Unified Representation and Evaluation of the Strength Hypotheses

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The equivalent stress hypotheses allow us to compare an arbitrary stress state with a uni-dimesional one. These hypotheses are empirical, so some plausibility criteria are needed. Diversity of such criteria shows that the hypotheses of strength are still being developed. However as a whole they can be systemised.

Six modelling principles are suggested, leading to a small number of "all-purpose" models. These models contain well-established hypotheses and can be applied to different material classes.

The various forms of the meridian being of practical relevance can be received for these models. Unlike the known models, here the rotational, triangular or hexagonal symmetry in the deviatoric plane can be obtained independently from the compressibility. Meaningful are only the models with trigonal symmetry. In order to compute the parameters of the models a simple optimisation routine as well as certain constraints are proposed.

1. Introduction

The concept of the equivalent stress is used since XVII century to deliver a "compact" form of certain relevant information about the current stress state and its limits. To make an appropriate model choice the available experimental data (e.g. compression σ_-, torsion τ_*, hydrostatic compression σ^{hyd}, etc.) are compared with the results of tension test σ_+:

$$d = \sigma_- / \sigma_+, \qquad k = \sqrt{3}\,\tau_* / \sigma_+, \qquad a^{hyd} = \sigma^{hyd} / \sigma_+ \qquad (1)$$

In this case the tensile stress is the same as the equivalent stress: $\sigma_+ = \sigma_{eq}$.

The hypotheses can be expressed as a surface Φ in the principal stress space σ_I, σ_{II} and σ_{III}. To describe this surface, different sets of invariants can be used, e.g.

- the axiatoric-deviatoric invariants

$$I_1 = \sigma_{kk}, \qquad I_2' = \sigma_{ij}' \sigma_{ji}' / 2, \qquad I_3' = \sigma_{ij}' \sigma_{jk}' \sigma_{ki}' / 3 \qquad (2)$$

- the cylindrical invariants (invariants due to NOVOZHILOV)

$$I_1 = \sigma_{kk}, \qquad \sigma_{\text{vM}} = \sqrt{3 I_2'}, \qquad \cos 3\theta = \frac{3\sqrt{3}}{2} \frac{I_3'}{I_2'^{3/2}} \qquad (3)$$

The universal appearance of the model Φ is as follows:

$$\Phi(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_{eq}) = \Omega(I_1, I_2^{'}, I_3^{'}, \sigma_{\mathrm{eq}}) = \Psi(I_1, \sigma_{\mathrm{vM}}, \theta, \sigma_{eq}), \quad (4)$$

Where Φ, Ω, and Ψ are suitable functions. The invariants can be combined according to certain rules discussed later.

2. Additional constraints

The development of hypotheses results in formulation of additional constraints. The necessary conditions are:

- triangular symmetry in the π-plane, while rotational or hexagonal symmetry is taken out of consideration as an inadequate idealisation of the material behaviour,
- restrictions for the plastic POISSON's ratio at tension $\nu_+^{\mathrm{pl}} \in]-1;\, 1/2]$,
- convexity (not always necessary for failure criteria),
- suitable approximation of the experimental data.

However, the above formulated conditions are not enough to make a choice of the model. The "missing" sufficient conditions are replaced by a number of plausibility requirements [5]. That leads to a reduction of the number of models, which can be chosen. Such requirements are:

- simple and safe application, comprehensible models,
- evident physical background, not only abstract mathematical construct,
- explicit (not numeric) solvability with respect to σ_{eq},
- low number of parameters,
- dimensionless parameters,
- continuously differentiable models, also for limit surfaces; continuous derivative at the angular point („rounded top" due to FRANKLIN [4]),
- as small as possible power of stresses, presumably $n \leq 6$,
- maximal area of the convex solution in the π-plane, what usually leads to singular edges (e.g. the Models of TRESCA, SCHMIDT-ISHLINSKY-HILL),
- reliability of the result.

The large number of such plausibility requirements shows that the "strength theories" (strength hypotheses) are still being developed. However, these models joined together give us the possibility to systemise them. The systematisation allows deriving of generalised models, which contain established hypotheses.

3. Modelling principles

The analysis of the existing models [1-6] affords extracting of six modelling principles:

- models are to be built up from planes (DRUCKER's hexagonal pyramid, HAYTHORNTHWAITE's hexagonal prism, MOHR-COULOMB's hexagonal pyramid, RANKINE, SAYIR's hexagonal prism, SCHMIDT-ISHLINSKI-HILL, ST. VENANT, TRESCA, YU). For instance by taking all the permutations of indices in the equation below one obtains the model due to KO

$$\sigma_{\mathrm{eq}} - \sigma_I - \sigma_{II} + 2\sigma_{III} = 0\,. \qquad (5)$$

- preferably quadratic or cubic equations should be used for simple computation of σ_{eq} (DRUCKER's cylinder, SAYIR's trigonal prism, SCHMIDT-ISHLINSKY-HILL, TRESCA, VON MISES):

cubic
$$3I_2^{'}\,\sigma_{\mathrm{eq}} + c_3\,I_3^{'} = \sigma_{\mathrm{eq}}^3\,(1 + 2\,c_3/3^3)\,, \qquad (6)$$

bicubic
$$\frac{b_6 I_2^{'3} + c_6\,I_3^{'2} + b_4 I_2^{'2}\,\sigma_{\mathrm{eq}}^2 + 3\,I_2^{'}\,\sigma_{\mathrm{eq}}^4}{b_6/3^3 + 2^2\,c_6/3^6 + b_4/3^2 + 1} = \sigma_{\mathrm{eq}}^6\,, \qquad (7)$$

triquadratic
$$(3I_2^{'})^3 + c_6\,I_3^{'2} + c_3\,\sigma_{\mathrm{eq}}^3\,I_3^{'} = \sigma_{\mathrm{eq}}^6\,(1 + 2\,c_3/3^3 + 2^2\,c_6/3^6)\,. \qquad (8)$$

- infinitely expandable sum normalized to the exponent $r = 1, 2, 3...$ (ALTENBACH, DRUCKER's cylinder, FREUDENTHAL, IYER, SPITZIG) should be applicable in the compressible case

$$\frac{a_1 I_1^{\,r} + (a_2 I_1 + b_2\,I_2^{'})^{r/2} + (a_3 I_1^3 + d_3 I_1\,I_2^{'} + c_3 I_3^{'})^{r/3} + ...}{a_1 + (a_2 + b_2/3)^{r/2} + (a_3 + d_3/3 + 2\,c_3/3)^{r/3} + ...} = \sigma_{\mathrm{eq}}^r \qquad (9)$$

or in the incompressible case

$$\frac{(b_2 I_2^{'})^{r/2} + (c_3\,I_3^{'})^{r/3} + (e_5\,I_2^{'} I_3^{'})^{r/5} + (b_6\,I_2^{'3} + c_6 I_3^{'2})^{r/6} ...}{(b_2/3)^{r/2} + (2c_3/3^3)^{r/3} + (2\,e_5/3^4)^{r/5} + (b_6/3^3 + 2^3 c_6/3^6)^{r/6} ...} = \sigma_{eq}^r \qquad (10)$$

- models should be based on the stress angle (DESAI, DRUCKER's cylinder, EHLERS, cam-clay model in the FEM-program ABAQUS):

$$(3I_2^{'})^{n/2}\,(1 + c_3\cos 3\theta + c_6\cos^2 3\theta) = \sigma_{\mathrm{eq}}^n\,(1 + c_3 + c_6)\,. \qquad (11)$$

- the junction of two models with different types of symmetry controlled by one parameter only should be possible (MOHR-COULOMB, HAYHURST, LIPATOV, YU), e.g. the model of HOEK-BRAUN:

$$[(\sigma_I - \sigma_{II})^2 - \sigma_{\mathrm{eq}}^2] + (d^2 - 1)(\sigma_I - \sigma_{eq})\sigma_{\mathrm{eq}} = 0\,, \qquad d \in [1; +\infty[\,, \qquad (12)$$

- combined models (z.B. BECKER, COWAN, PAUL, HUBER, KUHN, YU) should be built up by intersection of the basic ones.

Note that some models can reside in several categories.

4. Models for Incompressible Material Behaviour

The most important models for incompressible material behaviour are: SAYIR's hexagonal prism [1-3]

$$\sigma_I - \frac{1}{1+b}(b\,\sigma_{II} + \sigma_{III}) - \sigma_{eq} = 0\,, \qquad b \in [-\frac{1}{2};\,1] \tag{13}$$

with $1/2 \le d \le 2$, $k = \sqrt{3}\,d\,/(d+1)$:and HAYTHORNTHWAITE's hexagonal prisms

$$\left(\frac{3(2\,d\,\sigma_{eq})\,I_2^{'} + (-3^2)\;I_3^{'}}{1+2\,(-3^2)/3^3} - (2\,d\,\sigma_{eq})^3\right)\left(\frac{3\sigma_{eq}\;I_2^{'} + (3^2/2)\;I_3^{'}}{1+2\,(3^2/2)/3^3} - \sigma_{eq}^3\right) = 0 \tag{14}$$

with two regions $1/2 \le d < 1$, $k = 2\,d\,/\sqrt{3}$ and $1 \le d \le 2$, $k = 2/\sqrt{3}$. These restrictions arise from the convexity condition.

Each „classical" model for incompressible material behaviour (VON MISES, TRESCA, SCHMIDT-ISHLINSKY-HILL, dodecagonal prism due to SOKOLOVSKIJ) corresponds to a single point in the $k-d$ Diagram. All these models represent some very special cases of material behaviour and hence are not flexible enough to be fitted, if numerous measurements are available.

The models with hexagonal symmetry (DRUCKER's cylinder, bicubic model (7), hexagonal prism due to YU) restrict the material behaviour to the case $d=1$, which is often not founded by an experimental evidence. The triangular prism due to SAYIR, Eq. (6) does not have this drawback.

The model Eq. (10) contains the convex cylinders due to FREUDENTHAL and DRUCKER. The model due to SPITZIG is not convex. The latter three models are not suitable for a general application.

For applications the geomechanical model Eq. (11) with $n=6$ can be recommended. The convexity condition yields [3]:

$$c_6 = \frac{1}{4}(2+c_3)\,, \qquad c_6 = \frac{1}{4}(2-c_3)\,, \qquad c_6 = \frac{1}{\sqrt[3]{13}}\,c_3^2 - \frac{1}{3}\,. \tag{15}$$

The convexity region of the triquadratic model Eq. (8) in the $k-d$-space is "a bit smaller" than the convexity region of the geomechanical model with $n=6$. The computation of the equivalent stress σ_{eq} can be done explicitly according to Eq. (11), therefore the model can be recommenden for applications.

5. Compressible Generalisation

The models for incompressible material behaviour allow a compressible generalisation using the transformation of SAYIR:

$$\sigma_{eq} \to \frac{\sigma_{eq} - \gamma_1\, I_1}{1-\gamma_1} \; . \tag{16}$$

The models Eq. (7) and Eq. (8) can be extended to the compressible behaviour using the transformations [1, 3]

$$\sigma_{eq}^2 \to \frac{\sigma_{eq} - \gamma_1\, I_1}{1-\gamma_1}\,\frac{\sigma_{eq} - \gamma_2\, I_1}{1-\gamma_2} , \tag{17}$$

$$\sigma_{eq}^3 \to \frac{\sigma_{eq} - \gamma_1\, I_1}{1-\gamma_1}\,\frac{\sigma_{eq} - \gamma_2\, I_1}{1-\gamma_2}\,\frac{\sigma_{eq} - \gamma_3\, I_1}{1-\gamma_3} \, . \tag{18}$$

The parameters γ_i describe the position of the hydrostatic nodes ($I_2^{'} = I_3^{'} = 0$). They are subjects to certain restrictions, which depend on the material.

In order to apply the transformation rule Eq. (18) to the geomechanical model Eq. (11) the right hand side of the model should be represented in the form $(\sigma_{eq}^3)^2$. Thus the model becomes suitable for a large variety of isotropic materials:

$$(3I_2^{'})^3\,\frac{1+c_3\cos 3\theta + c_6\cos^2 3\theta}{1+c_3+c_6} = \left(\frac{\sigma_{eq} - \gamma_1\, I_1}{1-\gamma_1}\,\frac{\sigma_{eq} - \gamma_2\, I_1}{1-\gamma_2}\,\frac{\sigma_{eq} - \gamma_3\, I_1}{1-\gamma_3}\right)^2 . \tag{19}$$

Unlike the well-known and wide-used models (e.g MOHR-COULOMB, HAYHURST), the type of the symmetry in the π-plane (rotationally symmetric, triangular, hexagonal) is not coupled to the compressibility/incompressibility. The resulting rotationally symmetric model ($c_3 = c_6 = 0$) provides more possibilities of fitting than the quadratic model $3I_2^{'} = \sigma_{eq}^2$ with the transformation rule Eq. (17).

6. Simple Objective Function

The model (19) can be rewritten in the form

$$(3I_2^{'})^3\,\frac{1+c_3\cos 3\theta + c_6\cos^2 3\theta}{1+c_3+c_6} - \left(\frac{\sigma_{eq} - \gamma_1\, I_1}{1-\gamma_1}\,\frac{\sigma_{eq} - \gamma_2\, I_1}{1-\gamma_2}\,\frac{\sigma_{eq} - \gamma_3\, I_1}{1-\gamma_3}\right)^2 = 0 \, . \tag{20}$$

Let the left hand side of the equation (20) be denoted by Φ. For a set of measurements given in the principal stresses $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, $i = 1 \ldots n$ an objective function can be formulated:

$$\mathtt{f} = \sum_{i=1}^{n} \left| \Phi(c_3, c_6, \gamma_1, \gamma_2, \gamma_3, \sigma_I^i, \sigma_{II}^i, \sigma_{III}^i) \right|^m \tag{21}$$

with $m = 1, 2$ or alternatively

$$\mathtt{f} = \max_{i=1..n} | \Phi(c_3, c_6, \gamma_1, \gamma_2, \gamma_3, \sigma_I^i, \sigma_{II}^i, \sigma_{III}^i) | . \tag{22}$$

Other values of m are possible as well, but they are not helpful in applications.

Now the following optimisation problem can be formulated:

$$\mathtt{minimize}\ \mathtt{f}(c_3, c_6, \gamma_1, \gamma_2, \gamma_3). \tag{23}$$

A solution leads to a model, which approximates the measurements $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, $i = 1...n$. The optimisation problem Eq. (23) is subjected to certain constraints.

7. Restrictons

The analysis of the experimental results requires restricting the parameters of the model in order to obtain physically meaningful solutions. The restrictions arise from certain plausibility considerations as well as from the geometry of the models and result in:

- Convexity conditions in the π-plane. For example they yield the restriction Eq. (15) in the parameter space (c_3, c_6) for the models Eq. (20).
- The POISSON's ratio at tension lies in the interval $\nu_+^{pl} \in]-1; 1/2]$. This constraint is physically meaningful for failure as well as for yield behaviour. This constraint can be strengthened for yielding and for ductile failure:

  $$\nu_+^{el} \le \nu_+^{pl} \le 1/2 . \tag{24}$$

  For brittle failure the inequality

  $$-1 < \nu_+^{pl} \le \nu_+^{el} \tag{25}$$

  is valid [3].
- For the materials, which fail at three-dimensional compression (e.g. hard foams, ceramics, sintered materials), the POISSON's ratio at compression lies in the interval $\nu_-^{pl} \in]-1; 1/2]$. Further a stronger condition $\nu_-^{pl} \le \nu_-^{el}$ can be conjectured:

  For all the other materials (steel, lead, cast iron), which do not fail under three-dimensional compression, the POISSON's ratio lies in the interval $\nu_-^{pl} \in [1/2;+\infty[$. From the geometrical considerations it follows:

$$\nu_-^{el} \leq \nu_-^{pl} . \qquad (26)$$

- The position of the hydrostatic nodes a_-^{hyd} and a_+^{hyd} is not arbitrary, this results in restrictions imposed on the γ_i. If the material does not fail under three-dimensional compression, the following two combinations can be recommended for applications: $\gamma_1 = \gamma_2 \in [0;1]$, $\gamma_3 = 0$ or $\gamma_1 \in [0;1]$, $\gamma_2 = \gamma_3 = 0$. Closed models can be obtained with the following parameter values: $\gamma_1 = \gamma_2 \in [0;1]$, $\gamma_3 < 0$; $\gamma_1 \in [0;1]$, $\gamma_2 = \gamma_3 < 0$ or $\gamma_1 \in [0;1]$, $\gamma_2 < 0$, $\gamma_3 = 0$. All three γ_i can have different values; however fitting of the models becomes rather complicated.
  An upper boundary for the hydrostatic tensile stress is obtained from the cone potential (model of DRUCKER-PRAGER):

  $$a_+^{hyd} < \frac{1}{1-2\nu_+^{pl}} . \qquad (27)$$

  It can be expected for hard foams (cf. the normal stress hypothesis):

  $$a_+^{hyd} \leq 3 . \qquad (28)$$

  A restriction for the hydrostatic compression for hard foams can be formulated, it has a purely empirical character (cf. the normal stress hypotheses in the form of triangular dipyramid):

  $$a_-^{hyd} \leq \frac{2}{3} d . \qquad (29)$$

- Restrictions of the relations

  $$k > 0, \qquad d > 0, \qquad a_+^{hyd} > 1/3, \qquad a_-^{hyd} > 1/3\, d \qquad (30)$$

  are self-evident, but sometimes are useful to exclude unphysical solutions.
- In addition, mixed constraints can be introduced. For instance for foams with $\sigma_{eq} = 1$ and $\gamma_2 < 0$ the following inequality can be formulated:

  $$\gamma_2 \geq -\frac{1}{d} \qquad (31)$$

- For σ_+ there are two options:
  - with $\sigma_{eq} = \sigma_+$ the results of the tension test are distinguished,
  - the equivalent stress σ_{eq} is considered as a free parameter. If the statistical scatter in the measurements is large, the relation σ_{eq} / σ_+ has to be restricted, e.g. $\tilde{\sigma}_{eq} \in [0.9;1.1]$, otherwise unphysical solutions can arise.

8. Evaluation criteria

Consider a failure surface, which is given in the form Eq. (4), and a set of experimental data in the principal stress space $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, $i = 1 \ldots n$. The quality of approximation of the experimental data by the surface must be evaluated. Three evaluation criteria are discussed (Fig. 1).

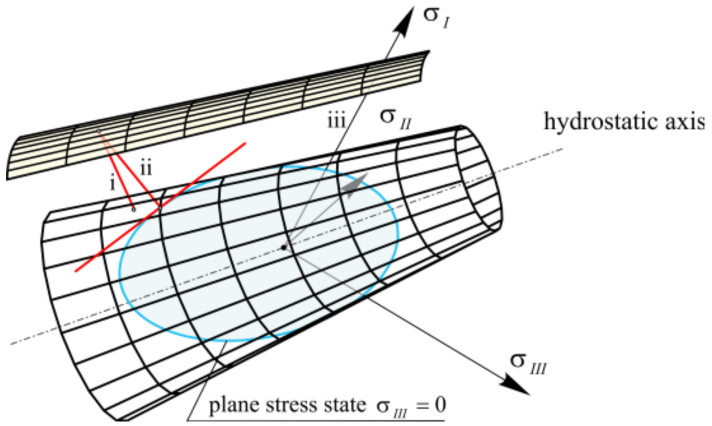

Fig. 1. Cone potential in the principal stress space. Three evaluation criteria are discussed:

i) minimal distance from the measurement to the surface,

ii) minimal distance from the measurement to the line of the plane stress space $\sigma_{III} = 0$,

iii) value of equivalent stress σ_{eq} is chosen so, that the point P lies on the surface Φ other model parameters are known and fixed.

The three criteria described below cannot be used to compute the parameters of the model because of their computational complexity. However, they still can be used to compare different optimisation criteria (i.e. objective functions Eq. (23)).

i) The quality of regression in the principal stress space is evaluated. For every measurement the distance to the failure surface in the principal stress space $(\sigma_I, \sigma_{II}, \sigma_{III})$ is computed and then averaged over all measurements. That means, that for each measurement $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, $i = 1 \ldots n$ the optimisation problem

$$\min_{(\sigma_I, \sigma_{II}, \sigma_{III})} [(\sigma_I - \sigma_I^i)^2 + (\sigma_{II} - \sigma_{II}^i)^2 + (\sigma_{III} - \sigma_{III}^i)^2], \quad \ldots..$$

$$\text{subject to } \Phi(\sigma_I, \sigma_{II}, \sigma_{III}) = 0 \tag{32}$$

must be solved. The solution is obtained using the LAGRANGE multiplier:

$$F(\sigma_I, \sigma_{II}, \sigma_{III}, \lambda) = (\sigma_I - \sigma_I^i)^2 + (\sigma_{II} - \sigma_{II}^i)^2 + (\sigma_{III} - \sigma_{III}^i)^2 - \lambda \Phi(\sigma_I, \sigma_{II}, \sigma_{III}).$$

Its stationary points are the solutions of the equation $\nabla F = 0$.

The latter equation has multiple solutions, however only the point $(z_I^i, z_{II}^i, z_{III}^i)$ on the failure surface, which has the minimal distance to the point $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, is needed. Since the number of different solutions is small, the point of minimal distance can be found by trial and error. Finally, the value f_{3D} of the estimation computes to

$$f_{3D} = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(\sigma_I^i - z_I^i)^2 + (\sigma_{II}^i - z_{II}^i)^2 + (\sigma_{III}^i - z_{III}^i)^2} \,. \tag{34}$$

ii) The second way to evaluate the quality of the approximation is to compute the minimal distance not in the principal stress space, but for a plane stress state (e.g. $\sigma_{III} = 0$). This criterion makes sense for special experimental data only. The optimisation problem (34) is reduced to

$$\min_{(\sigma_I, \sigma_{II}, \sigma_{III})} [(\sigma_I - \sigma_I^i)^2 + (\sigma_{II} - \sigma_{II}^i)^2], \qquad \text{subject to } \Phi(\sigma_I, \sigma_{II}, 0) = 0\,. \tag{35}$$

Further proceed analogous to the case i). For each measurement $(\sigma_I^i, \sigma_{II}^i, 0)$ find the point of minimal distance $(\sigma_I^i, \sigma_{II}^i, 0)$ on the curve $\Phi(\sigma_I, \sigma_{II}, 0) = 0$ and compute the value

$$f_{2D} = \frac{1}{n} \sum_{i=1}^{n} \sqrt{(\sigma_I^i - z_I^i)^2 + (\sigma_{II}^i - z_{II}^i)^2} \,. \tag{36}$$

iii) For a model built up upon the concept of equivalent stress, i.e.

$$\Phi(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_{eq}) = 0 \tag{37}$$

with $\sigma_{eq} = \sigma_+$, a simple evaluation criterion can be formulated. Whereas, for fitting of the model to the measurements, σ_{eq} is considered to be a parameter. The value of σ_{eq}, which was obtained from fitting, is denoted by σ_{eq}^*. In order to estimate the quality of approximation, for each measurement $(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i)$, $i = 1 \ldots n$ the value $\sigma_{eq} = \sigma_{eq}^i$ should be computed with the property, that the respective measurement belongs to the surface $\Phi(\sigma_I, \sigma_{II}, \sigma_{III}, \sigma_{eq}^i) = 0$. The equation $\Phi(\sigma_I^i, \sigma_{II}^i, \sigma_{III}^i, \sigma_{eq}^i) = 0$ must be solved with respect to σ_{eq}^i for each $i = 1 \ldots n$. The value of the estimation computes to

$$f_{eq} = \frac{1}{n} \sum_{i=1}^{n} \frac{\sigma_{eq}^i}{\sigma_{eq}^*} \tag{38}$$

The criterion i) is ubiquitous and can be applied to any model and any set of measurements. However, it is rather often the case, that the measurements are provided for a plane stress state, so the criterion ii) matches with the nature of the measurements much better than i). It is based on the assumption that the measurements, which belong to a plane stress state, are to be approximated by points on the model, which also belong to a plane stress state. The criterion iii) can be applied to the models based on the concept of equivalent stress only.

Further iii) can be expanded so that comparison is made for a distinguished stress state (e.g. torsion, compression, etc.).

11. Summary

The analysis of the stress hypotheses leads to six new modelling principles. According to that principles new formulations for the models were found, see Eq. (7), (8), (10), (11). However, the model Eq. (11) only satisfies the plausibility conditions. Compressible generalisation of Eq. (11) yields the model Eq. (20). The rotational, triangular or hexagonal symmetry in the deviatoric plane can be obtained independently from the compressibility.

The model Eq. (20) was chosen for the analysis of the measurements. The routine, which computes the values `f`, f_{3D}, f_{2D} and f_{eq}, was programmed in the CAS MATHEMATICA. The function `NMinimize` can be used in order to find the solution of the optimisation problem (23). The optimisation routine can be iterated in order to find the value $\nu_+^{pl} \in]-1;1/2]$ such that one of the values f_{3D}, f_{2D} or f_{eq} is minimal. The second condition in Eq. (15) was replaced by $c_6 \leq (1 - c_3 - 10^{-10} c_3^2)/4$, so that numerical computation becomes less complex.

For the measurements of a hard thermoplastic foam EPP-78 (closed surface Φ) and non-reinforced thermoplastic PMMA (surface with $a_-^{hyd} \to \infty$) different solutions were found, which could be compared using the criteria suggested here. Furthermore for the obtained solutions the values f_{3D}, f_{2D} and f_{eq} are computed and then such solutions are taken, for which two or three of the computed values are minimal (PARETO-solutions).

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