Impacts on thin elastic sheets

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We study transverse impacts of rigid objects on a free membrane. The thin elastic sheet is made of natural rubber. After impact, two distinct waves propagate in the sheet. First a tensile wave travels at the speed of sound leaving behind the wave front a stretched domain. Then a flexural wave propagates in the stretched area at a lower speed. In the stretched area geometrical confinement induces compressive circumferential stresses. They trigger a buckling instability giving rise to radial wrinkles. In this paper we report on an experimental and theoretical study of this dynamic wrinkling.

1 Introduction.

When a thin sheet is transversely impacted, it is pulled out of its plane. The out of plane displacement is often accompanied with a rich radial pattern. Two examples are particularly striking: the radial cracks seen on impacted glass plates [1] and the wrinkles seen on the sides of a tablecloth [2]. To understand how such patterns appear on an initially axisymmetric structure, it is important to understand the global response of the sheet. When a free standing thin elastic sheet is transversely impacted, the impactor drags material points out of the plane. This pulling action induces motion of the material points towards the center of the sheet [3, 4]. Thus compressive hoop stresses develop in the membrane. As seen on Figs (3) and (4), these stresses induce a buckling instability giving rise to radial wrinkles. In this work we study this dynamic wrinkling instability. First we discuss wave propagation in and on the membrane. Then we study the stability of the axisymmetric situation and we show that a wrinkling instability develops. We propose a model that allows us to compute the wavelength of this instability.

2 Experimental setup.

We use a gas gun to launch projectiles on a free standing thin elastic sheet (Fig. 1). Most of our experiments are performed with a steel cylindric impactor of radius r_i = 2.25 mm and mass 3.3 10^{-3} kg. The speed of the impactor can be adjusted between 0 and 30 m/s. The latex sheet stands initially on a netting stretched on an open frame. The frame is maintained by two electromagnets. When the gas gun is triggered, the electromagnets are switched off, the frame is violently pulled down by two rubber bands and the latex sheet falls under the action of gravity. The characteristic time of the free fall is much longer than any timescale in the problem and thus we consider that the impactor hits a stress free

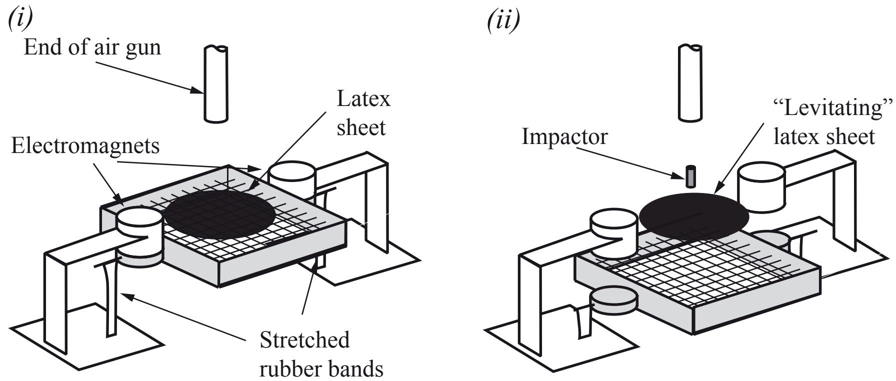


Fig1: Experimental setup

membrane at rest. The sheets were cut from natural rubber with different thicknesses (from 0.10 mm to 0.30 mm). The outer radius of the sheets is $r_0 = 60$ mm. The latex Young's modulus is $E = 1.5\ 10^6$ Pa and the sound speed in latex is $c = 45$ m/s. The typical timescale of the problem is $r_0/c = 1.3\ 10^{-3}$ s. We use a high speed camera to record the motion of the sheet after impact.

3 Wave propagation.

We now discuss our results and we also give an outline of the model that is detailed elsewhere [4]. Fig. 2i shows the motion of the material points on a meridian line after impact. Two waves can clearly be seen:

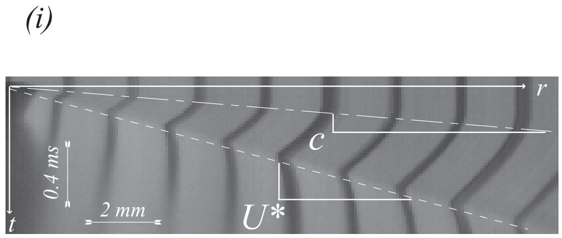
1. A tensile wave travels at the speed of sound in the membrane. At time t, the wavefront is located at the radius $r_t = r_i + ct$ where r_i is the radius of the impactor. Behind the wavefront material points move towards the impact point. We use the following *ansatz* for the radial displacement of a material point initially located at r

$$\zeta(r,t) = \alpha(r_i + ct)\left(\frac{r}{r_i + ct} - \frac{r_i + ct}{r}\right) \tag{1}$$

   This form corresponds to the quasi-static solution of the equation for the propagation of in plane disturbances in the membrane [5]. α is unknown but it will depend on the impactor's speed V, and it will be determined when matching this solution with the solution in the cone. The corresponding radial strain is

$$\varepsilon_r(r,t) = \frac{\partial \zeta}{\partial r} = \alpha\left[1 + \left(\frac{r_i + ct}{r}\right)^2\right] \tag{2}$$

2. A transverse wave travels at the speed of transverse disturbances in a membrane, namely $U = (\sigma_r / \rho)^{1/2}$. Behind the transverse wavefront, the membrane takes the form of a cone (Fig. 3). As seen on Fig. 2i, the base of the cone travels at constant speed. The position of the transverse wavefront in the



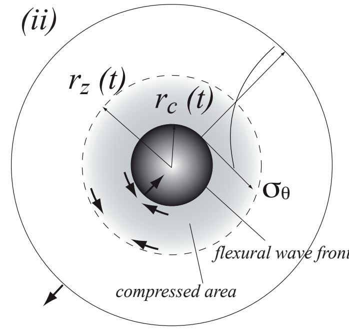


Fig 2: *(i) Motion of the material points on a meridian line. (ii) In plane stress field in the membrane*

material frame is $r_c = r_i + Ut$. We assume that the strain in the cone ε_c is uniform and is equal to the strain at $r = r_c$ as given by Eq. (2).

To find the coefficient α, we need to match the two domains. Assuming that the strain in the cone is uniform and equal to We use a simple geometric relation to compute the length of the meridian line between r_i and r_c namely we write Pythagoras' theorem

$$(1+\varepsilon_c)^2 r_c^2(t) = (Vt)^2 + (U^* t)^2 \tag{3}$$

where U^* is the speed of the base of the cone in the laboratory frame ($U^* t = r_c(t) + \zeta(r_c,t)$). The speed of the impactor V does not change much during the experiment (see discussion) and we will assume that it is constant.

We finally obtain in the limit $Ut >> r_i$

$$\alpha = \left(\frac{U}{c}\right)^4 \Big/ \left(1-\nu+(1+\nu)\left(\frac{U}{c}\right)^2\right) \tag{4}$$

and

$$\frac{U}{c} = \left(\frac{1-\nu}{4}\right)^{1/4}\left(\frac{V}{c}\right)^{1/2} \tag{5}$$

This result and in particular the scaling $U \sim V^{1/2}$ shows good agreement with experimental data [4].

Using Hooke's law, we obtain for the stresses in the stretched domain

$$\sigma_r(r,t) = \frac{E}{1-\nu^2}\left(\frac{V}{2c}\right)^2\left[1+\nu+(1-\nu)\left(\frac{r_i+ct}{r}\right)^2\right] \tag{6}$$

$$\sigma_\theta(r,t) = \frac{E}{1-\nu^2}\left(\frac{V}{2c}\right)^2\left[1+\nu-(1-\nu)\left(\frac{r_i+ct}{r}\right)^2\right] \tag{7}$$

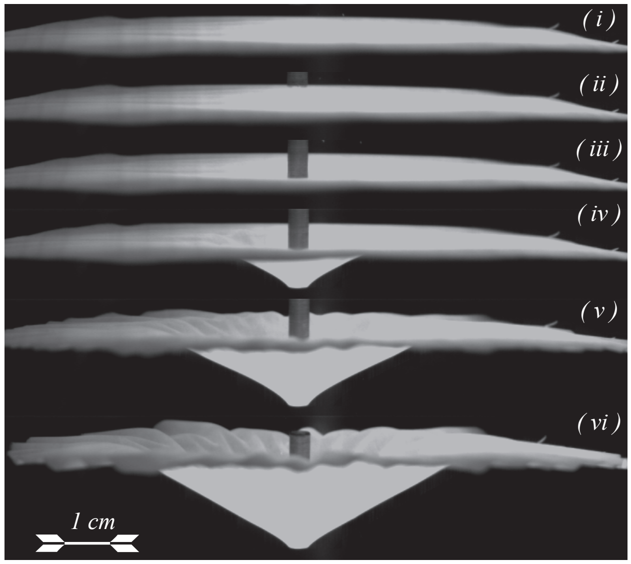


Fig 3: Side view of the impact of a latex membrane by a cylindric impactor. The impactor's speed is V= 3.7 m/s. The timestep between two frames is 0.50 ms.

σ_r is always positive and thus the material is stretched in the radial direction. However for $r < r_z$, with $r_z = [(1 - \nu)/(1 + \nu)]^{1/2}\ (r_i + ct)$, the hoop stress σ_θ is negative and thus circumferential compression occurs (Fig. 2ii). Similar stress fields occur in static configuration when an annular membrane is held at two different radii [6]. A similar situation is also encountered in the punching of metal plates [7] and in liquids [8]. In the present example, the difference between the wave speeds of longitudinal and transverse perturbations allows the development of an area with negative hoop stress. In a similar static situation, a circular membrane clamped at its outer radius, the flat stretched area does not exist. No wrinkling instability is observed in that case.

4 Wrinkling instability

Thin sheets cannot withstand compressive stresses. When compressive stresses are present, the sheet buckles. In the experiment, we observe radial wrinkles (Fig. 3 and 4) with a well defined wavelength in the stretched domain outside the cone. The number of wrinkles depends on the impactor's speed V. We use a quasistatic approach to study the development of a non axisymmetric transverse mode: at each time t, we consider the annulus bounded by r_c and r_z. We use the following *ansatz* for the transverse displacement

$$\xi(r,\theta,t) = \frac{(r-r_c)(r-r_z)}{(r_z-r_c)^2}\,\xi_n \sin(n\theta)\exp(\gamma_n t) \qquad (8)$$

The displacement is zero at r_c and r_z. We use Rayleigh's method to compute the growth rate γ_n of the displacement given by Eq. (8). The different energies are: the bending energy [4], the membrane energies associated with in plane stresses σ_r

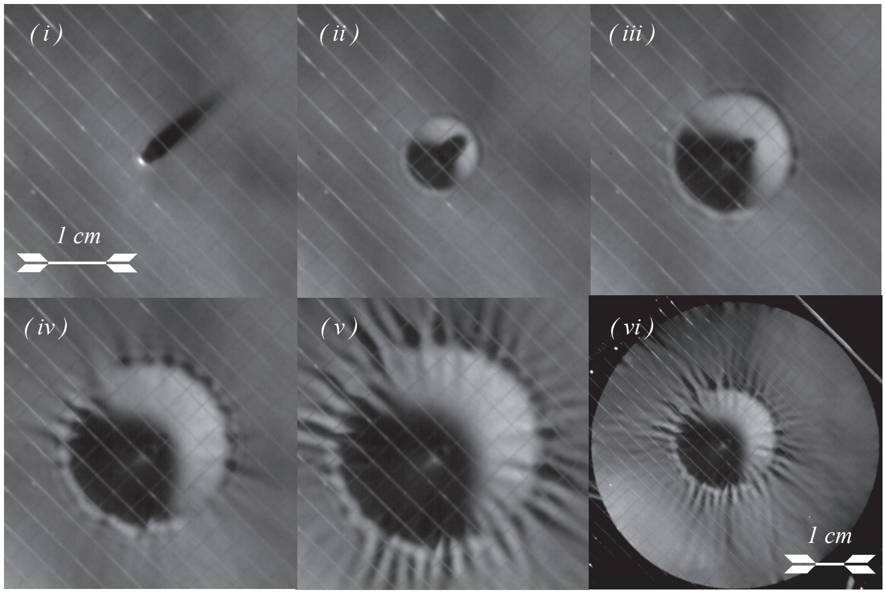


Fig 4: *Bottom view of the impact of a steel ball at speed V=5.2 m/s on a circular latex membrane. The timestep between each frame is 0.50 ms.*

and σ_θ and the kinetic energy. Minimizing with respect to ξ_n yields the dispersion relation and thus γ_n. At short times, all wavenumber n are stable and the γ_n are imaginary. At a critical time an instability occurs and γ_n becomes positive for a finite n. We use this critical wavenumber to obtain the critical wavelength $\lambda = 2\pi r_c/n$. It agrees with the wavelength of the radial pattern observed in the experiment (Fig. 5). It is in general not possible to derive a simple formula for the

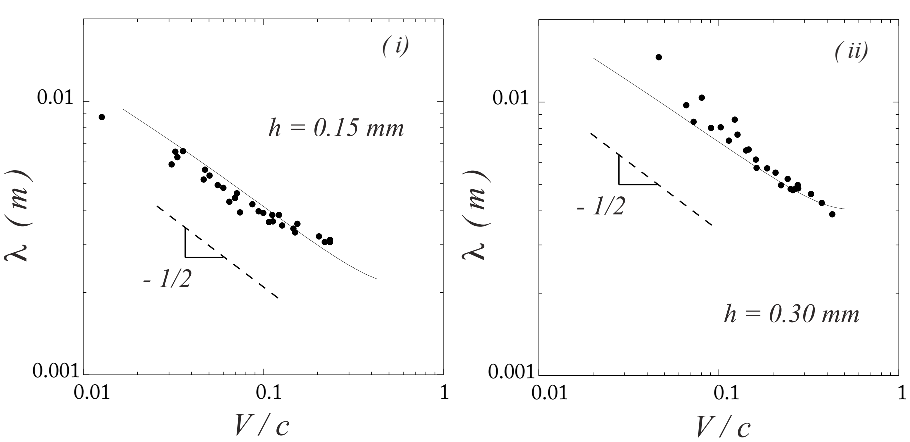


Fig 5: *Wavelength of the wrinkle pattern. The solid line shows the result of the model.*

selected wavelength. However a simple model in which the annulus is replaced by a beam (as in [7]) predicts a wavelength $\lambda \propto h\ (V/c)^{-1/2}$. This scaling is also observed in experiments (Fig. 5).

In our theory, the compressive hoop stress increases indefinitely with time. Thus if the development of the stress field is not perturbed by the limited size of the membrane, the instability will occur. This is an important difference between this problem and similar static problems [6-8] where a threshold depending on boundary conditions can be determined. In the present case, the limitation due to size occurs when the tensile front rebounds on the boundaries. Otherwise, like in our experiment, the nature of the boundary does not influence the dynamics. In the theory we also neglect the deceleration of the impactor that occurs if its mass is small compared to the mass of the membrane. Also, the bending rigidity of the membrane that has been neglected in the computation of the stress field may modify the dynamics for a thicker sheet such as a glass window. The link between the present work and the radial cracks pattern observed in fractured glass is still under investigation.

5 Summary

We have studied the transverse impact at moderate speed of a rigid impactor on a free elastic sheet. Two different waves, a tensile wave and a transverse wave propagate in and on the sheet at different speeds. In the area located between the two fronts, compressive hoop stresses induce a buckling instability that give rise to radial wrinkles.

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