A generalized dimensional analysis approach to fatigue crack growth

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Abstract
Barenblatt & Botvina have pioneered dimensional analysis arguments to show that Paris' power-law shows "incomplete similarity", and the Paris' parameters C and m are not true material constants. We generalize the approach to explore the functional dependencies of m and C on more dimensionless parameters than Barenblatt & Botvina, and more experimental results, for materials including both metals and concrete. We discuss the size-scale dependencies of m and C which are quite different for the two class of materials, but explain known empirical correlations between the constants C and m.

1 Introduction

More than 40 years ago, Paris and Erdogan [1] suggested using the elastic stress-intensity factor range, ΔK, to obtain the rate of crack advance per cycle, $\mathrm{d}a/\mathrm{d}N$, a proposal that received a strong opposition from the scientific Community (see [2]). Actually, other authors had proposed special laws, and Paris' main contribution was to release the exponent in a power-law form, with m as a free parameter, since he had collected experimental data where m seemed to vary between 3 and 4, according to the following equation:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = C\Delta K^m \tag{1}$$

suggesting C and m would be material "constants", since they did not appear to depend much on the load ratio R or on other factors.

The progress in subsequent years has seen a proliferation of "generalized laws", mainly to model the various observed deviations from the power-law regime. For example, considering Paris' law to hold in an intermediate region II, it is admitted that for low ΔK a region I exists, where there is a decrease in the crack growth rate until below a threshold stress-intensity factor, ΔK_{th}, long cracks do not propagate anymore. This threshold significantly depends on the material microstructure, environmental aspects, as well as on the loading ratio. Similarly, for high ΔK values, a region near critical conditions exists where cracks tend to accelerate with respect to region II. More still, when experiments were able to investigate "short cracks", it appeared that another, and complicated, deviation occurs with respect to Paris' law in its original form where the crack advancement is larger.

Despite these difficulties, the "crack propagation" approach had evolved in a philosophy called "damage tolerance" for which it is hoped to estimate crack

advancement so as to propose inspection intervals safe enough for cracks not to be catastrophic. However, also because of the deviations from the Paris' simple regime, no computational model is entirely satisfactory today, even in the opinion of Paris himself [2]. Indeed, research is still active on "damage tolerance in HCF" (High Cycle Fatigue) where most of the design approach is based on threshold and fatigue limits, returning in part to the original SN curves "empirical" approach, and not using Paris' type of laws (see for example a recent US Air Force initiative in the excellent reviews by Nicholas [3,4]).
It is clear that the observation of the strong power-law nature of crack propagation comes from the underlying self-similarity of crack propagation connected to the self-similarity of the crack geometry, when the crack length a is larger than the microstructural dimensions, yet smaller than any other dimensions. However, the constants of the power law would be true material properties only if the power were fixed by dimensional analysis, as elucidated by Barenblatt and Botvina [5] (BB in the following, see also [6]). For example, assuming the crack growth process stemming from a perfect plasticity mechanism, complete similarity would imply m=2 in Eq. (1), which is not observed if not as nearly a limit case. Abandoning complete similarity means introducing incomplete similarity, for which Paris' law parameter m depends *a priori* on all other possible dimensionless number of the problem. BB considered for example the size of the specimen as additional length scale, introducing the dimensionless number $Z = \sigma_y \sqrt{h} / K_{\mathrm{IC}}$, where σ_y is the tensile strength, K_{IC} is the fracture toughness and h is the specimen thickness. They "provisionally" suggested that m should be constant for Z less than about unity and then linearly increase with Z. A mechanistic interpretation suggests that large specimens imply more "static" modes of failure, as it is well-known that constraint at the crack tip is higher for large enough width and thickness of the specimen (see also the prescription in ASTM E399-90, (2002) for toughness measurement, as further remarked by Ritchie [7]). Ritchie [7] also made interesting further comparisons of data points using this approach. However, Ritchie's plot seems to suggest the slope in the linear increase with Z to be very different for different materials, and this suggested us a generalization of the BB's approach to look for more general dependencies on dimensionless quantities. In the mean time, we also add to the original data points of the BB's and Ritchie's papers, other fatigue data for concrete obtained by Bazant and coworkers [8,9] and recently reexamined by Spagnoli [10], and we also analyze the dependence of the other constant C, rather than just m.

2 BB’s generalized

Extending BB's analysis to cyclic material properties, such as threshold stress-intensity factor range, ΔK_{th}, the fatigue limit, $\Delta\sigma_{fl}$, and adding the Young's modulus, E, we have:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \Phi\left(\frac{\Delta K}{K_{\mathrm{IC}}}, \frac{\Delta K_{\mathrm{th}}}{K_{\mathrm{IC}}}, \frac{\Delta\sigma_{fl}}{\sigma_y}, \frac{E}{\sigma_y}, \frac{{\sigma_y}^2}{{K_{\mathrm{IC}}}^2} h, \frac{{\Delta\sigma_{fl}}^2}{{\Delta K_{\mathrm{th}}}^2} a; 1-R\right) = \\ = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \Phi\left(\Pi_i\right), \tag{2}$$

where Π_i $(i = 1,\ldots,6)$ are dimensionless numbers. Note that Π_5 corresponds to the square of the dimensionless number *Z* introduced by Barenblatt and Botvina [5] and to the inverse of the square of the brittleness number *s* introduced by Carpinteri [11,12]. The parameter Π_6 is responsible for the dependence of the fatigue phenomenon on the initial crack length, as recently pointed out by Spagnoli [10]. In fact, if we introduce the El Haddad length scale [13]:

$$a_0 = \frac{1}{\pi}\left(\frac{\Delta K_{\mathrm{th}}}{\Delta\sigma_{fl}}\right)^2 \tag{3}$$

The power-law dependence on ΔK, in Paris' law corresponds in BB's terminology to incomplete similarity, or self-similarity of the second kind in the parameter Π_1:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}}{\sigma_y}\right)^2 \left(\frac{\Delta K}{K_{\mathrm{IC}}}\right)^{\beta_1} \Phi_1\left(\Pi_2, \Pi_3, \Pi_4, \Pi_5, \Pi_6, \Pi_7\right) = \left(\frac{K_{\mathrm{IC}}^{2-\beta_1}}{\sigma_y^2}\right) \Delta K^{\beta_1} \Phi_1\left(\Pi_i\right), \tag{4}$$

However, additional incomplete similarity may hold for the parameters Π_5, Π_6 and Π_7, generally the transition occurring at $\Pi_i \cong 1$:

$$\frac{\mathrm{d}a}{\mathrm{d}N} = \left(\frac{K_{\mathrm{IC}}^{2-\beta_1}}{\sigma_y^2}\right) \Delta K^{\beta_1} \left(\pi\frac{h}{r_p}\right)^{\beta_2} \left(\pi\frac{a}{a_0}\right)^{\beta_3} \left(1-R\right)^{\beta_4} \Phi_2\left(\Pi_2, \Pi_3, \Pi_4\right), \tag{5}$$

where, again, the exponents β_i may depend on Π_i.

3 Analysis of the functional dependences of the Paris' law parameters

The main point raised by BB was that, because of incomplete similarity, Paris' law parameter m may depend on Π_5, which corresponds to the square of the brittleness number *Z*. Analyzing Aluminium alloys, 4340 steel and low-carbon steels, BB [5] supposed that the relationship between *m* and *Z* has three regimes: $m \cong \text{constant}$ for small *Z*, *m* linear with *Z* for $1 < Z < 2$ and again $m \cong \text{constant}$ for large *Z*. These data, along with the data for ASTM steels and for normal and high strength concretes (the data refer to a loading ratio *R*=0) are reconsidered here to examine the role played by the dimensionless parameter $\Pi_4 = E/\sigma_y = \varepsilon_y$, since we find that the slope of the linear relationship between *m* and *Z* progressively decreases from Aluminium alloys to steels. For low-carbon steels, *m* becomes nearly independent of *Z* and the slope becomes negative-valued for normal and high strength concretes (see Fig. 1).

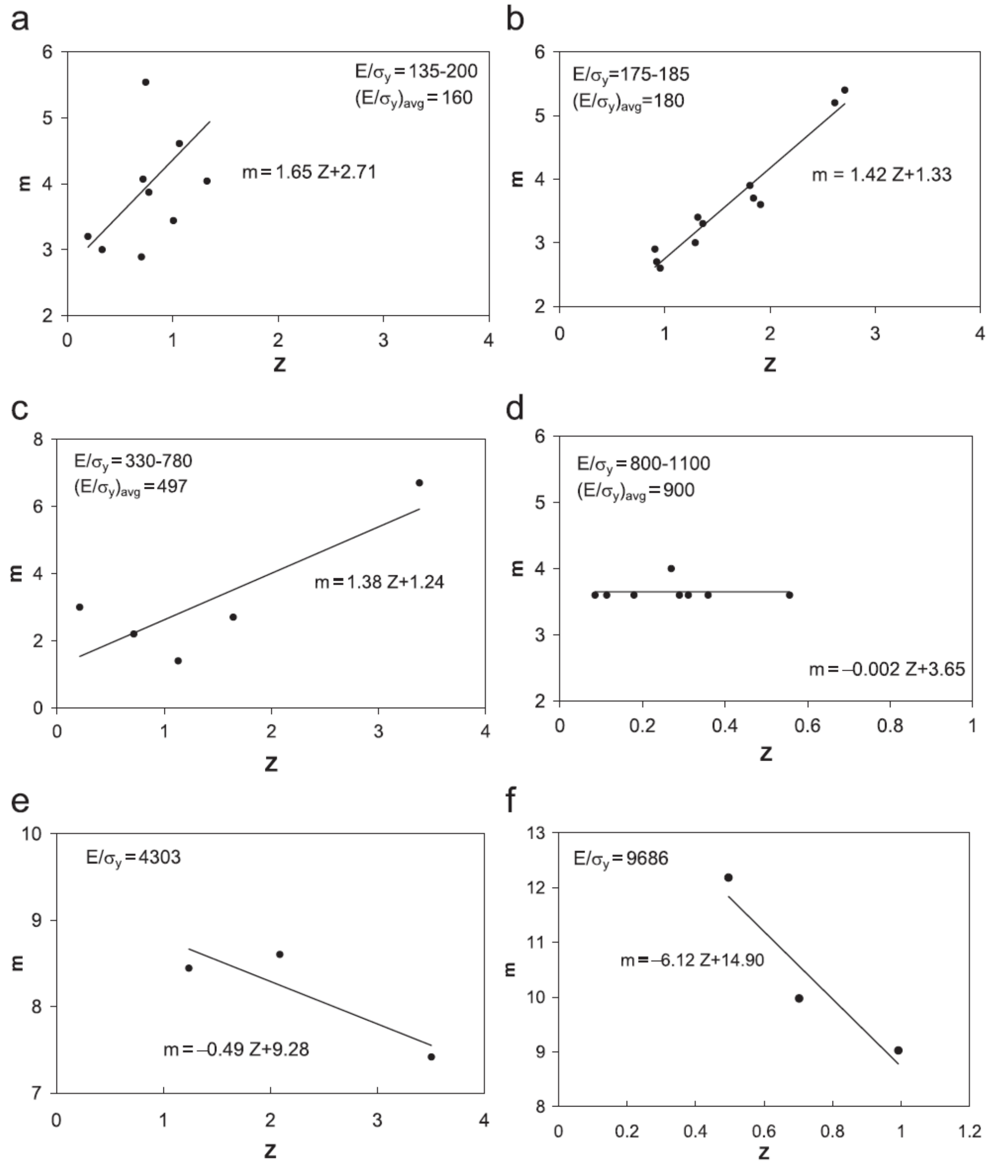

Figure 1. *Z*-dependence of the Paris' law parameter *m*. (a) Aluminium alloys [14,15]. (b) 4340 steel [16]. (c) ASTM steels [17]. (d) Low-carbon steel [18]. (e) High strength concrete (data from [9] reinterpreted by Spagnoli [10]). (f) Normal strength concrete (data from [8] reinterpreted by Spagnoli [10]).

The *Z*-dependence of the Paris' law parameter *C* can also be examined. This functional dependence has received a minor attention in the past as compared to that for m, although the variability of the parameter *C* is extremely important from the engineering point of view and its size-scale dependence may have strong consequences for the damage tolerance design of large scale structures. Dimensional analysis suggests two different scaling laws for C, depending on whether or not incomplete self-similarity in Π_5 takes place. For complete similarity, we have $\beta_2 = 0$ in Eq. (5) and the functional dependence of C on Z is no longer a power-law. On the other hand, if we have incomplete self-similarity in Π_5, then $\beta_2 \neq 0$ and a power-law dependence of C on Z can be obtained, i.e., $\log C \propto 2\beta_2 \log Z$.

To assess which of such conditions takes place, we can simply plot the available experimental data in the log C vs. Z and log C vs. log Z diagrams. The former plot would correspond to $C \propto 10^Z$, and therefore no power-law dependence between C and Z, whereas the latter gives the scaling $C \propto Z^{\beta_2}$ characteristic of incomplete self-similarity. A best fitting linear equation can be determined in both situations and the corresponding linear regression coefficient r^2 will provide their goodness of fit. The situation having r^2 closer to unity gives the best correlation. This procedure is performed in Fig. 2 for 4340 steel and ASTM steels. In this case, as can be seen, the log C vs. Z relationships present correlation coefficients higher than those for the log C vs. log Z representation. Therefore, this seems to exclude an incomplete self-similarity in Π_5 in metals.

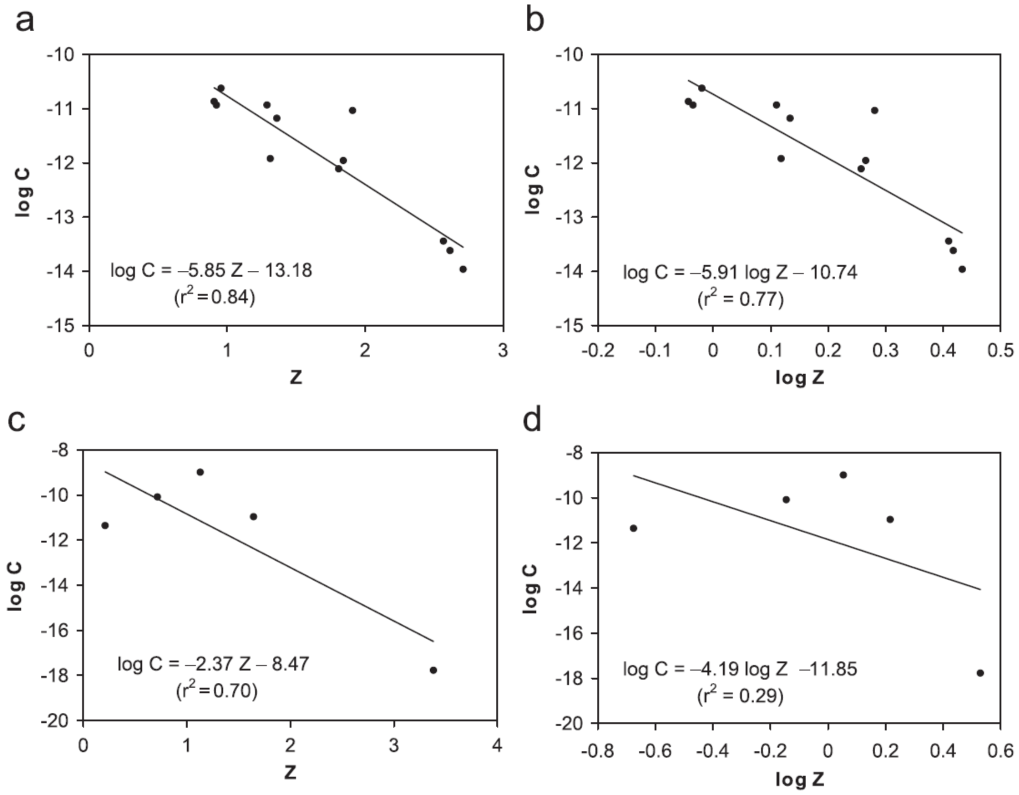

Figure 2. Assessment of incomplete self-similarity in Π_5 in metals (C evaluated using ΔK in MPa$\sqrt{m}$ and da/dN in m/cycle). (a) and (b) 4340steel [16]. (c) and (d) ASTM steels [17].

On the other hand, this cannot be considered as a universal result. In fact, for concrete, the situation is the opposite and the assumption of incomplete self-similarity in Π_5 gives the best correlation (see Fig. 3).

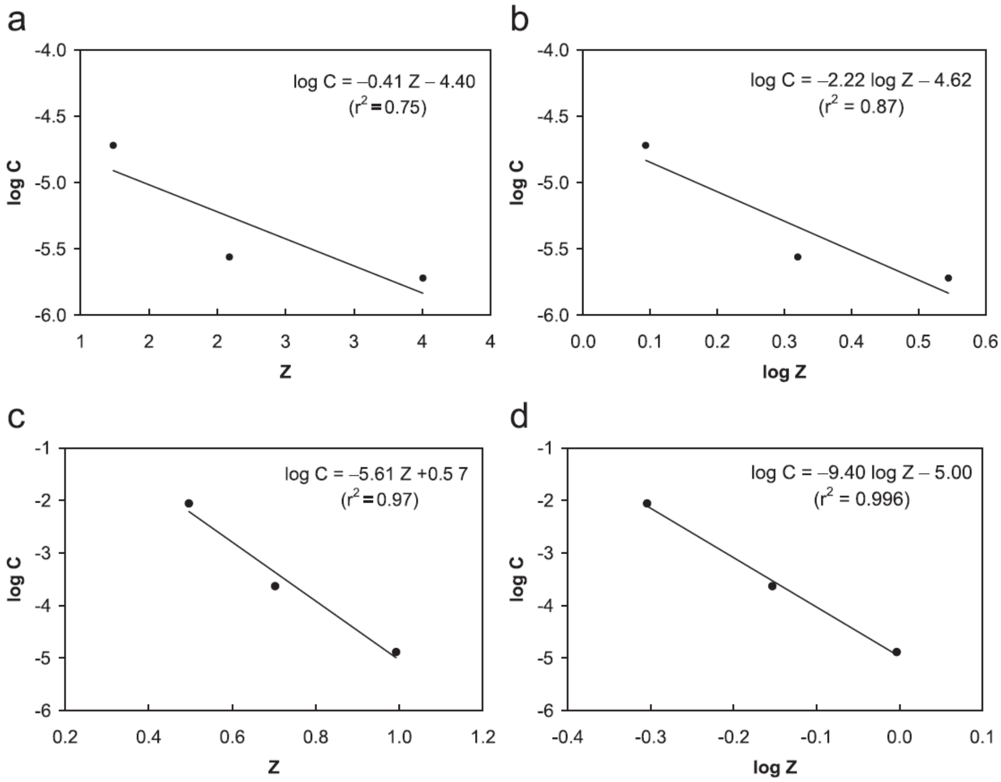

Figure 3. Assessment of incomplete self-similarity in Π_5 in concrete (C evaluated using ΔK in MPa$\sqrt{}$m and da/dN in m/cycle). (a) and (b) High strength concrete (data from [9] reinterpreted by Spagnoli [10]. (c) and (d) Normal strength concrete (data from [8] reinterpreted by Spagnoli [10]).

4 Correlations between the Paris' law parameters

It is useful at this point to step back and analyze the existing correlations between C and m. In fact, in view of the previous findings about the $m(Z)$ and $C(Z)$ relationships, it would be useful to see if eliminating Z between the two relations, the resulting correlation is more or less of the form obtained by previous Authors, who evidently were not looking specifically at size-scale effects. In fact, empirical correlations between C and m are usually obtained by considering a large series of samples with slightly different mechanical properties but with almost the same specimen size (see e.g. [19,20], among others).

As far as the previous findings about the $m(Z)$ and $C(Z)$ relationships are concerned, we have found the following functional dependencies in metals, where incomplete self-similarity in Π_5 does not apply:

$$\begin{aligned} m &= k_1 + k_2 Z, \\ \log C &= k_3 + k_4 Z, \end{aligned} \tag{6}$$

where k_i are best-fitting parameters. Determining Z from the first equation and introducing it in the second one, we find:

$$\log C = \left(k_3 - \frac{k_4 k_1}{k_2} \right) + \frac{k_4}{k_2} m, \tag{7}$$

which suggests a relationship between C and m very similar to that provided by the correlations in [19,20]. In fact, since the ratio k_4/k_2 is negative valued according to experimental data (see Figs. 1 and 2), log C is a decreasing function of m.

On the other hand, when the condition of incomplete self-similarity in Π_5 applies, like in concrete, then we have:

$$\begin{aligned} m &= k_1 + k_2 Z, \\ \log C &= k_5 + k_6 \log Z, \end{aligned} \tag{8}$$

Determining Z from the first equation and introducing it in the second one, we find, after some manipulation:

$$\log C = \left[k_5 + k_6 \log\left(-\frac{k_1}{k_2} \right) \right] - \frac{k_6}{k_1} m. \tag{9}$$

In this case, we note that the ratio k_6/k_1 is negative valued according to experimental data (see Figs. 1 and 3) and therefore log C should be an increasing function of m.

The experimental data regarding log C for 4340 steel, ASTM steels, high strength and low strength concretes previously analyzed are herein reported in Fig. 4 in terms of m. A best-fitting linear curve is shown with solid line in the same diagrams and the corresponding equations are written. Moreover, the correlations by Carpinteri and Paggi [19] and by Tanaka [20] are superimposed to the same diagrams whenever possible with dashed-dotted or dashed lines, respectively.

As argued before, the case of concrete appears quite singular in this respect, since incomplete self-similarity in Π_5 changes the slope of the log C vs. m relationship from negative to positive valued. In this case, the correlation by Carpinteri and

Paggi [19] would predict log *C* almost independent of *m* and is not able to capture the actual experimental trend.

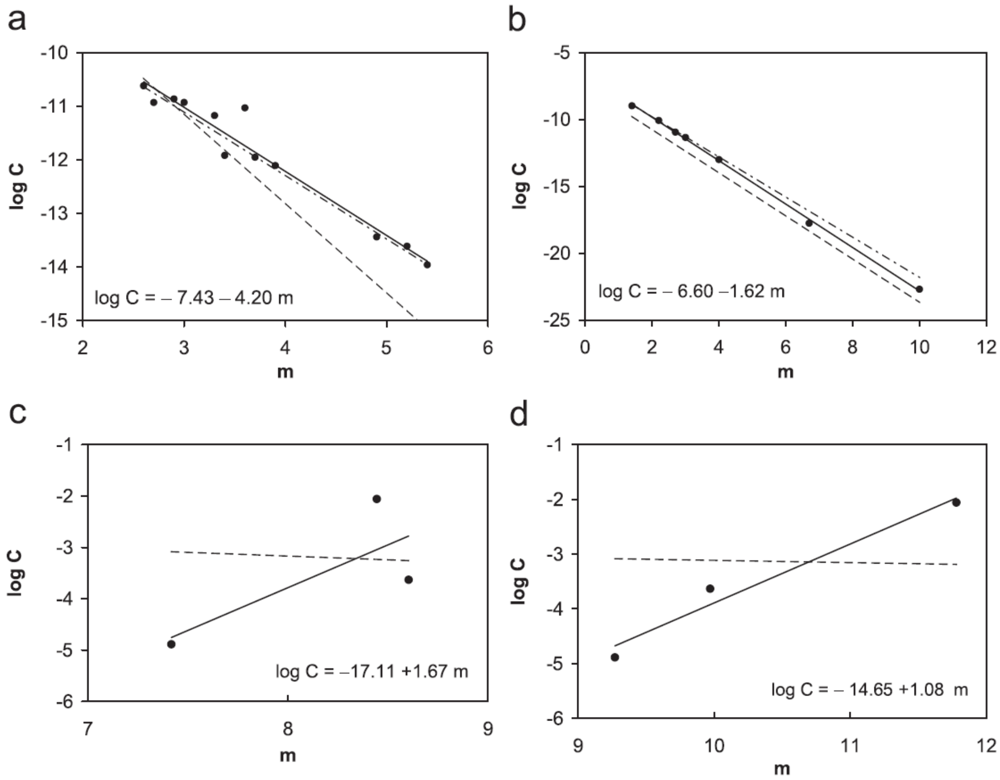

Figure 4. Correlations between *C* and *m* (*C* evaluated using ΔK in MPa√m and d*a*/d*N* in m/cycle). Dashed lines refer to the correlation by Carpinteri and Paggi [19], whereas dashed-dotted lines refer to the correlation by Tanaka [20] for steels. (a) 4340 steel. (b) ASTM steels. (c) High strength concrete. (d) Normal strength concrete.

5 Conclusions

If engineers have partly learned how to use deviations of Paris' regime, or to "adjust" *C* and *m*, for example by crack closure and effect of constraint, power-laws seem a guide, and no more than this. We should not expect them to be "laws", in the stricter and more specific sense of physical laws. Dimensional analysis and the concepts of complete and incomplete similarity help in elucidating various effects and various dependences on dimensionless parameters, but it is clear that this doesn't solve the problem of finding reliable estimates of crack growth.

Barenblatt and Botvina [5] concluded their paper with the hope that similar dimensional analysis arguments should be applied to the other power laws in fatigue, namely Basquin and Coffin-Manson, also to provide a unified framework, including also Hall-Petch relationships. While some of this work has been already done (see [21]), it appears that much more study is needed along these lines. So, while crack propagation criteria have for some time been perceived as less "empirical" than previous laws of fatigue (like those by Basquin and Coffin-Manson), it may appear that the contrary may well be true, since Basquin and Coffin-Manson's exponents tend to be less dependent on size effects. It should be in principle easy to check if this is indeed the case, since size effects on either fatigue limit or static strength are relatively well-known. Hence, it appears that the lesson of Barenblatt and Botvina should be reconsidered, and further extended.

Acknowledgements

The first author wishes to thank LMS-Ecole Polytechnique (Palaiseaux, Paris France) for inviting him to spend a wonderful and very productive sabbatical year there. M.P. and A.C. would like to acknowledge the financial support of the European Union to the Leonardo da Vinci Project ''Innovative Learning and Training on Fracture (ILTOF)'' I/06/B/F/PP-154069, where A.C. and M.P. are involved, respectively, as the Project Coordinator and the Scientific Secretary.

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