{2\alpha}(t-1)^{1/\alpha}+\{\sqrt{t}\sin\gamma+\sqrt{(1-t\cos^2\gamma)}\}^{2\alpha}}$$
$$\times\frac{-\{\sqrt{t}\sin\gamma-\sqrt{(1-t\cos^2\gamma)}\}^{2/\alpha}}{+\{\sqrt{t}\sin\gamma-\sqrt{(1-t\cos^2\gamma)}\}^{2/\alpha}}\cdot\frac{dt}{\sqrt{t}} \qquad (2.6)$$

If no unloading has occurred and the displacement is continuous the stress is uniform on straight lines through the plastic zone. If the stress on the crack surface is constant these lines pass through the crack tip and the stress is perpendicular to these lines, its magnitude being k. Thus at the boundary $\arg\tau + \arg(z - a)$ is zero, i.e. $\operatorname{Im}\tau(z - a)$ is zero. Also, on the continuation of the line of the crack, by symmetry τ is real so that $\operatorname{Im}\tau(z - a)$ is zero. On the perpendicular plane through the centre of the crack τ is real so that $\operatorname{Re}\tau(z - a) = a\tau$. On the crack itself $\operatorname{Re}\tau = -\lambda$ and hence Re $\tau\ (z-a) = -\lambda(z-a)$. Thus the imaginary part of $\tau(z-a)/\sqrt{(t-1)}$ is specified on the whole of the real axis in the t plane and hence since it is bounded it is determined except for an arbitrary real constant by the formula.

$$\frac{\tau(z-a)}{\sqrt{(t-1)}} = -\frac{a}{\pi}\int_{-\infty}^{0}\frac{\rho\, dr}{\sqrt{(1-r)}\,(r-t)} - \frac{\lambda(z-a)}{\pi}\int_{0}^{1}\frac{dr}{\sqrt{(1-r)}\,(r-t)}$$

$$= \frac{\lambda a}{\sqrt{(t-1)}} - \frac{2\lambda z}{\pi\sqrt{(t-1)}} - \frac{a}{\pi}\int_{-\infty}^{0}\frac{(\rho+\lambda)\, dr}{\sqrt{(1-r)}\,(r-t)} \qquad (2.7)$$

where r is the value of $t(\rho)$.
With the constant chosen so that the stress is real at the centre of the crack

$$z = a\,\frac{\tau+\lambda - \dfrac{t\sqrt{(t-1)}}{\pi}\displaystyle\int_{-\infty}^{0}\frac{(\rho+\lambda)\, dr}{\sqrt{(1-r)}\,(r-t)\tau}}{\tau+\dfrac{2\lambda}{\pi}\tan^{-1}\dfrac{1}{\sqrt{(t-1)}}} \qquad (2.8)$$

The length of the plastic zone is

$$a\,\frac{2\lambda\gamma-\sqrt{(k^2-\lambda^2)}\tan\gamma\displaystyle\int_{-\infty}^{0}\frac{\tan\left\{\dfrac{1}{a}\tan^{-1}\dfrac{\sin\gamma}{\sqrt{(\cos^2\gamma-1/r)}}\right\}dr}{r(\sqrt{(1-r)}\,(r\cos^2\gamma-1)}}{\pi(k+\lambda)-2\lambda\gamma} \qquad (2.9)$$

and the crack opening displacement is

$$\mathrm{Im}\,\frac{2}{\mu}\int_{T}^{1}\tau\,\frac{dz}{dt}\,dt \qquad (2.10)$$

In Figs. 2 and 3 lengths of plastic zone and the crack opening displacement are plotted from equations (2.5) and (2.9) and from equations (2.6) and (2.10) for various ratios of λ/k (the ratio of the stress on the crack to the yield stress) as a function of σ/k (the ratio of stress at infinity to yield stress).

Wedge shaped notch
Another case where both solutions may be obtained is for a symmetric pair of wedge shaped notches of half angle $(\pi/2)-a$ and depth a in a finite

slab under a stress σ at infinity. The stress is mapped on the t half plane by

$$t = \frac{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2\tau^{\pi/\alpha}}{(k^{2\pi/\alpha}-\sigma^{\pi/\alpha}\tau^{\pi/\alpha})(\tau^{\pi/\alpha}-\sigma^{\pi/\alpha})}$$

$$T = \frac{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2}{(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2}$$

$$\tau = \left\{\frac{(k^{\pi/\alpha}+{}^{\pi/\alpha})}{2\sigma^{\frac{1}{2}\pi/\alpha}}\sqrt{\left(1-\frac{1}{t}\right)} - \frac{1}{2\sigma^{\frac{1}{2}\pi/\alpha}}\sqrt{\left[(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2-\frac{1}{t}(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2\right]}\right\}^{2a/\pi} \qquad (3.1)$$

For simplicity we treat here only the case of a single notch in a half space.

If plasticity is confined to the line of the crack, an appropriate mapping of the z plane onto the t plane is

$$z = a + a\int_{1}^{t}\frac{dt}{t^{\alpha/\pi}(t-1)^{\frac{1}{2}-\alpha/\pi}}\Bigg/\int_{0}^{1}\frac{dt\sin a}{t^{\alpha/\pi}(1-t)^{\frac{1}{2}-\alpha/\pi}} \qquad (3.2)$$

Hence the size of the plastic zone is

$$a\left\{\int_{1}^{T}\frac{dt}{t^{\alpha/\pi}(t-1)^{\frac{1}{2}-\alpha/\pi}}\Bigg/\int_{0}^{1}\frac{dt\sin a}{t^{\alpha/\pi}(1-t)^{\frac{1}{2}-\alpha/\pi}}\right\} \qquad (3.3)$$

and the difference in displacement at the apex of the notch is

$$\mathrm{Im}\,\frac{\dfrac{2a}{\mu}\displaystyle\int_{1}^{T}\frac{\tau\, dt}{t^{\alpha/\pi}(t-1)^{\frac{1}{2}-\alpha/\pi}}}{\displaystyle\int_{0}^{1}\frac{dt\sin a}{t^{\alpha/\pi}(1-t)^{\frac{1}{2}-\alpha/\pi}}}$$

$$= \frac{\dfrac{2ak}{\mu\sin a}\displaystyle\int_{1}^{T}\frac{\sin\left\{\dfrac{2a}{\pi}\cot^{-1}\dfrac{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})\sqrt{(t-1)}}{\sqrt{\{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2-t(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2\}}}\right\}dt}{t^{\alpha/\pi}(t-1)^{\frac{1}{2}-\alpha/\pi}}}{\displaystyle\int_{0}^{1}\frac{dt}{t^{\alpha/\pi}(1-t)^{\frac{1}{2}-\alpha/\pi}}} \qquad (3.4)$$

As shown by Hult and McClintock [1] for a single notch and generalised by Rice [3] for a symmetric pair of notches, in the case of no unloading the boundary conditions on $(z-a)\tau$ are the same as for a single crack, so that

$$\frac{\tau(z-a)}{\sqrt{(t-1)t}} = \frac{a}{\pi}\int_{-\infty}^{0}\frac{\rho\, dr}{r(r-t)\sqrt{(1-r)}} \qquad (3.5)$$

The size of plastic zone is

$$\frac{2a\,(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2 k^{\pi/\alpha}\sigma^{\pi/\alpha}}{2^{2\alpha/\pi}\;\pi k\sigma\,(k^{\pi/\alpha}-\sigma^{\pi/\alpha})}$$

$$\times \int_{-\infty}^{0} \frac{\left\{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})\sqrt{\left(1-\frac{1}{r}\right)}-\sqrt{\left[(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2-\frac{1}{r}(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2\right]}\right\}^{2\alpha/\pi}}{r\sqrt{(1-r)}\{(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2 r-(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2\}}\,dr \qquad (3.6)$$

and the difference in displacement at the apex of the notch is

$$\text{Im}\ \frac{2}{\mu}\int_{T}^{1} \tau\,\frac{dz}{dt}\,dt = \frac{2\alpha a(k^{\pi/\alpha}+\sigma^{\pi/\alpha})}{2^{2\alpha/\pi}\,\pi^2\mu\sigma}$$

$$\times \int_{1}^{T}\int_{-\infty}^{0} \frac{\left\{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})\sqrt{\left(1-\frac{1}{r}\right)}-\sqrt{\left[(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2-\frac{1}{r}(k^{\pi/\alpha}+\sigma^{\pi/\alpha})\right]}\right\}^{2\alpha/\pi}}{r\sqrt{(1-r)}\,(r-t)\sqrt{\{(k^{\pi/\alpha}+\sigma^{\pi/\alpha})^2-t\,(k^{\pi/\alpha}-\sigma^{\pi/\alpha})^2\}}}\,dr\,dt \qquad (3.7)$$

In Fig. 4 and 5 respectively the plastic zone size and the crack opening displacement are plotted as functions of the ratio of applied stress to yield stress σ/k for various angles α when calculated from equations (3.3) and (3.4) and from equations (3.6) and (3.7).

In a previous paper [5] we presented equations for the critical opening displacement and plastic zone size for the cases of equal collinear and parallel cracks under a stress at infinity. Computation of these results has now been completed and there is good agreement between the two methods in both cases, but space does not allow them to be presented here.

Conclusions

The methods of Hult and McClintock [1] and Kostrov and Nikitin [4] for the solution of anti plane strain problems of a non work hardening elastic plastic material have been extended to more complicated geometries and loadings including the case of loading on the surface of the crack.

The usual parameters which are considered as critical for failure, the plastic zone size and crack opening displacement, have been calculated for a number of geometries and loadings on the alternative assumptions of plasticity confined to the plane of the crack, or plasticity occurring in centred fans at the crack tips corresponding to monotonic loading.

As there is no significant qualitative difference in the results, it is suggested that the effect of prior load history on the criterion for failure is not significant and that it is not inappropriate to take into account the effect of plasticity in calculations for a particular geometry and loading by considering any convenient statically admissable stress distribution.

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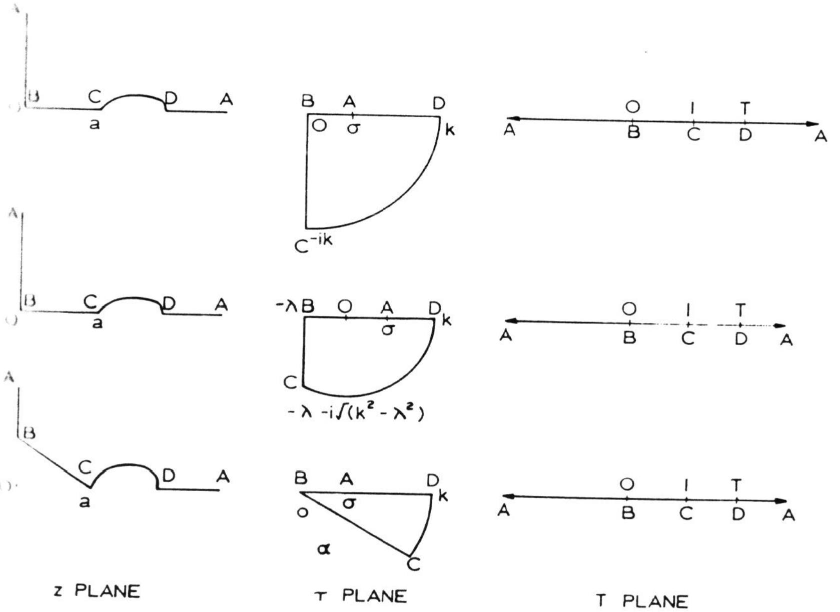

Fig. 1.

Effect of shape of plastic zone on zone size and crack opening

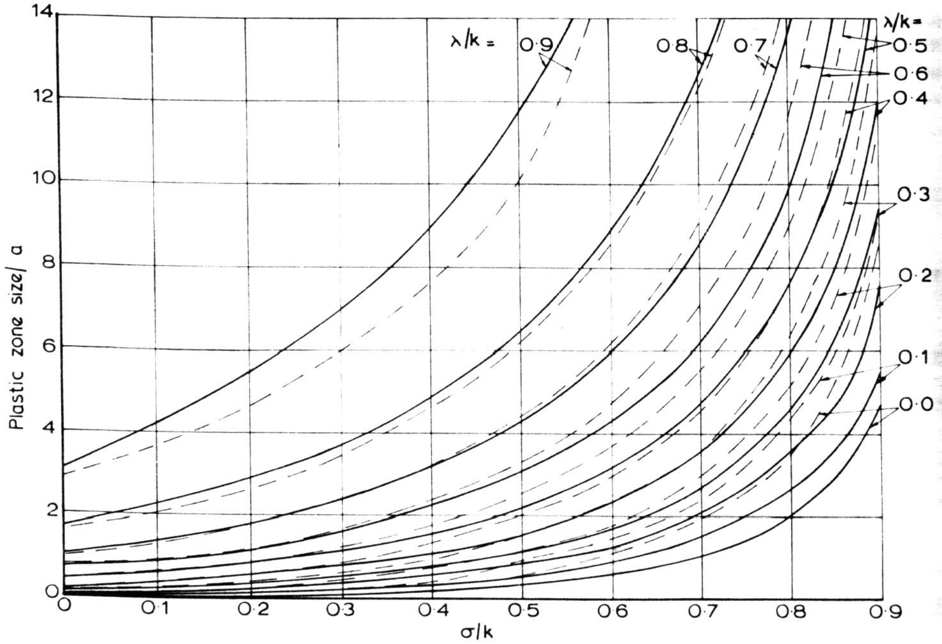

Fig. 2.

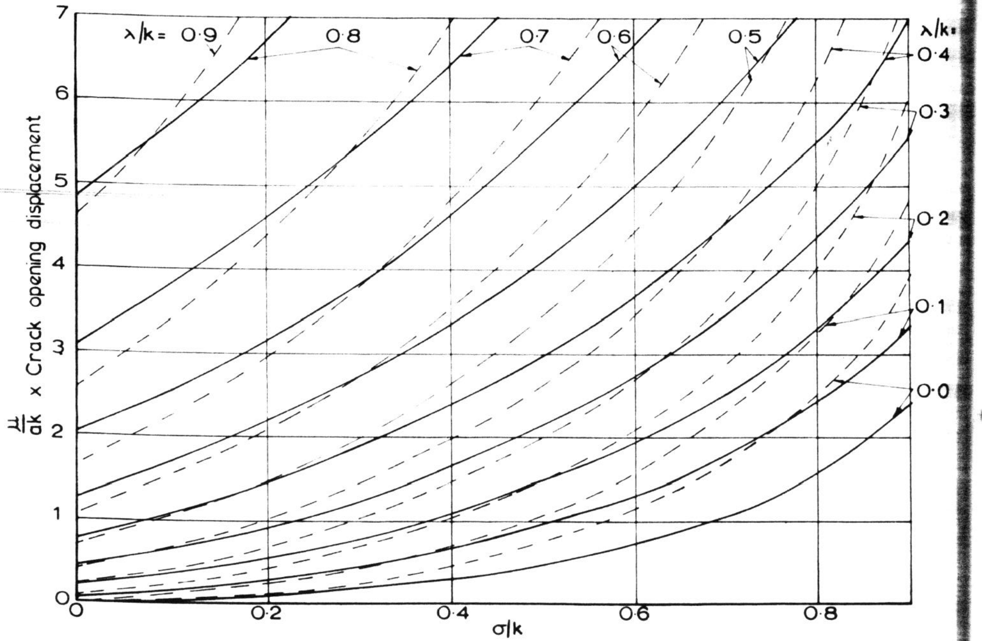

Fig. 3.

Effect of shape of plastic zone on zone size and crack opening

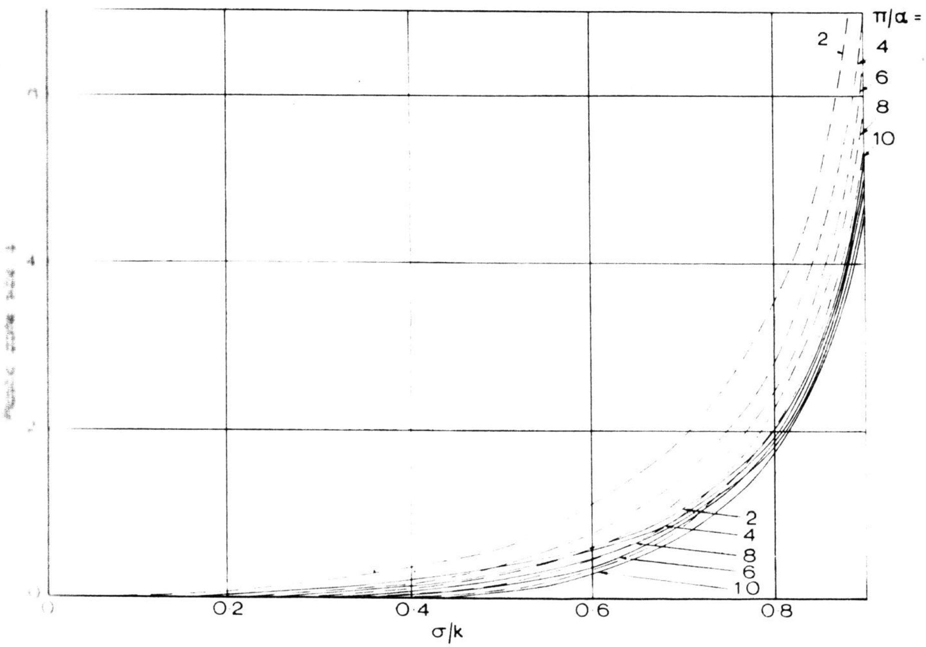

Fig. 4.

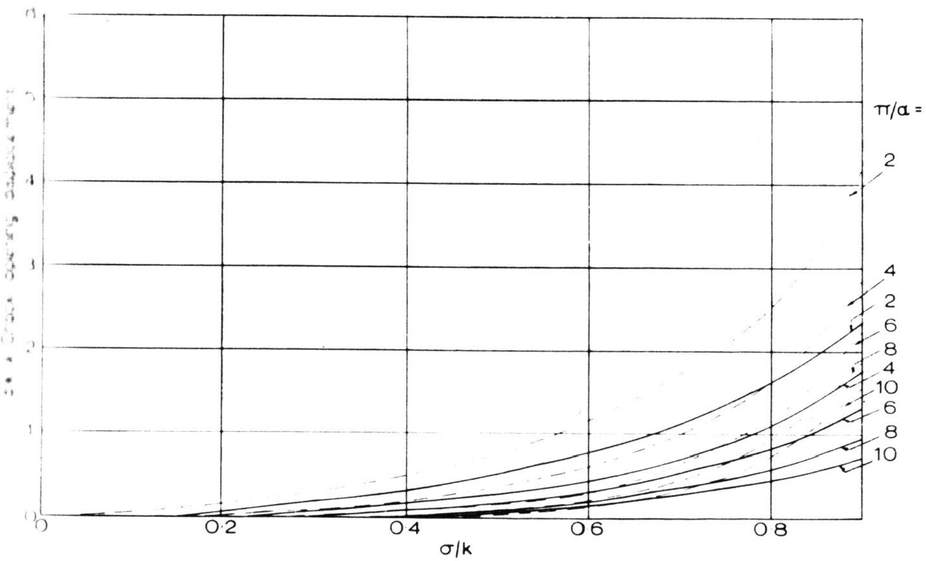

Fig. 5.