Stress intensity factors and the weight function for an elliptical crack.

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1. Introduction

Fatigue crack growth analysis requires accurate calculation of *stress intensity factors* (SIF) for planar cracks subjected to complex two-dimensional stress fields. Moreover, it is important to have a general method, which enables to calculate SIF for a given crack configuration under arbitrary applied stress field. The most efficient method is the weight function approach, proposed by Bueckner [1] and Rice [2].

The weight function represents the SIF induced by a pair of splitting unit forces. Hence, it depends only on the cracked body geometry. If the weight function for a given geometrical configuration is known, SIF due to any arbitrary applied stress field can be determined by integrating over the crack domain the product of the weight function and the applied stress field.

Unfortunately, for planar cracks the exact analytical weight function has been obtained only for a circular crack (Sneddon [3]). So far, an elliptical crack has been studied analytically only for polynomial types of loading (Kassir and Sih [4], Shah and Kobyashi [5], Nishioka and Atluri [6]).

We use the method of simultaneous dual integral equations (Atroshchenko et al.[7]) to obtain the weight function for an elliptical crack in the infinite elastic body. According to the method, the SIF is sought in the form of expansion into Fourier series and the governing integral equation for unknown crack

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opening displacement is transformed into the system of linear algebraic equations.

Consequently, we use the derived weight function to obtained the SIF induced by complex exponential stress field in a welded structure.

2. The weight function for an elliptical crack.

First we consider the boundary value problem for an elliptical crack (with semiaxes a and b, $b \leq a$) occupying the open domain $S : \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} < 1$ in the plane $z = 0$ in the infinite elastic body with elastic constants E and ν and opened up by the normal applied stress field $p(x, y)$ (Fig.1). As it was shown in [8] this

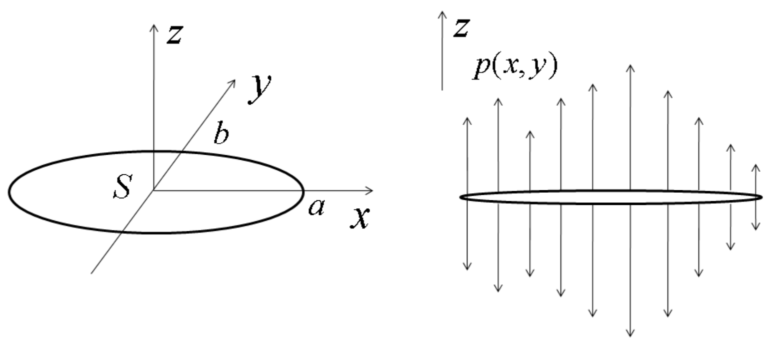

Figure 1: Crack position and arbitrary applied stress field.

problem can be transformed into the integral equation for the unknown crack opening displacement $w(x, y, 0)$ at any arbitrary point $Q(x, y) \in S$:

$$\Delta_{xy} \iint\limits_{S} \frac{w(\xi, \eta, 0)\, d\xi\, d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2}} = -\sigma(x, y),$$

where $\sigma(x,y) = \dfrac{4\pi(1-\nu^2)}{E}\,p\,(x,y)$ and Δ_{xy} is the two-dimensional Laplace operator:

$$\Delta_{xy} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

After the crack opening displacement has been found, SIF at the corresponding point of the crack contour $Q'(x' = a\cos\varphi, y' = b\sin\varphi)$ can be obtained as [9]:

$$K(\varphi) = \frac{E}{4(1-\nu^2)}\sqrt{\frac{\pi}{2}}\lim_{s\to 0}\frac{w(x,y,0)}{\sqrt{s}},$$

where s is the distance from the point $Q(x,y)$ to the crack front, i.e. the point $Q'(x',y')$ (Fig.2).

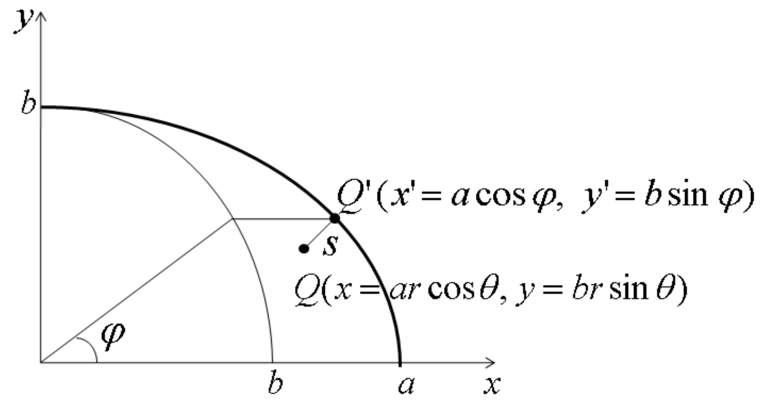

Figure 2: Distance to the crack front.

For convenience, we introduce the elliptical system of coordinates: $x = a\,r\cos\theta$, $y = b\,r\sin\theta$, then $S : r < 1$, $0 \le \theta \le 2\pi$. We also expand the normalized load into Fourier series

$$\sigma(x,y) = \sigma(r,\theta) = \frac{\sigma_0^c(r)}{2} + \sum_{n=1}^{\infty}\left(\sigma_n^c(r)\cos n\theta + \sigma_n^s(r)\sin n\theta\right).$$

As it was shown in [7] the SIF can be obtained in the following form:

$$K(\varphi) = \pi\sqrt{b}\left(\frac{b^2}{a^2}\cos^2\varphi + \sin^2\varphi\right)^{1/4}$$
$$\times\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}(-\boldsymbol{i})^n(-1)^k[A^c_{kn}\cos n\varphi + A^s_{kn}\sin n\varphi].$$

where unknown coefficients A^c_{km} can be found from the system of liner algebraic equations:

$$\sum_{m=0}^{\infty}\sum_{k=0}^{\infty} C^c_{nmki}A^c_{km} = B^c_{ni}, \tag{1}$$

where

$$C^c_{nmki} = \begin{cases} \alpha^c_{nm}, & m+2k = n+2i \\ 0, & \text{otherwise} \end{cases}$$

$$\alpha^c_{nm} = \int\limits_0^{2\pi}\sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\,\cos n\psi\cos m\psi\, d\psi,$$

$$B^c_{ni} = \frac{2(n+2i+\frac{3}{2})\Gamma(n+i+1)}{(-\boldsymbol{i})^n\Gamma(1+n)\Gamma(i+3/2)}\times$$
$$\int\limits_0^1 \sigma^c_n(r)r^{n+1}\sqrt{1-r^2}\,\mathfrak{F}_i(n+\tfrac{3}{2}, n+1, r^2)dr,$$

Here $\mathfrak{F}_i(n+\frac{3}{2}, n+1, r^2)$ is Jacobi Polynomial of degree k in r^2, defined by the hypergeometric function:

$$\mathfrak{F}_i(n+\tfrac{3}{2}, n+1, r^2) = {}_2F_1(-k, k+n+3/2, n+1, r^2).$$

Analogous system can be constructed for coefficients A^s_{kn} changing $\sigma^c_n(r)$ to $\sigma^s_n(r)$ and α^c_{nm} to

$$\alpha^s_{nm} = \int\limits_0^{2\pi} \sqrt{\frac{b^2}{a^2}\cos^2\psi + \sin^2\psi}\, \sin n\psi \sin m\psi \, d\psi.$$

To obtain the weight function corresponding to the constant force $P = 1$ applied at the point $Q_0(x_0 = a\,r_0\cos\theta_0, y_0 = b\,r_0\sin\theta_0) \in S$ we first derive the SIF induced by the following stress field (Fig.3):

$$\sigma(r,\theta) = \begin{cases} \dfrac{P}{4\,a\,b\,\varepsilon_1\varepsilon_2\,r_0}, & (r,\theta)\in\Omega \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\Omega = \begin{cases} r_0 - \varepsilon_1 \le r \le r_0 + \varepsilon_1, \\ \theta_0 - \varepsilon_2 \le \theta \le \theta_0 + \varepsilon_2 \end{cases}$$

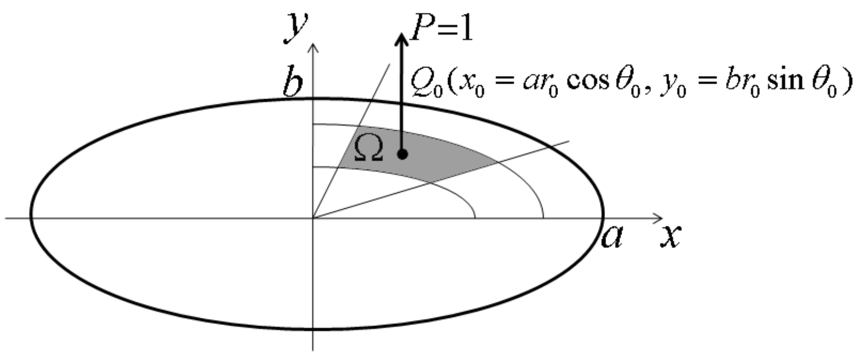

Figure 3: Point load is modeled as a constant pressure distributed over the small region.

After solving the system (1) and finding the limit as $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ we obtain the weight function for an elliptical crack $W(\varphi; r_0, \theta_0)$.

Consequently, SIF induced by an arbitrary applied stress field $\sigma(r_0, \theta_0)$ can be derived as:

$$K(\varphi) = \int_0^{2\pi} \int_0^1 W(\varphi; r_0, \theta_0)\sigma(r_0, \theta_0) a\, b\, r_0 dr_0 d\theta_0$$

3. Numerical results.

Using the derived weight function we calculate the SIF for an elliptical crack with semi-axes $a = 1$, $b = 0.5$ induced by the applied stress:

$$p(x, y) = \sigma_0 e^{-\frac{1}{2}\left[1-\left(\frac{x-\xi}{a}\right)^2\right]} \left[1 - \left(\frac{x-\xi}{a}\right)^2\right]$$

for three different cases: $\eta = \frac{\xi}{a} = 0$, $\eta = 0.5$, $\eta = 1$ (Fig.4).

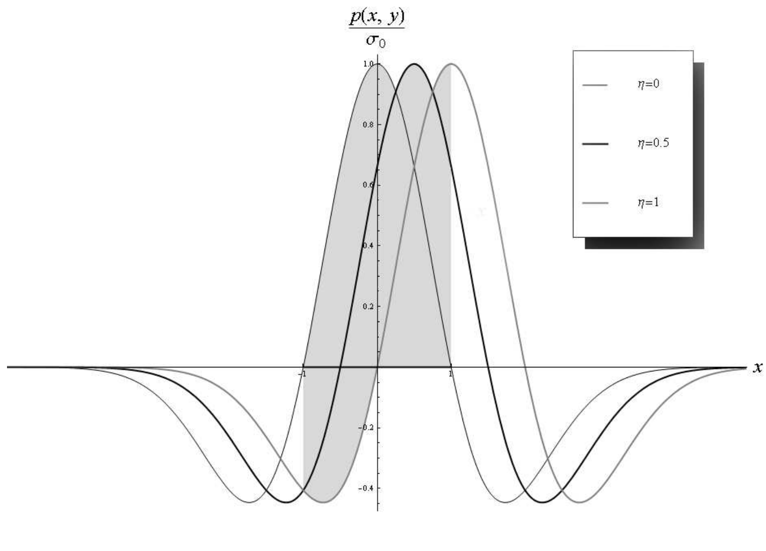

Figure 4: Applied stress field.

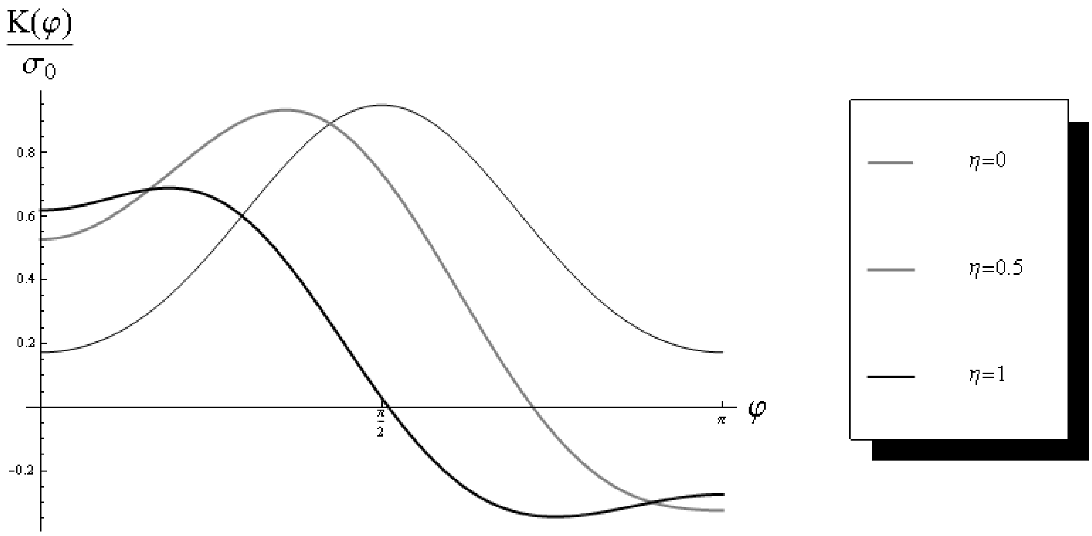

Figure 5: Variation of SIF along the contour.

The variation of SIF along the contour of a crack is shown in Fig.5.

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